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Fractional Hermite-Hadamard Type Inequalities for Functions Whose Derivatives are s-Preinvex

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Abstract
In this paper, we establish a new fractional integral identity, and then we derive some new fractional Hermite-Hadamard type inequalities for functions whose derivatives are s-preinvex.

Keywords: *integral inequality; s-preinvex function; Hölder inequality, power mean inequality.*

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1. Introduction

It is well known that convexity plays an important and central role in many areas, such as economic, finance, optimization, and games theory. Due to its diverse applications this concept has been extended and generalized in several directions. One of the most significant is that introduced by Hanson [4], called invex functions. Many authors have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems [12, 13, 22, 26].

It is well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard integral inequality

$$f\left(\tfrac{a+b}{2}\right)\leq \tfrac{1}{b-a}\int_a^b f(x)dx\leq \tfrac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a real convex function on the finite interval $[a,b]$. If the function f is concave, then (1.1) holds in the reverse direction (see [11]).

The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants have appeared in the literature one can see [2, 3, 5, 7–9, 11, 12, 16–21], and references therein.

Kirmaci et al. [6] presented some results connected with inequality (1.1)

$$\left|\tfrac{1}{b-a}\int_a^b f(x)dx-f\left(\tfrac{a+b}{2}\right)\right|\leq \tfrac{b-a}{8}\left(\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|\right).$$

Recently, Sarikaya et al [23], gave the fractional analogue of (1.1)

$$f\left(\tfrac{a+b}{2}\right)\leq \tfrac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(J_{a^+}^{\alpha}f\right)(b)+\left(J_{b^-}^{\alpha}f\right)(a)\right]\leq \tfrac{f(a)+f(b)}{2}. \tag{1.2}$$

Zhu et al [27] established the following result connected with inequality (1.2).

$$\left|\tfrac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(J_{a^+}^{\alpha}f\right)(b)+\left(J_{b^-}^{\alpha}f\right)(a)\right]-f\left(\tfrac{a+b}{2}\right)\right|$$

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$$\leq \frac{b-a}{4(1+\alpha)} \left(|f'(a)| + |f'(b)|\right) \left(\alpha + 3 - \frac{1}{2^{\alpha-1}}\right).$$

Motivated by the above results, in this paper we establish a new fractional integral identity and derive some new fractional Hermite-Hadamard type inequalities for functions whose derivatives are s-preinvex.

2. Preliminaries

In this sections we recall some definitions and lemmas

Definition 2.1. [21] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.2. [1] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 2.3. [26] A set $K \subseteq \mathbb{R}^n$ is said an invex with respect to the bifunction $\eta : K \times K \to \mathbb{R}^n$, if for all $x, y \in K$, we have

$$x + t\eta(y,x) \in K.$$

In what follows we assume that $K \subseteq \mathbb{R}$ be an invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$.

Definition 2.4. [26] A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if

$$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y)$$

holds for all $x, y \in K$ and all $t \in [0,1]$.

Definition 2.5. [7] A nonnegative function $f : K \subset [0,\infty) \to \mathbb{R}$ is said to be s-preinvex in the second sense with respect to η for some fixed $s \in (0,1]$, if

$$f(x + t\eta(y,x)) \leq (1-t)^s f(x) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 2.6. [5] Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$, is the Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

The incomplete beta function is given by

$$B_t(x,y) = \int_0^t \theta^{x-1}(1-\theta)^{y-1} d\theta, \quad 0 < t < 1.$$

Lemma 2.1. *[20] For any $0 \leq a < b$ in $\mathbb{R}$ and fixed $p \geq 1$, we have*

$$(b-a)^p \leq b^p - a^p.$$

3. Main results

We start with the following identity

Lemma 3.1. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$, and assume that $f'\in L([a,a+\eta(b,a)])$. Then the following equality holds*

$$\begin{aligned}&f\left(\tfrac{2a+\eta(b,a)}{2}\right)-\tfrac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f(a)+J^{\alpha}_{a^+}f(a+\eta(b,a))\right)\\&=\tfrac{\eta(b,a)}{2}\left(\int_0^1 kf'(a+t\eta(b,a))\,dt+\int_0^1(t^{\alpha}-(1-t)^{\alpha})f'(a+t\eta(b,a))\,dt\right),\end{aligned}\tag{3.1}$$

where

$$k=\begin{cases}1 \text{ if } 0\le t<\frac{1}{2},\\-1 \text{ if } \frac{1}{2}\le t<1.\end{cases}\tag{3.2}$$

Proof. Let

$$I_1=\int_0^1 kf'(a+t\eta(b,a))\,dt,\tag{3.3}$$

and

$$I_2=\int_0^1(t^{\alpha}-(1-t)^{\alpha})f'(a+t\eta(b,a))\,dt,\tag{3.4}$$

where k is defined by (3.2).

Clearly, we have

$$I_1=\tfrac{2}{\eta(b,a)}f\left(\tfrac{2a+\eta(b,a)}{2}\right)-\tfrac{1}{\eta(b,a)}\left(f(a)+f(a+\eta(b,a))\right).\tag{3.5}$$

Now, by integration by parts, I_2 gives

$$\begin{aligned}I_2=&\tfrac{1}{\eta(b,a)}f(a+\eta(b,a))+\tfrac{1}{\eta(b,a)}f(a)\\&-\tfrac{\alpha}{\eta(b,a)}\left(\int_0^1 t^{\alpha-1}f(a+t\eta(b,a))\,dt+\int_0^1(1-t)^{\alpha-1}f(a+t\eta(b,a))\,dt\right).\end{aligned}\tag{3.6}$$

Making the change of variable $u=a+t\eta(b,a)$, (3.6) becomes

$$\begin{aligned}I_2=&\tfrac{1}{\eta(b,a)}f(a+\eta(b,a))+\tfrac{1}{\eta(b,a)}f(a)-\tfrac{\alpha}{(\eta(b,a))^{\alpha+1}}\\&\times\left(\int_a^{a+\eta(b,a)}(u-a)^{\alpha-1}f(u)\,du+\int_a^{a+\eta(b,a)}(a+\eta(b,a)-u)^{\alpha-1}f(u)\,du\right)\\=&\tfrac{1}{\eta(b,a)}f(a+\eta(b,a))+\tfrac{1}{\eta(b,a)}f(a)-\tfrac{\alpha\Gamma(\alpha)}{(\eta(b,a))^{\alpha+1}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f(a)+J^{\alpha}_{a^+}f(a+\eta(b,a))\right)\\=&\tfrac{1}{\eta(b,a)}\left(f(a+\eta(b,a))+f(a)\right)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha+1}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f(a)+J^{\alpha}_{a^+}f(a+\eta(b,a))\right).\end{aligned}\tag{3.7}$$

Summing (3.6) and (3.7), and then multiplying the resulting equality by $\frac{\eta(b,a)}{2}$, we obtain the desired result. □

In what follows we note by

$$\lambda_{\alpha,s}=\tfrac{1-\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}-B_{\frac{1}{2}}(\alpha+1,s+1),\tag{3.8}$$

$$\mu_{\alpha,s}=B_{\frac{1}{2}}(s+1,\alpha+1)-\tfrac{\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1},\tag{3.9}$$

$$\lambda_{\alpha,1}=\tfrac{1}{\alpha+2}-\tfrac{1}{\alpha+1}\left(\tfrac{1}{2}\right)^{\alpha+1},\tag{3.10}$$

$$\mu_{\alpha,1} = \frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1}, \tag{3.11}$$

$$\lambda_{1,s} = \frac{1}{(s+1)(s+2)}\left(s + \left(\frac{1}{2}\right)^{1+s}\right), \tag{3.12}$$

and

$$\mu_{1,s} = \frac{1}{(s+1)(s+2)}\left(\frac{1}{2}\right)^{1+s}. \tag{3.13}$$

Theorem 3.1. *Let $f : [a, a+\eta(b,a)] \subset [0,\infty) \to \mathbb{R}$ be a positive function on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$ and $f \in L[a, a+\eta(b,a)]$. If $|f'|$ is s-preinvex function, where $s \in (0,1]$, then the following fractional inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-} f(a) + J^{\alpha}_{a^+} f(a+\eta(b,a))\right)\right| \\ &\leq \frac{\eta(b,a)}{2}\left(\frac{1}{s+1} + \frac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1, s+1) + B_{\frac{1}{2}}(s+1,\alpha+1)\right)(|f'(a)| + |f'(b)|),\end{aligned} \tag{3.14}$$

where $B_{\frac{1}{2}}(.,.)$ is the incomplete beta function.

Proof. From Lemma 3.1, properties of modulus, and s-preinvexity of $|f'|$, we have

$$\begin{aligned}&\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-} f(a) + J^{\alpha}_{a^+} f(a+\eta(b,a))\right)\right| \\
&\leq \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}} |f'(a+t\eta(b,a))|\,dt - \int_{\frac{1}{2}}^{1} |f'(a+t\eta(b,a))|\,dt\right. \\
&\left. + \int_0^{\frac{1}{2}} ((1-t)^{\alpha} - t^{\alpha})|f'(a+t\eta(b,a))|\,dt + \int_{\frac{1}{2}}^{1} (t^{\alpha} - (1-t)^{\alpha})|f'(a+t\eta(b,a))|\,dt\right) \\
&\leq \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}} ((1-t)^s|f'(a)| + t^s|f'(b)|)\,dt + \int_{\frac{1}{2}}^{1} ((1-t)^s|f'(a)| + t^s|f'(b)|)\,dt\right. \\
&\left. + \int_0^{\frac{1}{2}} ((1-t)^{\alpha} - t^{\alpha})((1-t)^s|f'(a)| + t^s|f'(b)|)\,dt + \int_{\frac{1}{2}}^{1} (t^{\alpha} - (1-t)^{\alpha})((1-t)^s|f'(a)| + t^s|f'(b)|)\,dt\right) \\
&= \frac{\eta(b,a)}{2}\left(\left(\frac{|f'(a)|+|f'(b)|}{s+1}\right) + \left(\frac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1,s+1) + B_{\frac{1}{2}}(s+1,\alpha+1)\right)|f'(a)|\right. \\
&\left. + \left(\left(B_{\frac{1}{2}}(s+1,\alpha+1) - B_{\frac{1}{2}}(\alpha+1,s+1) + \frac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1}\right)|f'(b)|\right)\right),\end{aligned}$$

which is the desired result. □

Corollary 3.1. *In Theorem 3.1 if we take $s = 1$, we obtain the following fractional midpoint inequality for preinvex functions*

$$\begin{aligned}&\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-} f(a) + J^{\alpha}_{a^+} f(a+\eta(b,a))\right)\right| \\ &\leq \frac{\eta(b,a)}{4(\alpha+1)}\left(\alpha + 3 - \left(\frac{1}{2}\right)^{\alpha-1}\right)(|f'(a)| + |f'(b)|).\end{aligned}$$

Corollary 3.2. *In Theorem 3.1 if we choose $\eta(b,a) = b-a$, we obtain the following fractional midpoint inequality for s-convex functions*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{b^-} f(a) + J^{\alpha}_{a^+} f(b)\right)\right| \\ &\leq \frac{b-a}{2}\left(\frac{1}{s+1} + \frac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1,s+1) + B_{\frac{1}{2}}(s+1,\alpha+1)\right)(|f'(a)| + |f'(b)|).\end{aligned}$$

Moreover if we take $\eta(b,a) = b-a$ and $s = 1$, we obtain Theorem 2.3 from [27].

Corollary 3.3. *In Theorem 3.1 if we take $\alpha = 1$, we obtain the following midpoint inequality for s-preinvex functions*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(t)\,dt \right| \le \frac{\eta(b,a)}{(s+1)(s+2)} \left(s+1+\left(\frac{1}{2}\right)^{s+1}\right) \left(|f'(a)| + |f'(b)|\right).$$

Moreover if we choose $s = 1$, we obtain the following midpoint inequality for preinvex functions

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(t)\,dt \right| \le \frac{3\eta(b,a)}{8} \left(|f'(a)| + |f'(b)|\right).$$

Corollary 3.4. *In Theorem 3.1 if we choose $\eta(b,a) = b-a$ and $\alpha = 1$, we obtain the following midpoint inequality for s-convex functions*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^{b} f(t)\,dt \right| \le \frac{b-a}{(s+1)(s+2)} \left(s+1+\left(\frac{1}{2}\right)^{s+1}\right) \left(|f'(a)| + |f'(b)|\right).$$

Moreover if we choose $s = 1$, we obtain the following midpoint inequality for convex functions

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^{b} f(t)\,dt \right| \le \frac{3(b-a)}{8} \left(|f'(a)| + |f'(b)|\right).$$

Theorem 3.2. *Let $f : [a, a+\eta(b,a)] \subset [0,\infty) \to \mathbb{R}$ be a positive function on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$ and $f \in L[a, a+\eta(b,a)]$. If $|f'|^q$ is s-preinvex function, where $s \in (0,1]$ and $q \ge 1$, then the following fractional inequality holds*

$$\begin{aligned}
&\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha} \left(J^\alpha_{(a+\eta(b,a))^-} f(a) + J^\alpha_{a^+} f(a+\eta(b,a))\right) \right| \\
&\le \frac{\eta(b,a)}{2} \left(\left(\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(a)|^q + \left(\frac{1}{2}\right)^{s+1}|f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\frac{\left(\frac{1}{2}\right)^{s+1}|f'(a)|^q + \left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(b)|^q}{s+1} \right)^{\frac{1}{q}} + \frac{\left(1-\left(\frac{1}{2}\right)^\alpha\right)^{1-\frac{1}{q}}}{(\alpha+1)^{1-\frac{1}{q}}} \\
&\quad \times \left. \left(\left(\lambda_{\alpha,s}|f'(a)|^q + \mu_{\alpha,s}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\mu_{\alpha,s}|f'(a)|^q + \lambda_{\alpha,s}|f'(b)|^q\right)^{\frac{1}{q}} \right) \right),
\end{aligned}$$

where $\lambda_{\alpha,s}$ and $\mu_{\alpha,s}$ are defined as in (3.8) and (3.9) respectively.

Proof. From Lemma 3.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha} \left(J^\alpha_{(a+\eta(b,a))^-} f(a) + J^\alpha_{a^+} f(a+\eta(b,a))\right) \right| \\
&\le \frac{\eta(b,a)}{2} \left(\left(\int_0^{\frac{1}{2}} |f'(a+t\eta(b,a))|^q\,dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{1} |f'(a+t\eta(b,a))|^q\,dt \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right) dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right) |f'(a+t\eta(b,a))|^q\,dt \right)^{\frac{1}{q}} \\
&\quad + \left. \left(\int_{\frac{1}{2}}^{1} \left(t^\alpha - (1-t)^\alpha\right) dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^{1} \left(t^\alpha - (1-t)^\alpha\right) |f'(a+t\eta(b,a))|^q\,dt \right)^{\frac{1}{q}} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\eta(b,a)}{2}\left(\left(\left(\int_0^{\frac{1}{2}} |f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}\right.\right. \\
&\left(\frac{1}{(\alpha+1)^{1-\frac{1}{q}}}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{1-\frac{1}{q}}\left(\left(\int_0^{\frac{1}{2}} ((1-t)^{\alpha}-t^{\alpha})\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}\right.\right. \\
&\left.\left.\left.+\left(\int_{\frac{1}{2}}^1 (t^{\alpha}-(1-t)^{\alpha})\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}\right)\right)\right). \qquad (3.15)
\end{aligned}
$$

Since $|f'|^q$ is s-preinvex function, (3.15) gives

$$
\begin{aligned}
&\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f(a)+J^{\alpha}_{a^+}f(a+\eta(b,a))\right)\right| \\
&\le \frac{\eta(b,a)}{2}\left(\left(\left(\int_0^{\frac{1}{2}}(1-t)^s|f'(a)|^q+t^s|f'(b)|^q\,dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(1-t)^s|f'(a)|^q+t^s|f'(b)|^q\,dt\right)^{\frac{1}{q}}\right.\right. \\
&+\frac{\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{1-\frac{1}{q}}}{(\alpha+1)^{1-\frac{1}{q}}}\left(\left(|f'(a)|^q\left(\int_0^{\frac{1}{2}}(1-t)^{\alpha+s}\,dt-\int_0^{\frac{1}{2}}t^{\alpha}(1-t)^s dt\right)\right.\right. \\
&\left.\left.+\ |f'(b)|^q\left(\int_0^{\frac{1}{2}}t^s(1-t)^{\alpha}\,dt-\int_0^{\frac{1}{2}}t^{\alpha+s}dt\right)\right)^{\frac{1}{q}}\right)+\left(|f'(a)|^q\left(\int_{\frac{1}{2}}^1 t^{\alpha}(1-t)^s dt-\int_{\frac{1}{2}}^1(1-t)^{\alpha+s}\,dt\right)\right. \\
&\left.\left.+\ |f'(b)|^q\left(\int_{\frac{1}{2}}^1 t^{\alpha+s}dt-\int_{\frac{1}{2}}^1 t^s(1-t)^{\alpha}\,dt\right)\right)^{\frac{1}{q}}\right) \\
&= \frac{\eta(b,a)}{2}\left(\left(\left(\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(a)|^q+\left(\frac{1}{2}\right)^{s+1}|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left(\frac{1}{2}\right)^{s+1}|f'(a)|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right.\right. \\
&+\frac{\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{1-\frac{1}{q}}}{(\alpha+1)^{1-\frac{1}{q}}}\left(\left(|f'(a)|^q\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}-B_{\frac{1}{2}}(\alpha+1,s+1)\right)\right.\right. \\
&\left.\left.+\ |f'(b)|^q\left(B_{\frac{1}{2}}(s+1,\alpha+1)-\frac{\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}\right)\right)^{\frac{1}{q}}\right) \\
&\left.+\left(|f'(a)|^q\left(B_{\frac{1}{2}}(s+1,\alpha+1)-\frac{\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}\right)+|f'(b)|^q\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}-B_{\frac{1}{2}}(\alpha+1,s+1)\right)\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

which is the required result. □

Corollary 3.5. *In Theorem 3.2 if we take $s=1$, we obtain the following fractional midpoint inequality for preinvex functions*

$$
\begin{aligned}
&\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f(a)+J^{\alpha}_{a^+}f(a+\eta(b,a))\right)\right| \\
&\le \frac{\eta(b,a)}{2}\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}+\frac{\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{1-\frac{1}{q}}}{(\alpha+1)^{1-\frac{1}{q}}}\right. \\
&\left.\times\left(\left(\lambda_{\alpha,1}|f'(a)|^q+\mu_{\alpha,1}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\mu_{\alpha,1}|f'(a)|^q+\lambda_{\alpha,1}|f'(b)|^q\right)^{\frac{1}{q}}\right)\right),
\end{aligned}
$$

where $\lambda_{\alpha,1}$ *and* $\mu_{\alpha,1}$ *are defined as in* (3.10) *and* (3.11) *respectively.*

Corollary 3.6. *In Theorem 3.2 if we choose* $\eta(b,a) = b - a$, *we obtain the following fractional midpoint inequality for s-convex functions*

$$\begin{aligned}&\left|f\left(\tfrac{a+b}{2}\right) - \tfrac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{b^-}^{\alpha}f(a) + J_{a^+}^{\alpha}f(b)\right)\right| \\ &\leq \tfrac{b-a}{2}\left(\left(\tfrac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(a)|^q+\left(\frac{1}{2}\right)^{s+1}|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right. \\ &+\left(\tfrac{\left(\frac{1}{2}\right)^{s+1}|f'(a)|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}} + \tfrac{\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{1-\frac{1}{q}}}{(\alpha+1)^{1-\frac{1}{q}}} \\ &\left.\times\left(\left(\lambda_{\alpha,s}|f'(a)|^q + \mu_{\alpha,s}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\mu_{\alpha,s}|f'(a)|^q + \lambda_{\alpha,s}|f'(b)|^q\right)^{\frac{1}{q}}\right)\right),\end{aligned}$$

where $\lambda_{\alpha,s}$ *and* $\mu_{\alpha,s}$ *are defined as in* (3.8) *and* (3.9) *respectively. Moreover if we take* $\eta(b,a) = b - a$ *and* $s = 1$, *we obtain the following fractional midpoint inequality for convex functions*

$$\begin{aligned}&\left|f\left(\tfrac{a+b}{2}\right) - \tfrac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{b^-}^{\alpha}f(a) + J_{a^+}^{\alpha}f(b)\right)\right| \\ &\leq \tfrac{b-a}{2}\left(\left(\tfrac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}} + \left(\tfrac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right. \\ &\left.+\tfrac{\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)^{1-\frac{1}{q}}}{(\alpha+1)^{1-\frac{1}{q}}}\left(\left(\lambda_{\alpha,1}|f'(a)|^q + \mu_{\alpha,1}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\mu_{\alpha,1}|f'(a)|^q + \lambda_{\alpha,1}|f'(b)|^q\right)^{\frac{1}{q}}\right)\right),\end{aligned}$$

where $\lambda_{\alpha,1}$ *and* $\mu_{\alpha,1}$ *are defined as in* (3.10) *and* (3.11) *respectively.*

Corollary 3.7. *In Theorem 3.2 if we take* $\alpha = 1$, *we obtain the following midpoint inequality for s-preinvex functions*

$$\begin{aligned}&\left|f\left(\tfrac{2a+\eta(b,a)}{2}\right) - \tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(t)\,dt\right| \\ &\leq \tfrac{\eta(b,a)}{2}\left(\left(\tfrac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(a)|^q+\left(\frac{1}{2}\right)^{s+1}|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right. \\ &+\left(\tfrac{\left(\frac{1}{2}\right)^{s+1}|f'(a)|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}} + \left(\tfrac{1}{2}\right)^{2\left(1-\frac{1}{q}\right)} \\ &\left.\times\left(\left(\lambda_{1,s}|f'(a)|^q + \mu_{1,s}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\mu_{1,s}|f'(a)|^q + \lambda_{1,s}|f'(b)|^q\right)^{\frac{1}{q}}\right)\right),\end{aligned}$$

where $\lambda_{1,s}$ *and* $\mu_{1,s}$ *are defined as in* (3.12) *and* (3.13) *respectively. Moreover if we choose* $s = 1$, *we obtain the following midpoint inequality for preinvex functions*

$$\begin{aligned}&\left|f\left(\tfrac{2a+\eta(b,a)}{2}\right) - \tfrac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J_{(a+\eta(b,a))^-}^{\alpha}f(a) + J_{a^+}^{\alpha}f(a+\eta(b,a))\right)\right| \\ &\leq \tfrac{\eta(b,a)}{2}\left(\left(\tfrac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}} + \left(\tfrac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right. \\ &\left.+\tfrac{1}{4}\left(\left(\tfrac{5|f'(a)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}} + \left(\tfrac{|f'(a)|^q+5|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Corollary 3.8. *In Theorem 3.2 if we choose* $\eta(b,a) = b - a$ *and* $\alpha = 1$, *we obtain the following midpoint inequality for s-convex functions*

$$\left|f\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \tfrac{b-a}{2}\left(\left(\tfrac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(a)|^q+\left(\frac{1}{2}\right)^{s+1}|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{1}{2}\right)^{2\left(1-\frac{1}{q}\right)}$$
$$\times\left(\left(\lambda_{1,s}\left|f'(a)\right|^q+\mu_{1,s}\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(\mu_{1,s}\left|f'(a)\right|^q+\lambda_{1,s}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right)\right),$$

where $\lambda_{1,s}$ *and* $\mu_{1,s}$ *are defined as in* (3.12) *and* (3.13) *respectively. Moreover if we choose* $s=1$*, we obtain the following midpoint inequality for convex functions*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{2}\left(\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{1}{4}\left(\left(\frac{5\left|f'(a)\right|^q+\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+5\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\right)\right).$$

Theorem 3.3. *Let* $f:[a,a+\eta(b,a)]\subset[0,\infty)\to\mathbb{R}$ *be a positive function on* $[a,a+\eta(b,a)]$ *with* $\eta(b,a)>0$ *and* $f\in L[a,a+\eta(b,a)]$*. If* $|f'|^q$ *is s-preinvex function, where* $s\in(0,1]$*, and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then the following fractional inequality holds*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha}\left(J^\alpha_{(a+\eta(b,a))^-}f(a)+J^\alpha_{a^+}f(a+\eta(b,a))\right)\right|$$
$$\leq\frac{\eta(b,a)}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^{\alpha p}\right)^{\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}}\right)\left(\left(\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(a)\right|^q+\left(\frac{1}{2}\right)^{s+1}\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Proof. From Lemma 3.1, properties of modulus, and Hölder inequality, and Lemma 2.1, we have

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha}\left(J^\alpha_{(a+\eta(b,a))^-}f(a)+J^\alpha_{a^+}f(a+\eta(b,a))\right)\right|$$
$$\leq\frac{\eta(b,a)}{2}\left(\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{1}{2}}^1(t^\alpha-(1-t)^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right)$$
$$\leq\frac{\eta(b,a)}{2}\left(\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{2}}((1-t)^{\alpha p}-t^{\alpha p})\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{1}{2}}^1(t^{\alpha p}-(1-t)^{\alpha p})\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$=\frac{\eta(b,a)}{2}\left(\left(\left(\int_0^{\frac{1}{2}}\left|f'\left(a+t\eta\left(b,a\right)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left|f'\left(a+t\eta\left(b,a\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.\right.$$
$$\left.\left.+\frac{1}{(\alpha p+1)^{\frac{1}{p}}}\left(1-\left(\tfrac{1}{2}\right)^{\alpha p}\right)^{\frac{1}{p}}\left(\left(\int_0^{\frac{1}{2}}\left|f'\left(a+t\eta\left(b,a\right)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left|f'\left(a+t\eta\left(b,a\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right)\right)\right). \tag{3.16}$$

Since $|f'|$ is s-preinvex function, (3.16) gives

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f(a)+J^{\alpha}_{a^+}f(a+\eta\left(b,a\right))\right)\right|$$
$$\leq\frac{\eta(b,a)}{2}\left(1+\frac{1}{(\alpha p+1)^{\frac{1}{p}}}\left(1-\left(\tfrac{1}{2}\right)^{\alpha p}\right)^{\frac{1}{p}}\right)$$
$$\left(\left(\int_0^{\frac{1}{2}}\left((1-t)^s\left|f'\left(a\right)\right|^q+t^s\left|f'\left(b\right)\right|^q\right)dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left((1-t)^s\left|f'\left(a\right)\right|^q+t^s\left|f'\left(b\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right)$$
$$=\frac{\eta(b,a)}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^{\alpha p}\right)^{\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}}\right)\left(\left(\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(a)\right|^q+\left(\frac{1}{2}\right)^{s+1}\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right),$$

which is the desired result. □

Corollary 3.9. *In Theorem 3.3 if we take $s=1$, we obtain the following fractional midpoint inequality for preinvex functions*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f(a)+J^{\alpha}_{a^+}f(a+\eta\left(b,a\right))\right)\right|$$
$$\leq\frac{\eta(b,a)}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^{\alpha p}\right)^{\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}}\right)\left(\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right).$$

Corollary 3.10. *In Theorem 3.3 if we choose $\eta\left(b,a\right)=b-a$, we obtain the following fractional midpoint inequality for s-convex functions*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{b^-}f(a)+J^{\alpha}_{a^+}f(b)\right)\right|\leq\frac{b-a}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^{\alpha p}\right)^{\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}}\right)\left(\left(\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(a)\right|^q+\left(\frac{1}{2}\right)^{s+1}\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Moreover if we take $\eta\left(b,a\right)=b-a$ and $s=1$, we obtain the following fractional midpoint inequality for convex functions

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{b^-}f(a)+J^{\alpha}_{a^+}f(b)\right)\right|\leq\frac{b-a}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^{\alpha p}\right)^{\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}}\right)\left(\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right).$$

Corollary 3.11. *In Theorem 3.3 if we take $\alpha=1$, we obtain the following midpoint inequality for s-preinvex functions*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f\left(t\right)dt\right|\leq\frac{\eta(b,a)}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^{p}\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\right)\left(\left(\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(a)\right|^q+\left(\frac{1}{2}\right)^{s+1}\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\Bigg).$$

Moreover if we choose $s=1$, we obtain the following midpoint inequality for preinvex functions

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)}f(t)\,dt\right|\leq\frac{\eta(b,a)}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^p\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\right)\left(\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right).$$

Corollary 3.12. *In Theorem 3.3 if we choose $\eta(b,a)=b-a$ and $\alpha=1$, we obtain the following midpoint inequality for s-convex functions*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(t)\,dt\right|\leq\frac{b-a}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^p\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\right)\Bigg(\left(\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(a)\right|^q+\left(\frac{1}{2}\right)^{s+1}\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

$$+\left(\frac{\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\Bigg).$$

Moreover if we choose $s=1$, we obtain the following midpoint inequality for convex functions

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(t)\,dt\right|\leq\frac{b-a}{2}\left(1+\frac{\left(1-\left(\frac{1}{2}\right)^p\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\right)\left(\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right).$$

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