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On the Quadra Fibona-Pell and Hexa Fibona-Pell-Jacobsthal Sequences

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Abstract
In this paper, we consider the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas sequences. We introduce the quadra Fibona-Pell,Fibona-Jacobsthal and Pell-Jacobsthal and the hexa Fibona-Pell-Jacobsthal sequences whose compounds are the Fibonacci, Pell and Jacobsthal sequences. We derive the Binet-like formulas, the generating functions and the exponential generating functions of these sequences. Also, we obtain some binomial identities for them.

Keywords: *Fibonacci sequence; Lucas sequence; Pell sequence; Jacobsthal sequence; Binet like formula; generating function; exponential generating function.*

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1. Introduction

Special numbers and the corresponding recurrence relations and their generalizations have many applications to every field of science and they have many interesting properties [8, 9, 10]. One application of second order linear recurrences occurs in graph theory [12]. Second order linear recurrences related to Fibonacci and Lucas numbers and their generalizations are investigated in [6], [7], [11], [14]. Fourth order linear recurrences and their generalizations are studied in [3], [4], [13], [15].

In [3] and [4] various fourth order linear recurrences and their polynomials are defined and studied.

In [15] the author define the quadrapell numbers and quadrapell polynomials as fourth order linear recurrences.

In [13] the author define the quadra Fibona-Pell integers sequences and she gives some algebraic identities.

In the present work we consider fourth and sixth orders linear recurrences and we define the quadra Fibona-Pell, Fibona-Jacobsthal and Pell-Jacobsthal and the hexa Fibona-Pell-Jacobsthal sequences. We give some properties of them.

The Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences $\{F_n\}$, $\{L_n\}$, $\{P_n\}$, $\{p_n\}$, $\{J_n\}$ and $\{j_n\}$ are defined by two order recurrences for $n \geq 0$, respectively,

$$F_{n+2} = F_{n+1} + F_n,$$

$$L_{n+2} = L_{n+1} + L_n,$$

$$P_{n+2} = 2P_{n+1} + P_n,$$

$$p_{n+2} = 2p_{n+1} + p_n,$$

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$$J_{n+2} = J_{n+1} + 2J_n,$$

$$j_{n+2} = j_{n+1} + 2j_n,$$

with the initial conditions are given as follow, respectively,

$$F_0 = 0, \quad \text{and} \quad F_1 = 1,$$

$$L_0 = 2, \quad \text{and} \quad L_1 = 1,$$

$$P_0 = 0, \quad \text{and} \quad P_1 = 1,$$

$$p_0 = 2, \quad \text{and} \quad p_1 = 1,$$

$$J_0 = 0, \quad \text{and} \quad J_1 = 1,$$

$$j_0 = 2, \quad \text{and} \quad j_1 = 1.$$

The first few members of this sequences are given as follow, respectively,

n	0	1	2	3	4	5	6	7	8	9	10	11	...
F_n	0	1	1	2	3	5	8	13	21	34	55	89	...
L_n	2	1	3	4	7	11	18	29	47	76	123	199	...
P_n	0	1	2	5	12	29	70	169	408	985	2378	5741	...
p_n	2	1	4	9	22	53	128	309	746	1801	4348	10497	...
J_n	0	1	1	3	5	11	21	43	85	171	341	683	...
j_n	2	1	5	7	17	31	65	127	257	511	1025	2047	...

Table 1. The first few members of this sequences

The recurrences involve the characteristic equations, respectively,

$$x^2 - x - 1 = 0,$$

$$y^2 - 2y - 1 = 0,$$

$$z^2 - z - 2 = 0.$$

The roots of the equations are as follows, respectively,

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

$$\gamma = 1+\sqrt{2} \quad \text{and} \quad \delta = 1-\sqrt{2}$$

$$\lambda = 2 \quad \text{and} \quad \mu = -1.$$

Then the following equalities follow directly from Vieta's formulas, respectively,

$$\alpha + \beta = 1, \quad \alpha - \beta = \sqrt{5}, \quad \alpha\beta = -1,$$

$$\gamma + \delta = 2, \quad \gamma - \delta = 2\sqrt{2}, \quad \gamma\delta = -1,$$

$$\lambda + \mu = 1, \quad \lambda - \mu = 3, \quad \lambda\mu = -2.$$

Moreover, the Binet formulas for the Fibonacci, Lucas, Pell, Pell-lucas, Jacobsthal and Jacobsthal-Lucas sequences are, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

$$L_n = \alpha^n + \beta^n,$$

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta},$$

$$p_n = \gamma^n + \delta^n,$$

$$J_n = \frac{\lambda^n - \mu^n}{\lambda - \mu},$$

$$j_n = \lambda^n + \mu^n.$$

The generating functions for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences are, respectively,

$$G_F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2},$$

$$G_L(x) = \sum_{n=0}^{\infty} L_n x^n = \frac{2 - x}{1 - x - x^2},$$

$$G_P(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - 2x - x^2},$$

$$G_p(x) = \sum_{n=0}^{\infty} p_n x^n = \frac{2 - 3x}{1 - 2x - x^2},$$

$$G_J(x) = \sum_{n=0}^{\infty} J_n x^n = \frac{x}{1 - x - 2x^2},$$

$$G_j(x) = \sum_{n=0}^{\infty} j_n x^n = \frac{2 - x}{1 - x - 2x^2}.$$

The exponential generating functions for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences are, respectively,

$$E_F(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n,$$

$$E_L(x) = e^{\alpha x} + e^{\beta x} = \sum_{n=0}^{\infty} \frac{L_n}{n!} x^n,$$

$$E_P(x) = \frac{e^{\gamma x} - e^{\delta x}}{\gamma - \delta} = \sum_{n=0}^{\infty} \frac{P_n}{n!} x^n,$$

$$E_p(x) = e^{\gamma x} + e^{\delta x} = \sum_{n=0}^{\infty} \frac{p_n}{n!} x^n,$$

$$E_J(x) = \frac{e^{\lambda x} - e^{\mu x}}{\lambda - \mu} = \sum_{n=0}^{\infty} \frac{J_n}{n!} x^n,$$

$$E_j(x) = e^{\lambda x} + e^{\mu x} = \sum_{n=0}^{\infty} \frac{j_n}{n!} x^n.$$

The Fibonacci, Pell and Jacobsthal sequences and identities in the above passage are available in [1],[2],[5],[8] and [9].

2. New Sequences

In this section we aim to obtain new sequences have the roots of the characteristic equations of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences. Then we will examine the situation of these new sequences in different initial conditions, find the Binet-like formulas and reach the generating functions. Similar investigations were given in [13, 15, 16]. In [15] the quadra pell numbers are defined and some properties are given. In [13] the Fibona-Pell integer sequence is defined and some algebraic identities are obtained. In [16] the Quadra Lucas-Jacobsthal Numbers were investigated.

2.1 The Quadra Fibona-Pell Sequence

Definition 2.1. The quadra Fibona-Pell sequence $\{FP_n\}_{n\geq 0}$ is defined by a fourth order recurrence;

$$FP_{n+4} = 3FP_{n+3} - 3FP_{n+1} - FP_n \tag{2.1}$$

with the different initial conditions $FP_0 = 0$, $FP_1 = 0$, $FP_2 = 1$, $FP_3 = 3$.

The first few members of this sequence are given as follow ;

n	0	1	2	3	4	5	6	7	8	9	10	...
FP_n	0	0	1	3	9	24	62	156	387	941	1512	...

Table 2. The first few members of the quadra Fibona-Pell sequence

If we take the different initial conditions, we generate the certain number sequences as follows;

n	0	1	2	3	Numbers
FP_n	0	1	1	2	Fibonacci numbers
FP_n	2	1	3	4	Lucas numbers
FP_n	0	1	2	5	Pell numbers
FP_n	2	1	4	9	Pell-Lucas numbers

Table 3. The first few members of the different initial conditions

The characteristic equation associated to the recurrence relation is

$$r^4 - 3r^3 + 3r + 1 = 0. \tag{2.2}$$

The roots of the equations are as follows

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad \gamma = 1+\sqrt{2} \quad \text{and} \quad \delta = 1-\sqrt{2}.$$

Then the following equalities follow directly from Vieta's formulas

$$\alpha+\beta+\gamma+\delta = 3 \quad \text{and} \quad \alpha\beta\gamma\delta = 1.$$

Theorem 2.1. *The Binet-like formula for the quadra Fibona-Pell sequence is*

$$FP_n = a_1\alpha^n + a_2\beta^n + a_3\gamma^n + a_4\delta^n,$$

where,

$$a_1 = \frac{3-(\beta+\gamma+\delta)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)},$$

$$a_2 = \frac{3-(\alpha+\gamma+\delta)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},$$

$$a_3 = \frac{3-(\alpha+\beta+\delta)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},$$

$$a_4 = \frac{3-(\alpha+\beta+\gamma)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},$$

such that α, β, γ and δ are the roots of the characteristic equation of the quadra Fibona-Pell sequence.

Theorem 2.2. *The generating function for the quadra Fibona-Pell sequence is*

$$G_{FP}(x) = \sum_{n=0}^{\infty} FP_n x^n = \frac{x^2}{1-3x+3x^3+x^4}.$$

Proof. The proof can be given in a similar way of the proof of Theorem 2.14. □

The generating function of the quadra Fibona-Pell sequence is the multiplication of the generating function of the Fibonacci and Pell sequence as seen following,

$$G_F(x)G_P(x) = \left(\frac{x}{1-x-x^2}\right)\left(\frac{x}{1-2x-x^2}\right) = \frac{x^2}{1-3x+3x^3+x^4} = G_{FP}(x)$$

Theorem 2.3. *The exponential generating function for the quadra Fibona-Pell sequence is*

$$E_{FP}(x) = a_1e^{\alpha x} + a_2e^{\beta x} + a_3e^{\gamma x} + a_4e^{\delta x} = \sum_{n=0}^{\infty} \frac{FP_n}{n!}x^n.$$

Proof. The proof can be given in a similar way of the proof of Theorem 2.15. □

Theorem 2.4. *The sum of the first n terms of FP_n is*

$$\sum_{i=0}^{n} FP_i = \frac{FP_n + 4FP_{n-1} + 4FP_{n-2} + FP_{n-3} + 1}{2}, \quad n \geq 3.$$

Proof. The proof can be given in a similar way of the proof of Theorem 2.16. □

2.2 The Quadra Fibona-Jacobsthal Sequence

Definition 2.2. The quadra Fibona-Jacobsthal sequence $\{FJ_n\}_{n\geq 0}$ is defined by a fourth order recurrence;

$$FJ_{n+4} = 2FJ_{n+3} + 2FJ_{n+2} - 3FJ_{n+1} - 2FJ_n \tag{2.3}$$

with the different initial conditions $FJ_0 = 0$, $FJ_1 = 0$, $FJ_2 = 1$, $FJ_3 = 2$.

The first few members of this sequence are given as follow ;

n	0	1	2	3	4	5	6	7	8	9	10	...
FJ_n	0	0	1	2	6	13	30	64	137	286	594	...

Table 4. The first few members of the quadra Fibona-Jacobsthal sequence

If we take the different initial conditions, the certain number sequences are generated as follows;

n	0	1	2	3	Numbers
FJ_n	0	1	1	2	Fibonacci numbers
FJ_n	2	1	3	4	Lucas numbers
FJ_n	0	1	1	3	Jacobsthal numbers
FJ_n	2	1	5	7	Jacobsthal-Lucas numbers

Table 5. The first few members of the different initial conditions

The characteristic equation associated to the recurrence relation is

$$p^4 - 2p^3 - 3p^2 + 4p + 2 = 0. \tag{2.4}$$

The roots of the equations are as follows

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad \lambda = 2 \quad \text{and} \quad \mu = -1.$$

Then the following equalities follow directly from Vieta's formulas

$$\alpha + \beta + \lambda + \mu = 2 \quad \text{and} \quad \alpha\beta\lambda\mu = 2.$$

Theorem 2.5. *The Binet-like formulas for the quadra Fibona-Jacobsthal sequence is*

$$FJ_n = b_1\alpha^n + b_2\beta^n + b_3\lambda^n + b_4\mu^n,$$

where,

$$b_1 = \frac{2-(\beta+\lambda+\mu)}{(\alpha-\beta)(\alpha-\lambda)(\alpha-\mu)},$$
$$b_2 = \frac{2-(\alpha+\lambda+\mu)}{(\beta-\alpha)(\beta-\lambda)(\beta-\mu)},$$
$$b_3 = \frac{2-(\alpha+\beta+\mu)}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\mu)},$$
$$b_4 = \frac{2-(\alpha+\beta+\lambda)}{(\mu-\alpha)(\mu-\beta)(\mu-\lambda)},$$

such that α, β, λ and μ are the roots of the characteristic equation of the quadra Fibona-Jacobsthal sequence.

Theorem 2.6. *The generating function for the quadra Fibona-Jacobsthal sequence is*

$$G_{FJ}(x) = \sum_{n=0}^{\infty} FJ_n x^n = \frac{x^2}{1-2x-2x^2+3x^3+2x^4}.$$

Proof. The proof can be given in a similar way of the proof of Theorem 2.14. □

The generating function of the quadra Fibona-Jacobsthal sequence is the multiplication of the generating function of the Fibonacci and Jacobsthal sequence as seen following,

$$G_F(x)G_J(x) = \left(\frac{x}{1-x-x^2}\right)\left(\frac{x}{1-x-2x^2}\right) = \frac{x^2}{1-2x-2x^2+3x^3+2x^4} = G_{FJ}(x).$$

Theorem 2.7. *The exponential generating function for the quadra Fibona-Jacobsthal sequence is*

$$E_{FJ}(x) = b_1 e^{\alpha x} + b_2 e^{\beta x} + b_3 e^{\lambda x} + b_4 e^{\mu x} = \sum_{n=0}^{\infty} \frac{FJ_n}{n!} x^n.$$

Proof. The proof can be given in a similar way of the proof of Theorem 2.15. □

Theorem 2.8. *The sum of the first n terms of FP_n is*

$$\sum_{i=0}^{n} FJ_i = \frac{FJ_n + 3FJ_{n-1} + 5FJ_{n-2} + 2FJ_{n-3} + 1}{2}, \quad n \geq 3.$$

Proof. The proof can be given in a similar way of the proof of Theorem 2.16. □

2.3 The Quadra Pell-Jacobsthal Sequence

Definition 2.3. The quadra Pell-Jacobsthal sequence $\{PJ_n\}_{n\geq 0}$ is defined by a fourth order recurrence;

$$PJ_{n+4} = 3PJ_{n+3} + PJ_{n+2} - 5PJ_{n+1} - 2PJ_n \tag{2.5}$$

with the different initial conditions $PJ_0 = 0$, $PJ_1 = 0$, $PJ_2 = 1$, $PJ_3 = 3$.

The first few members of this sequence are given as follow ;

n	0	1	2	3	4	5	6	7	8	9	10	...
FJ_n	0	0	1	3	10	28	77	203	526	1340	3377	...

Table 6. The first few members of the quadra Pell-Jacobsthal sequence

If we take the different initial conditions, we obtain the certain number sequences as follows;

n	0	1	2	3	Numbers
PJ_n	0	1	2	5	Pell numbers
PJ_n	2	1	4	9	Pell-Lucas numbers
PJ_n	0	1	1	3	Jacobsthal numbers
PJ_n	2	1	5	7	Jacobsthal-Lucas numbers

Table 7. The first few members of the different initial conditions

The characteristic equation associated to the recurrence relation is

$$q^4 - 3q^3 - q^2 + 5q + 2 = 0. \tag{2.6}$$

The roots of the equations are as follows

$$\gamma = 1+\sqrt{2}, \quad \delta = 1-\sqrt{2}, \quad \lambda = 2 \quad \text{and} \quad \mu = -1.$$

Then the following equalities follow directly from Vieta's formulas

$$\gamma + \delta + \lambda + \mu = 3 \quad \text{and} \quad \gamma\delta\lambda\mu = 2.$$

Theorem 2.9. *The Binet-like formula for the quadra Pell-Jacobsthal sequence is*

$$PJ_n = c_1\gamma^n + c_2\delta^n + c_3\lambda^n + c_4\mu^n,$$

where,

$$c_1 = \frac{3-(\delta+\lambda+\mu)}{(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)},$$

$$c_2 = \frac{3-(\gamma+\lambda+\mu)}{(\delta-\gamma)(\delta-\lambda)(\delta-\mu)},$$

$$c_3 = \frac{3-(\gamma+\delta+\mu)}{(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)},$$

$$c_4 = \frac{3-(\gamma+\delta+\lambda)}{(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},$$

such that γ, δ, λ and μ are the roots of the characteristic equation of the quadra Pell-Jacobsthal sequence.

Theorem 2.10. *The generating function for the quadra Pell-Jacobsthal sequence is*

$$G_{PJ}(x) = \sum_{n=0}^{\infty} PJ_n x^n = \frac{x^2}{1-3x-x^2+5x^3+2x^4}.$$

Proof. The proof can be given in a similar way of the proof of Theorem 2.14. □

The generating function of the quadra Pell-Jacobsthal sequence is the multiplication of the generating function of the Pell and Jacobsthal sequence as seen following,

$$G_P(x)G_J(x) = \left(\frac{x}{1-2x-x^2}\right)\left(\frac{x}{1-x-2x^2}\right) = \frac{x^2}{1-3x-x^2+5x^3+2x^4} = G_{PJ}(x)$$

Theorem 2.11. *The exponential generating function for the quadra Pell-Jacobsthal sequence is*

$$E_{PJ}(x) = c_1e^{\gamma x} + c_2e^{\delta x} + c_3e^{\lambda x} + c_4e^{\mu x} = \sum_{n=0}^{\infty} \frac{PJ_n}{n!}x^n.$$

Proof. The proof can be given in a similar way of the proof of Theorem2.15. □

Theorem 2.12. *The sum of the first n terms of PJ_n is*

$$\sum_{i=0}^{n} PJ_i = \frac{-PJ_{n+4}+2PJ_{n+3}+3PJ_{n+2}-2PJ_{n+1}+1}{4}, \quad n \geq 0.$$

Proof. The proof can be given in a similar way of the proof of Theorem 2.16. □

2.4 The Hexa Fibona-Pell-Jacobsthal Sequence

Definition 2.4. The hexa Fibona-Pell-Jacobsthal sequence K_n are defined as follows;

$$K_{n+6} = 4K_{n+5} - K_{n+4} - 9K_{n+3} + 2K_{n+2} + 7K_{n+1} + 2K_n \tag{2.7}$$

with the different initial conditions $K_0 = 0$, $K_1 = 0$, $K_2 = 0$, $K_3 = 1$, $K_4 = 4$, $K_5 = 15$.

The first few members of this sequence are given as follow;

n	0	1	2	3	4	5	6	7	8	...
K_n	0	0	0	1	4	15	47	139	389	...

Table 8. The first few members of the hexa Fibona-Pell-Jacobsthal sequence

If we take the different initial conditions, the certain number sequences are generated as follows;

n	0	1	2	3	4	5	Numbers
K_n	0	1	1	2	3	5	Fibonacci numbers
K_n	2	1	3	4	7	11	Lucas numbers
K_n	0	1	2	5	12	29	Pell numbers
K_n	2	1	4	9	22	53	Pell-Lucas numbers
K_n	0	1	1	3	5	11	Jacobsthal numbers
K_n	2	1	5	7	17	31	Jacobsthal-Lucas numbers

Table 9. The first few members of the different initial conditions

The characteristic equation associated to the recurrence relation is

$$t^6 - 4t^5 + t^4 + 9t^3 - 2t^2 - 7t - 2 = 0. \tag{2.8}$$

The roots of the equations are as follows

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad \gamma = 1+\sqrt{2}, \quad \delta = 1-\sqrt{2}, \quad \lambda = 2 \quad \text{and} \quad \mu = -1.$$

Then the following equalities follow directly from Vieta's formulas

$$\alpha + \beta + \gamma + \delta + \lambda + \mu = 4 \quad \text{and} \quad \alpha\beta\gamma\delta\lambda\mu = -2$$

Theorem 2.13. *The Binet-like formula for the hexa Fibona-Pell-Jacobsthal sequence is*

$$K_n = d_1\alpha^n + d_2\beta^n + d_3\gamma^n + d_4\delta^n + d_5\lambda^n + d_6\mu^n.$$

where, the coefficients d_i's are uniquely defined by the following relations,

$$\begin{aligned}
&d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0\\
&d_1\alpha + d_2\beta + d_3\gamma + d_4\delta + d_5\lambda + d_6\mu = 0\\
&d_1\alpha^2 + d_2\beta^2 + d_3\gamma^2 + d_4\delta^2 + d_5\lambda^2 + d_6\mu^2 = 0\\
&d_1\alpha^3 + d_2\beta^3 + d_3\gamma^3 + d_4\delta^3 + d_5\lambda^3 + d_6\mu^3 = 1\\
&d_1\alpha^4 + d_2\beta^4 + d_3\gamma^4 + d_4\delta^4 + d_5\lambda^4 + d_6\mu^4 = 4\\
&d_1\alpha^5 + d_2\beta^5 + d_3\gamma^5 + d_4\delta^5 + d_5\lambda^5 + d_6\mu^5 = 15
\end{aligned}$$

such that α, β, γ, δ, λ and μ are the roots of the characteristic equation of the hexa Fibona-Pell-Jacobsthal sequence.

Proof. Assume that

$$K_n = x_1\alpha^n + x_2\beta^n + x_3\gamma^n + x_4\delta^n + x_5\lambda^n + x_6\mu^n.$$

where α, β, γ, δ, λ and μ are roots of the characteristic equation of the hexa Fibona-Pell-Jacobsthal sequence and d_i's are un-known parameters. Talking $n = o, 1, 2, 3, 4, 5$ we have the system of lineer equations below

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0\\
&x_1\alpha + x_2\beta + x_3\gamma + x_4\delta + x_5\lambda + x_6\mu = 0\\
&x_1\alpha^2 + x_2\beta^2 + x_3\gamma^2 + x_4\delta^2 + x_5\lambda^2 + x_6\mu^2 = 0\\
&x_1\alpha^3 + x_2\beta^3 + x_3\gamma^3 + x_4\delta^3 + x_5\lambda^3 + x_6\mu^3 = 1\\
&x_1\alpha^4 + x_2\beta^4 + x_3\gamma^4 + x_4\delta^4 + x_5\lambda^4 + x_6\mu^4 = 4\\
&x_1\alpha^5 + x_2\beta^5 + x_3\gamma^5 + x_4\delta^5 + x_5\lambda^5 + x_6\mu^5 = 15
\end{aligned}$$

By the simplicity of the roots of the hexa Fibona-Pell-Jacobsthal sequence, the determinant of the system of the linear equations above is different from zero. Hence, the system of the linear equations above has uniquely solition, namely, $d_1, \ldots, d_6$. This completes the proof of the theorem. □

Theorem 2.14. *The generating function for the hexa Fibona-Pell-Jacobsthal sequence is*

$$G_K(x) = \sum_{n=0}^{\infty} K_n x^n = \frac{x^3}{1 - 4x + x^2 + 9x^3 - 2x^4 - 7x^5 - 2x^6}.$$

Proof. Let

$$G_K(x) = \sum_{n=0}^{\infty} K_n x^n = K_0 + K_1 x + K_2 x^2 + K_3 x^3 + \cdots + K_n x^n + \ldots$$

be the generating function of the hexa Fibona-Pell-Jacobsthal sequence. Multiply both of side of the equality by the term $-4x$, x^2 $9x^3$, $-2x^4$, $-7x^5$ and $-2x^6$, respectively, such as

$$\begin{aligned}
-4xG_K(x) &= -4K_0x - 4K_1x^2 - 4K_2x^3 - 4K_3x^4 - \cdots - 4K_nx^{n+1} + \ldots \\
x^2G_K(x) &= K_0x^2 + K_1x^3 + K_2x^4 + K_3x^5 + \cdots + K_nx^{n+2} + \ldots \\
9x^3G_K(x) &= 9K_0x^3 + 9K_1x^4 + 9K_2x^5 + 9K_3x^6 + \cdots + 9K_nx^{n+3} + \ldots \\
-2x^4G_K(x) &= -2K_0x^4 - 2K_1x^5 - 2K_2x^6 - 2K_3x^7 - \cdots - 2K_nx^{n+4} + \ldots \\
-7x^5G_K(x) &= -7K_0x^5 - 7K_1x^6 - 7K_2x^7 - 7K_3x^8 - \cdots - 7K_nx^{n+5} + \ldots \\
-2x^6G_K(x) &= -2K_0x^6 - 2K_1x^7 - 2K_2x^8 - 2K_3x^9 - \cdots - 2K_nx^{n+6} + \ldots
\end{aligned}$$

Then, we write Let's $T = (1 - 4x + x^2 + 9x^3 - 2x^4 - 7x^5 - 2x^6)G_K(x)$.

$$\begin{aligned}
T = {} & K_0 + (K_1 - 4K_0)x + (K_2 - 4K_1 + K_0)x^2 + (K_3 - 4K_2 + K_1 + 9K_0)x^3 + \\
& + (K_4 - 4K_3 + K_2 + 9K_1 - 2K_0)x^4 + (K_5 - 4K_4 + K_3 + 9K_2 - 2K_1 - 7K_0)x^5 \\
& + (K_6 - 4K_5 + K_4 + 9K_3 - 2K_2 - 7K_1 - 2K_0)x^6 + \ldots \\
& + (K_n - 4K_{n-1} + K_{n-2} + 9K_{n-3} - 2K_{n-4} - 7K_{n-5} - 2K_{n-6})x^n + \ldots
\end{aligned}$$

Now, by using the initial conditions of the hexa Fibona-Pell-Jacobsthal sequence and

$$K_n - 4K_{n-1} + K_{n-2} + 9K_{n-3} - 2K_{n-4} - 7K_{n-5} - 2K_{n-6} = 0,$$

we obtain that

$$G_K(x) = \sum_{n=0}^{\infty} K_n x^n = \frac{x^3}{1 - 4x + x^2 + 9x^3 - 2x^4 - 7x^5 - 2x^6}.$$

Thus, the proof is completed. □

We note that the generating function of the hexa Fibona-Pell-Jacobsthal sequence is the multiplication of the generating functions of the Fibonacci, Pell and Jacobsthal sequences as seen following

$$\begin{aligned}
G_F(x)G_P(x)G_J(x) &= \left(\frac{x}{1 - x - x^2}\right)\left(\frac{x}{1 - 2x - x^2}\right)\left(\frac{x}{1 - x - 2x^2}\right) \\
&= \frac{x^3}{1 - 4x + x^2 + 9x^3 - 2x^4 - 7x^5 - 2x^6} \\
&= G_K(x).
\end{aligned}$$

Theorem 2.15. *The exponential generating function for the Fibona-Pell-Jacobsthal sequence is*

$$E_K(x) = d_1e^{\alpha x} + d_2e^{\beta x} + d_3e^{\gamma x} + d_4e^{\delta x} + d_5e^{\lambda x} + d_6e^{\mu x} = \sum_{n=0}^{\infty} \frac{K_n}{n!} x^n.$$

Proof. We know that,

$$e^{\alpha x} = \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{n!}, \quad e^{\beta x} = \sum_{n=0}^{\infty} \frac{\beta^n x^n}{n!}, \quad e^{\gamma x} = \sum_{n=0}^{\infty} \frac{\gamma^n x^n}{n!}, \quad e^{\delta x} = \sum_{n=0}^{\infty} \frac{\delta^n x^n}{n!},$$
$$e^{\lambda x} = \sum_{n=0}^{\infty} \frac{\lambda^n x^n}{n!} \quad \text{and} \quad e^{\mu x} = \sum_{n=0}^{\infty} \frac{\mu^n x^n}{n!}$$

Multiplying each side of the identities, respectively, by d_1, d_2, d_3, d_4, d_5 and d_6 and adding of them, we obtain that

$$\begin{aligned} E_K(x) &= d_1 e^{\alpha x} + d_2 e^{\beta x} + d_3 e^{\gamma x} + d_4 e^{\delta x} + d_5 e^{\lambda x} + d_6 e^{\mu x} \\ &= \sum_{n=0}^{\infty} (d_1 \alpha^n + d_2 \beta^n + d_3 \alpha^n + d_4 \beta^n + d_5 \lambda^n + d_6 \mu^n) \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{K_n}{n!} x^n. \end{aligned}$$

□

Theorem 2.16. *The sum of the first n terms of K_n is*

$$\sum_{i=0}^{n} K_i = \frac{K_{n+6} - 3K_{n+5} - 2K_{n+4} + 7K_{n+3} + 5K_{n+2} - 2K_{n+1} - 1}{4}, \quad n \geq 0.$$

Proof. We know that

$$K_{n+6} = 4K_{n+5} - K_{n+4} - 9K_{n+3} + 2K_{n+2} + 7K_{n+1} + 2K_n$$

So,

$$2K_n + 2K_{n+1} = K_{n+6} - 4K_{n+5} + K_{n+4} + 9K_{n+3} - 2K_{n+2} - 5K_{n+1}.$$

Applying to the identity above, we deduce that

$$\begin{aligned} &2K_0 + 2K_1 = K_6 - 4K_5 + K_4 + 9K_3 - 2K_2 - 5K_1, \\ &2K_1 + 2K_2 = K_7 - 4K_6 + K_5 + 9K_4 - 2K_3 - 5K_2, \\ &2K_2 + 2K_3 = K_8 - 4K_7 + K_6 + 9K_5 - 2K_4 - 5K_3, \\ &\ldots, \\ &2K_{n-1} + 2K_n = K_{n+5} - 4K_{n+4} + K_{n+3} + 9K_{n+2} - 2K_{n+1} - 5K_n, \\ &2K_n + 2K_{n+1} = K_{n+6} - 4K_{n+5} + K_{n+4} + 9K_{n+3} - 2K_{n+2} - 5K_{n+1} \end{aligned}$$

If we sum of both of sides of the identities above, we obtain,

$$\begin{aligned} 4(K_0 + K_1 + K_2 + \cdots + K_n) + 2K_{n+1} - 2K_0 = K_{n+6} - 3K_{n+5} - 2K_{n+4} \\ + 7K_{n+3} + 5K_{n+2} - 1. \end{aligned}$$

Hence, we get the desired result. □

3. Conclusions

In this paper, we define new compound sequences as Fibonacci, Lucas, Pell and Pell-Lucas (Quadra Fibona-Pell), Fibonacci, Lucas, Jacobsthal and Jacobsthal-Lucas (Quadra Fibona-Jacobsthal), Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas (Quadra Pell-Jacobsthal) and Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas (Hexa Fibona-Pell-Jacobsthal) sequence. We prove that their characteristic equation is a multiplication of the characteristic equations of Fibonacci, Pell and Jacobsthal. We showed that by certain initial conditions from these sequences we derive all of compound sequences: Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas, Pell and Pell-Lucas. We gave the Binet-like formula for quadra Fibona-Pell, quadra Fibona-Jacobsthal, quadra Pell-Jacobsthal and hexa Fibona-Pell-Jacobsthal sequences. Finally, we obtain their generating functions. Also, we see that the generating functions of these sequences arise from the multiplication of the generating functions of Fibonacci, Pell and Jacobsthal sequences.

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