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ON RICCI PSEUDO-SYMMETRIC SUPER QUASI-EINSTEIN HERMITIAN MANIFOLDS

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Abstract. The present paper deals the study of a Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold and a holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold.

1. Introduction

An even dimensional differentiable manifold M^n is said to be a Hermitian manifold if a complex structure J of type (1, 1) and a pseudo-Riemannian metric g of the manifold satisfy [13, 20]

$$J^2 = -I, \tag{1.1}$$

and

$$g(JX, JY) = g(X, Y), \tag{1.2}$$

where $X, Y \in \chi(M)$ and $\chi(M)$ is Lie algebra of vector fields on the manifold.

The notion of an Einstein manifold was introduced by Albert Einstein in differential geometry and mathematical physics. An Einstein manifold is a Riemannian or pseudo-Riemannian manifold $(M^n, g)(n \geq 2)$ in which Ricci tensor be a scalar multiple of the Riemannian metric i.e.

$$S(X, Y) = \alpha g(X, Y), \tag{1.3}$$

where S denote the Ricci tensor of the manifold $(M^n, g)(n \geq 2)$ and α is a non-zero scalar. According to [3], equation (1.3) is called the Einstein metric condition. An Einstein manifold plays a important role in the study of Riemannian geometry and general theory of relativity.

From the equation (1.3), we get

$$r = n\alpha. \tag{1.4}$$

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In 2000, M. C. Chaki and R.K. Maity [8] introduced a new type of a non-flat Riemannian manifold whose non-zero Ricci tensor satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.5}$$

and they called it a quasi-Einstein manfold, where α, β are scalars such that $\beta \neq 0$ and A is a non-zero 1-form associated with unit vector field ρ defined by $g(X,\rho) = A(X)$, for every vector field X. ρ is also called generator of the manifold. An n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.
Contraction of the equation (1.5), gives

$$r = \alpha\, n + \beta. \tag{1.6}$$

From the equations (1.2) and (1.5), we can easily write

$$\begin{aligned} &S(X,\rho) = (\alpha+\beta)A(X), \quad S(\rho,\,\rho) = (\alpha+\beta), \\ &g(J\rho,\,\rho) = 0 \quad and \quad S(J\rho,\,\rho) = 0. \end{aligned} \tag{1.7}$$

A quasi-Einstein manifold came in existence during the study of exact solutions of Einstein fields equations as well as considerations of a quasi-umbilical hypersurfaces of semi-Euclidean space. The Walker-space time is an example of a quasi-Einstein manifold. Also a quasi-Einstein manifolds can be taken as a model of the perfect fluid space time in general theory of relativity [17].

In 2001, M. C. Chaki [9] introduced the notion of generalised quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a generalised quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma C(X)C(Y), \tag{1.8}$$

where α, β and γ are scalars such that $\beta \neq 0$, $\gamma \neq 0$ and A, C are non-vanishing 1-forms associated with two orthogonal unit vectors ρ and μ by

$$\begin{aligned} &g(X,\rho) = A(X),\, g(X,\mu) = C(X), \\ &g(\rho,\,\rho) = g(\mu,\,\mu) = 1. \end{aligned} \tag{1.9}$$

An n-dimensional generalised quasi-Einstein manifold is denoted by $G(QE)_n$.
After contraction of equation (1.8), we get

$$r = \alpha\, n + \beta + \gamma. \tag{1.10}$$

From the equations (1.2), (1.8) and (1.9), we can easily write

$$\begin{aligned} &S(X,\,\rho) = (\alpha+\beta)A(X),\, S(X,\,\mu) = (\alpha+\gamma)C(X), S(\mu,\,\mu) = \alpha+\gamma, \\ &S(\rho,\,\rho) = \alpha+\beta,\;\, g(J\rho,\,\rho) = g(J\mu,\mu) = 0,\;\, and \quad S(J\mu,\,\mu) = S(J\rho,\,\rho) = 0. \end{aligned} \tag{1.11}$$

Also in 2004, M. C. Chaki [10] introduced the notion of super quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a super quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)C(Y) + C(X)A(Y)] + \delta D(X,Y), \tag{1.12}$$

where $\alpha,\ \beta,\ \gamma$ and δ are non-zero scalars, A, C are non-vanishing 1-forms defined as (1.9) and $\rho,\ \mu$ are orthogonal unit vector fields, D is symmetric tensor of type (0, 2) with zero trace which satisfies the condition

$$D(X,\rho) = 0,\ \forall\ X. \tag{1.13}$$

An n-dimensional super quasi-Einstein manifold is denoted by $S(QE)_n$.
From the equations (1.2), (1.9), (1.12) and (1.13), we can easily write

$$\begin{aligned} &S(X,\,\rho) = (\alpha+\beta)A(X) + \gamma C(X),\ S(X,\,\mu) = \alpha C(X) + \gamma A(X),\\ &S(\mu,\,\mu) = \alpha + \delta D(\mu,\mu),\ S(\rho,\,\rho) = \alpha+\beta+\delta D(\rho,\rho),\ \ g(J\rho,\,\rho) = g(J\mu,\mu) = 0,\\ &S(J\mu,\,\mu) = \gamma A(J\mu) + \delta D(J\mu,\mu),\ \ S(J\rho,\,\rho) = \gamma C(J\rho) + \delta D(J\rho,\rho). \end{aligned} \tag{1.14}$$

In 2009, A. A. Shaikh [1] introduced the notion of pseudo quasi-Einstein manifold. A semi-Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a pseudo quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \delta D(X,Y), \tag{1.15}$$

where $\alpha,\ \beta,$ and δ are non-zero scalars and A is a non-zero 1-form defined by $g(X,\rho) = A(X)$. ρ denotes the unit vector called the generator of the manifold and D is symmetric tensor of type (0, 2) with zero trace defined as (1.13). An n-dimensional pseudo quasi-Einstein manifold is denoted by $P(QE)_n$.
From the equations (1.2), (1.9) (1.13) and (1.15), we can easily write

$$\begin{aligned} &S(X,\,\rho) = (\alpha+\beta)A(X),\ S(\rho,\,\rho) = \alpha+\beta,\\ &g(J\rho,\,\rho) = 0\ \textit{and}\ S(J\rho,\,\rho) = \delta D(J\rho,\,\rho). \end{aligned} \tag{1.16}$$

2. Semi-symmetric and Ricci pseudo-symmetric manifold

Let (M^n, g) be a Riemannian manifold and ∇ be the Levi-Civita connection on (M^n, g) then, a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). The locally symmetric manifold has been studied by different geometers through different aproaches and different notion have been developed e.g., a semi-symmetric manifold by *Szabò* [18], recurrent manifold by Walker [2], conformally recurrent manifold by Adati and Miyazawa [16].

According to Z. I. *Szabò*[18], if the manifold M satisfies the condition

$$(R(X,Y).R)(U,V)W = 0, \quad X,Y,U,V,W \in \chi(M) \tag{2.1}$$

for all vector fields X and Y, then the manifold is called a semi-symmetric manifold. For a (0, k)- tensor field T on M, $k \geq 1$ and a symmetric (0, 2)-tensor field A on

M, the $(0, k+2)$-tensor fields R.T and Q(A, T) are defined by

$$(R.T)(X_1, ...X_k; X, Y) = -T(R(X,Y)X_1, X_2, ...X_k) \\ - .... - T(X_1, ...X_{k-1}, R(X,Y)X_k), \tag{2.2}$$

and

$$Q(A,T)(X_1, ...X_k; X, Y) = -T((X \wedge_A Y)X_1, X_2, ...X_k) \\ - .... - T(X_1, ...X_{k-1}, (X \wedge_A Y)X_k), \tag{2.3}$$

where $X \wedge_A Y$ is the endomorphism given by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{2.4}$$

Definition 2.1. ([19]) An n-dimensional Riemannian manifold (M^n, g) is said to be Ricci pseudo-symmetric if and only if the tensors R.S and Q(g, S) are linearly dependent, i.e.

$$(R(X,Y).S)(Z,W) = L_S Q(g,S)(Z,W; X,Y), \tag{2.5}$$

holds on U_S, where $U_S = [x \in M : S \neq \frac{r}{n} g \ \ at \ \ x]$ and L_S is a certain function on U_S.

The above developments allow to several authors for the generaliseation of the notion of quasi Einstein manifolds. In this process generalized quasi-Einstein manifolds are studied by Prakasha and Venkatesha [7] and N(k)-quasi Einstein manifolds are studied by [6, 11]. In 2012, S. K. Hui and R. S. Lemence [15] discussed generalised quasi-Einstein manifold addmitting a W_2- curvature tensor and they proved that if a W_2- curvature tensor satisfies $W_2.S = 0$, then either the associated scalars β and γ are equal or the curvature tensor R satisfies a definite condition. D. G. Prakasha and H. Venkatesha [7] studied some results on generalised quasi-Einstein manifolds and they proved that in generalised quasi-Einstein manifold if a conharmonic curvature tensor satisfies $L.S = 0$, then either M is a nearly quasi-Einstein manifold $N(QE)_n$ or the curvature tensor R satisfies a definite condition. Recently B. B. Chaturvedi and B. K. Gupta [4] studied Bochner Ricci semi-symmetric Hermitian manifold and B. K. Gupta, B. B. Chaturvedi and M. A. Lone [5] studied Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold. We have gone through the above developments in quasi-Einstein manifold $(QE)_n$, generalised quasi- Einstein manifold $G(QE)_n$, a super quasi-Einstein manifold and decide to study Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold and holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold.

3. Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold

The notion of Bochner curvature tensor was introduced by S. Bochner [14]. The Bochner curvature tensor B is defined by

$$\begin{aligned}B(Y,Z,U,V) =&R(Y,Z,U,V) - \frac{1}{2(n+2)}\Big\{S(Y,V)g(Z,U) - S(Y,U)g(Z,V)\\&+ g(Y,V)S(Z,U) - g(Y,U)S(Z,V) + S(JY,V)g(JZ,U)\\&- S(JY,U)g(JZ,V) + S(JZ,U)g(JY,V) - g(JY,U)S(JZ,V)\\&- 2S(JY,Z)g(JU,V) - 2g(JY,Z)S(JU,V)\Big\}\\&+ \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\\&+ g(JZ,U)g(JY,V) - g(JY,U)g(JZ,V) - 2g(JY,Z)g(JU,V)\Big\},\end{aligned} \tag{3.1}$$

where r is a scalar curvature of the manifold.
In a Hermitian manifold a Bochner curvature tensor satisfies the condition

$$B(X,Y,U,V) = -B(X,Y,V,U). \tag{3.2}$$

We introduce the following:

Definition 3.1. A Hermitian manifold is said to be a super quasi-Einstein Hermitian manifold if it satisfies the equation (1.12). Throughout this paper, we denote the super quasi-Einstein Hermitian manifold by $S(QEH)_n$.

Definition 3.2. An even dimensional Hermitian manifold (M^n, g) is said to be a Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if and only if the tensors B.S and Q(g, S) are linearly dependent i.e.

$$(B(X,Y).S)(Z,U) = L_S Q(g,S)(Z,U;X,Y), \tag{3.3}$$

holds on U_S, where $U_S = [x \in M : S \neq \frac{r}{n} g \ \ at \ \ x]$ and L_S is a certain function on U_S. If we take a Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold, then from equation (3.3) and (1.12), we have

$$\begin{aligned}&S(B(X,Y)Z,U) + S(Z,B(X,Y)U)\\&= L_S[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)].\end{aligned} \tag{3.4}$$

Using equation (1.12) in equation (3.4), we get

$$\begin{aligned}
&\alpha[g(B(X,Y)Z,U)+g(Z,B(X,Y)U)]\\
&+\beta[A(B(X,Y)Z)A(U)+A(B(X,Y)U)A(Z)]\\
&+\gamma[A(B(X,Y)Z)C(U)+A(U)C(B(X,Y)Z)\\
&+A(Z)C(B(X,Y)U)+A(B(X,Y)U)C(Z)]\\
&+\delta[D(B(X,Y)Z,U)+D(Z,B(X,Y)U)]\\
&=L_S\Big(\beta[g(Y,Z)A(X)A(U)-g(X,Z)A(Y)A(U)\\
&+g(Y,U)A(X)A(Z)-g(X,U)A(Y)A(Z)]\\
&+\gamma[g(Y,Z)[(A(X)C(U)+A(U)C(X)]-g(X,Z)[A(Y)C(U)+A(U)C(Y)]\\
&+g(Y,U)[A(X)C(Z)+A(Z)C(X)]-g(X,U)[A(Y)C(Z)+A(Z)C(Y)]]\\
&+\delta[g(Y,Z)D(X,U)-g(X,Z)D(Y,U)+g(Y,U)D(X,Z)-g(X,U)D(Y,Z)]\Big).
\end{aligned} \tag{3.5}$$

Using equation (3.2) in equation (3.5) we infer

$$\begin{aligned}
&\beta[A(B(X,Y)Z)A(U)+A(B(X,Y)U)A(Z)]\\
&+\gamma[A(B(X,Y)Z)C(U)+A(U)C(B(X,Y)Z)\\
&+A(Z)C(B(X,Y)U)+A(B(X,Y)U)C(Z)]\\
&+\delta[D(B(X,Y)Z,U)+D(Z,B(X,Y)U)]\\
&=L_S\Big\{\beta[g(Y,Z)A(X)A(U)-g(X,Z)A(Y)A(U)\\
&+g(Y,U)A(X)A(Z)-g(X,U)A(Y)A(Z)]\\
&+\gamma[g(Y,Z)[(A(X)C(U)+A(U)C(X)]-g(X,Z)[A(Y)C(U)+A(U)C(Y)]\\
&+g(Y,U)[A(X)C(Z)+A(Z)C(X)]-g(X,U)[A(Y)C(Z)+A(Z)C(Y)]]\\
&+\delta[g(Y,Z)D(X,U)-g(X,Z)D(Y,U)\\
&+g(Y,U)D(X,Z)-g(X,U)D(Y,Z)]\Big\}.
\end{aligned} \tag{3.6}$$

Now putting $U=Z=\rho$, we have

$$2\gamma\Big\{B(X,Y,\rho,\mu)-L_S[C(X)A(Y)-C(Y)A(X)]\Big\}=0. \tag{3.7}$$

This implies either $\gamma=0$ or

$$B(X,Y,\rho,\mu)=L_S[C(X)A(Y)-C(Y)A(X)]. \tag{3.8}$$

If $\gamma=0$, then from equation (1.12), we obtain

$$S(X,Y)=\alpha g(X,Y)+\beta A(X)A(Y)+\delta D(X,Y). \tag{3.9}$$

This is the condition of a pseudo quasi-Einstein manifold.
Thus we conclude:

Theorem 3.3. *A Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold is either a Bochner Ricci pseudo-symmetric pseudo quasi-Einstein Hermitian manifold or*

$$B(X,Y,\rho,\mu) = L_S[C(X)A(Y) - C(Y)A(X)].$$

From equation (3.7), we can also conclude:

Corollary 3.1. *In a Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if $\gamma \neq 0$ then $B(X,Y,\rho,\mu) = 0$ if and only if the vector fields ρ and μ corresponding to 1-forms A and C respectively are codirectional.*

4. Bochner flat Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold with $(B(X,Y).S)(Z,U) = L_S Q(g,S)(Z,U;X,Y)$

If we take a Bochner flat curvature tensor then from equation (3.1), we have

$$\begin{aligned} R(Y,Z,U,V) = \frac{1}{2(n+2)}\Big\{&S(Y,V)g(Z,U) - S(Y,U)g(Z,V) \\ &+ g(Y,V)S(Z,U) - g(Y,U)S(Z,V) + S(JY,V)g(JZ,U) \\ &- S(JY,U)g(JZ,V) + S(JZ,U)g(JY,V) - g(JY,U)S(JZ,V) \\ &- 2S(JY,Z)g(JU,V) - 2g(JY,Z)S(JU,V)\Big\} \\ &- \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V) \\ &+ g(JZ,U)g(JY,V) - g(JY,U)g(JZ,V) - 2g(JY,Z)g(JU,V)\Big\}. \end{aligned} \tag{4.1}$$

From equations (2.5) and (4.1), we infer

$$\begin{aligned} &\frac{1}{(2n+4)}\Big\{S(QY,V)g(Z,U) - g(Y,U)S(QZ,V) + S(QJY,V)g(JZ,U) \\ &- g(JY,U)S(JQZ,V) - 2g(JY,Z)S(JQU,V) \\ &+ S(QY,U)g(Z,V) - g(Y,V)S(QZ,U) + S(QJY,U)g(JZ,V) \\ &- g(JY,V)S(JQZ,U) - 2g(JY,Z)S(JQV,U)\Big\} \\ &- \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)S(Y,V) - g(Y,U)S(Z,V) + g(JZ,U)S(JY,V) \\ &- g(JY,U)S(JZ,V) + g(Z,V)S(Y,U) - g(Y,V)S(Z,U) \\ &+ g(JZ,V)S(JY,U) - g(JY,V)S(JZ,U)\Big\} \\ &= L_S[g(Z,U)S(Y,V) - g(Y,U)S(Z,V) + g(Z,V)S(Y,U) - g(Y,V)S(Z,U)]. \end{aligned} \tag{4.2}$$

If we take λ be an eigen value of Q and JQ corresponding to eigen vectors X and JX respectively then $QX = \lambda X$ and $QJX = \lambda JX$ i.e. $S(X,U) = \lambda g(X,U)$ (where the manifold is not Einstein) and hence

$$S(QX,U) = \lambda S(X,U) \quad and \quad S(QJX,U) = \lambda S(JX,U). \tag{4.3}$$

Using equation (4.3) in equation (4.2), we infer

$$\begin{aligned}
&\left(\frac{\lambda}{(2n+4)} - \frac{r}{(2n+2)(2n+4)}\right)\Big\{S(Y,V)g(Z,U) - g(Y,U)S(Z,V) \\
&+ S(Y,U)g(Z,V) - g(Y,V)S(Z,U) + S(JY,V)g(JZ,U) \\
&- S(JZ,V)g(JY,U) + S(JY,U)g(JZ,V) - g(JY,V)S(JZ,U)\Big\} \\
&= L_S[g(Z,U)S(Y,V) - g(Y,U)S(Z,V) + g(Z,V)S(Y,U) - g(Y,V)S(Z,U)].
\end{aligned} \tag{4.4}$$

Now putting $V = U = \rho$, we get

$$\begin{aligned}
&\left(\frac{\lambda}{(2n+4)} - \frac{r}{(2n+2)(2n+4)}\right)\Big([S(Y,\rho)g(Z,\rho) - g(Y,\rho)S(Z,\rho) \\
&+ S(JY,\rho)g(JZ,\rho) - S(JZ,\rho)g(JY,\rho)]\Big) \\
&= L_S[g(Z,\rho)S(Y,\rho) - g(Y,\rho)S(Z,\rho)].
\end{aligned} \tag{4.5}$$

Now using equations (1.9) and (1.14) in equation (4.5), we have

$$\begin{aligned}
&\gamma\left(\left(\frac{\lambda}{(2n+4)} - \frac{r}{(2n+2)(2n+4)}\right) - L_S\right)[C(Y)A(Z) - C(Z)A(Y)] \\
&= \gamma\left(\frac{\lambda}{(2n+4)} - \frac{r}{(2n+2)(2n+4)}\right)[A(JY)C(JZ) - A(JZ)C(JY)].
\end{aligned} \tag{4.6}$$

If we take $\lambda = \frac{r}{(2n+2)}$ and $\gamma \neq 0$, then

$$A(Z)C(Y) = A(Y)C(Z). \tag{4.7}$$

If we take $\lambda = \frac{r}{(2n+2)}$ and $\gamma \neq 0$, then from equations (1.2) and (1.9), equation (4.7) imply $g(Z,\rho)\,g(Y,\mu) = g(Y,\rho)\,g(Z,\mu)$, therefore we can say that the vector fields ρ and μ corresponding to 1-forms A and C respectively are codirectional.
Thus we conclude:

Theorem 4.1. *In a Bochner flat Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if $\frac{r}{(2n+2)}$ is an eigen value of the Ricci operator Q and JQ and $\gamma \neq 0$ then the vector fields ρ and μ corresponding to 1-forms A and C respectively are codirectional.*

5. Holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold

The holomorphically projective curvature tensor is defined by [20]

$$\begin{aligned} P(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ + S(JX,Z)g(JY,W) - S(JY,Z)g(JX,W)]. \end{aligned} \tag{5.1}$$

This tensor has the following properties

$$P(X,Y,Z,W) = -P(Y,X,Z,W), \qquad P(JX,JY,Z,W) = P(X,Y,Z,W). \tag{5.2}$$

Now we introduce the following:

Definition 5.1. An even dimensional Hermitian manifold (M^n, g) is said to be a holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if the holomorphically projective curvature tensor of the manifold satisfies $P.S = 0$, i.e.

$$(P(X, Y).S)(Z,W) = L_S Q(g,S)(Z,W;X,Y). \tag{5.3}$$

for all $X, Y, Z, W \in \chi(M^n)$.

If we take a holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold, then from the equations (1.12) and (5.1), we have

$$\begin{aligned} &\alpha[P(X,Y,Z,W) + P(X,Y,W,Z)] \\ &+ \beta[A(P(X,Y)Z)A(W) + A(Z)A(P(X,Y)W)] \\ &+ \gamma[A(P(X,Y)Z)C(W) + A(W)C(P(X,Y)Z) \\ &+ A(Z)C(P(X,Y)W) + C(Z)A(P(X,Y)W)] \\ &+ \delta[D(P(X,Y)Z,W) + D(Z,P(X,Y)W)] \\ &= L_S[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)]. \end{aligned} \tag{5.4}$$

Now putting $Z = W = \rho$ in equation (5.4) and using equation (1.14), we have

$$\begin{aligned} &(\alpha+\beta)P(X,Y,\rho,\rho) + \gamma P(X,Y,\rho,\mu) \\ &= \gamma L_S[A(Y)C(X) - A(X)C(Y)]. \end{aligned} \tag{5.5}$$

Using $Z = W = \rho$ in equation (5.1), we have

$$\begin{aligned} P(X,Y,\rho,\rho) = -\frac{\gamma}{n-2}[C(Y)A(X) - A(Y)C(X) \\ + C(JX)A(JY) - C(JY)A(JX)]. \end{aligned} \tag{5.6}$$

Similarly putting $Z = \rho$ and $W = \mu$ in equation (5.1), we get

$$\begin{aligned} P(X,Y,\rho,\mu) = R(X,Y,\rho,\mu) - \frac{(\alpha+\beta)}{n-2}[A(Y)C(X) - C(Y)A(X) \\ + C(JX)A(JY) - C(JY)A(JX)]. \end{aligned} \tag{5.7}$$

Using equations (5.6) and (5.7) in (5.5), we get

$$\gamma[R(X,Y,\rho,\mu) - L_S[A(Y)C(X) - A(X)C(Y)]] = 0, \tag{5.8}$$

this implies that either $\gamma = 0$ or $R(X,Y,\rho,\mu) = L_S[A(Y)C(X) - A(X)C(Y)]$. If $\gamma = 0$ then from equation (1.12), we get the condition of a pseudo quasi-Einstein manifolds.
Thus we can conclude:

Theorem 5.2. *A holomorphically projectively Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold is either a holomorphically projective Ricci semi-symmetric pseudo quasi-Einstein Hermitian manifold or*

$$R(X,Y,\rho,\mu) = L_S[A(Y)C(X) - A(X)C(Y)].$$

Corollary 5.1. *In a holomorphically projectively Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if $\gamma \neq 0$ then $R(X,Y,\rho,\mu) = 0$ if and only if the vector fields ρ and μ corresponding to one form A and C respectively are codirectional.*

Putting $Z = \rho$ and $W = \mu$ in equation (5.4), we get

$$\begin{aligned} &\alpha[P(X,Y,\rho,\mu) + P(X,Y,\mu,\rho)] + \beta P(X,Y,\mu,\rho) \\ &+ \gamma[P(X,Y,\rho,\rho) + P(X,Y,\mu,\mu)] = \beta L_S[A(X)C(Y) - A(Y)C(X)]. \end{aligned} \tag{5.9}$$

Putting $Z = U = \mu$ in (5.1), we get

$$\begin{aligned} &P(X,Y,\mu,\mu) \\ &= -\frac{\gamma}{n-2}[A(Y)C(X) - C(Y)A(X) + C(JY)A(JX) - C(JX)A(JY)]. \end{aligned} \tag{5.10}$$

Adding equations (5.6) and (5.10), we get

$$P(X,Y,\mu,\mu) + P(X,Y,\rho,\rho) = 0, \tag{5.11}$$

from equations (5.9) and (5.11), we have

$$\alpha[P(X,Y,\rho,\mu) + P(X,Y,\mu,\rho)] + \beta P(X,Y,\mu,\rho) = \beta L_S[A(X)C(Y) - A(Y)C(X)]. \tag{5.12}$$

From equations (5.1), (5.7) and (5.12), we have

$$\beta[R(X,Y,\mu,\rho) - L_S[A(X)C(Y) - A(Y)C(X)]] = 0. \tag{5.13}$$

This implies either $\beta = 0$ or $R(X,Y,\mu,\rho) = L_S[A(X)C(Y) - A(Y)C(X)]$.

Theorem 5.3. *In a holomorphically projective Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold if $\beta \neq 0$ then $R(X,Y,\mu,\rho) = 0$ if and only if the vector fields ρ and μ corresponding to one form A and C respectively are codirectional.*

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