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On the Difference Sequence Space $\ell_p(\hat{T}^q)$

Pınar Zengin Alp* and Merve İlkhan

Abstract
In this study, we introduce a new matrix $\hat{T}^q = (\hat{t}^q_{nk})$ by

$$\hat{t}^q_{nk} = \begin{cases} \frac{q_n}{Q_n} t_n & , \quad k = n \\ \frac{q_k}{Q_n} t_k - \frac{q_{k+1}}{Q_n} \frac{1}{t_{k+1}} & , \quad k < n \\ 0 & , \quad k > n. \end{cases}$$

where $t_k > 0$ for all $n \in \mathbb{N}$ and $(t_n) \in c \backslash c_0$. By using the matrix $\hat{T}^q$, we introduce the sequence space $\ell_p(\hat{T}^q)$ for $1 \le p \le \infty$. In addition, we give some theorems on inclusion relations associated with $\ell_p(\hat{T}^q)$ and find the α-, β-, γ- duals of this space. Lastly, we analyze the necessary and sufficient conditions for an infinite matrix to be in the classes $(\ell_p(\hat{T}^q), \lambda)$ or $(\lambda, \ell_p(\hat{T}^q))$, where $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$.

Keywords: *sequence spaces; matrix transformations; Schauder basis; $\alpha-, \beta-, \gamma-$duals.*

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**Corresponding author*

1. Introduction and preliminaries

Let ω denote the set of all real or complex sequences and λ and μ be subsets of ω. We shall use $\sup_k$ instead of $\sup_{k\in\mathbb{N}}$ and $\sum_k$ instead of $\sum_{k=0}^{\infty}$, where $\mathbb{N} = \{0, 1, 2, ...\}$ to provide convenience. Also, if $u = (u_k)_{k=0}^{\infty} \in \omega$, we simply denote it by $u = (u_k)$. Further, $e = (1, 1, ...)$ and $e^{(k)}$ is the sequence whose kth term is 1 and the other terms are 0, that is, $e^{(k)} = (e_0^{(k)}, e_1^{(k)}, ..., e_k^{(k)}, ...) = (0, 0, ..., 1, ...)$. Any vector subspace of ω is called a *sequence space*. By ℓ_∞, c, c_0 and ℓ_p $(1 \le p < \infty)$, we denote the spaces of all bounded, convergent, null sequences and $p-$absolutely convergent series, respectively.

λ with a linear topology is called a *K-space* provided each of the maps $p_n : \lambda \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$, where $\mathbb{C}$ is the set of all complex numbers. If a K-space λ is a complete metric space, it is said to be an *FK-space*. A normed FK-space is defined as a *BK-space*, hence, a BK-space is a Banach sequence space. For instance, the sequence space ℓ_∞ is a BK-space with the norm given by $\|u\|_{\ell_\infty} = \sup_k |u_k|$. Further, ℓ_p is a complete $p-$normed space with respect to the usual p-norm defined by

$$\|u\|_{\ell_p} = \sum_k |u_k|^p \;\; (0 < p < 1)$$

and ℓ_p is a BK-space with respect to ℓ_p-norm defined by

$$\|u\|_{\ell_p} = \left(\sum_k |u_k|^p \right)^{1/p} \quad (1 \le p < \infty).$$

Let $B = (b_{nk})$ be an infinite matrix of real or complex numbers b_{nk}, where $n, k \in \mathbb{N}$. Then B defines a matrix mapping from λ into μ and we write $B : \lambda \to \mu$ if for every sequence $u = (u_k) \in \lambda$, the sequence $Bu = (B_n(u))$, the

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$B-$transform of u, is in μ, where

$$B_n(u) = \sum_k b_{nk} u_k \quad (n \in \mathbb{N}). \tag{1.1}$$

By (λ, μ), we denote the class of all infinite matrices that map λ into μ. Hence $A \in (\lambda, \mu)$ if and only if the series $\sum_k b_{nk} u_k$ converges for each $n \in \mathbb{N}$ and every $u \in \lambda$, and $Bu \in \mu$ for all $u \in \lambda$. If λ and μ are two arbitrary Banach spaces, then $\mathcal{B}(\lambda, \mu)$ denotes the set of all bounded linear operators from λ into μ.

The *matrix domain* λ_B of an infinite matrix B is defined by

$$\lambda_B = \{u = (u_k) \in \omega : Bu \in \lambda\}$$

which is also a sequence space.

In the literature, there are many papers related to new sequence spaces constructed by means of the matrix domain of a special triangle. See, for example [1]-[20]. For more information about matrix domains of triangles, one can see [21].

A sequence (β_n) in normed space λ is called a *Schauder basis* for λ if for every $u \in \lambda$ there is a unique sequence (α_n) of scalars such that $u = \sum_n \alpha_n \beta_n$, i.e.,

$$\lim_{m\to\infty} \|u - \sum_{n=0}^{m} \alpha_n \beta_n\| = 0.$$

By cs_0, cs and bs, we denote the set of all convergent to zero, convergent and bounded series, respectively, that is, $cs_0 = \left\{u = (u_k) \in \omega : \left(\sum_{k=0}^{n} u_k\right)_{n=0}^{\infty} \in c_0\right\}$, $cs = \{u = (u_k) \in \omega : (\sum_{k=0}^{n} u_k)_{n=0}^{\infty} \in c\}$ and $bs = \{u = (u_k) \in \omega : (\sum_{k=0}^{n} u_k)_{n=0}^{\infty} \in \ell_\infty\}$, and we define the norm on cs_0, cs and bs by $\|u\|_{cs_0} = \|u\|_{cs} = \|u\|_{bs} = \sup_n |\sum_{k=0}^{n} u_k|$. For all $z \in \omega$, we write $z^{-1} * \mu = \{x \in \omega : xz = (x_k z_k) \in \mu\}$. The set $Z = M(\lambda, \mu) = \cap_{u\in\lambda} u^{-1} * \mu = \{a \in \omega : au \in \mu \text{ for all } u \in \lambda\}$ is called the multiplier space of λ and μ. In the special case, where $\mu = \ell_1$, $\mu = cs$ or $\mu = bs$, the multiplier spaces $\lambda^\alpha = M(\lambda, \ell_1)$, $\lambda^\beta = M(\lambda, cs)$ and $\lambda^\gamma = M(\lambda, bs)$ are called the α-, β- and γ- *duals* of λ.

Throughout this paper, we assume that $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

The difference operator $\Delta : \omega \to \omega$ is defined by $\Delta u = (\Delta u_k) = (u_k - u_{k-1})$ or $\Delta u = (\Delta u_k) = (u_{k-1} - u_k)$ for all $u = (u_k) \in \omega$. When λ is a sequence space, the matrix domain λ_Δ is called the difference sequence space. For the first time, Kızmaz [22] gave the notion of difference sequence spaces as

$$\lambda(\Delta) = \{u = (u_k) \in \omega : (u_k - u_{k-1}) \in \lambda\}$$

for $\lambda = \ell_\infty, c$ and c_0. After Kızmaz, Et and Çolak [23] defined the generalized difference sequence spaces

$$\ell_\infty(\Delta^m) = \{u = (u_k) \in \omega : \Delta^m u \in \ell_\infty\},$$

$$c(\Delta^m) = \{u = (u_k) \in \omega : \Delta^m u \in c\}$$

and

$$c_0(\Delta^m) = \{u = (u_k) \in \omega : \Delta^m u \in c_0\},$$

where $m \in \mathbb{N}$, $\Delta^m u = (\Delta^m u_k) = (\Delta^{m-1} u_k - \Delta^{m-1} u_{k+1})$ and so that

$$\Delta^m u_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} u_{k+i}.$$

The difference space

$$bv_p = \{u = (u_k) \in \omega : (u_k - u_{k-1}) \in \ell_p\} \quad (0 < p < \infty)$$

was studied by Altay and Başar [24] for $0 < p < 1$ and in the case $1 \leq p \leq \infty$ Başar and Altay [25], and Çolak et al [26]. Recently, for $\lambda \in \{\ell_p, c_0, c, \ell_\infty\}$ $(1 \leq p < \infty)$, Kirişçi and Başar [4] introduced the generalized difference sequence space

$$\widehat{\lambda} = \{u = (u_k) :\in \omega : B(r,s)u = ((B(r,s)u)_k) \in \lambda\},$$

where $B(r,s)u$ is the sequence defined by $(B(r,s)u)_k = ru_k + su_{k-1}$ for all $k \in \mathbb{N}$ and $r, s \in \mathbb{R}\backslash\{0\}$.

In [27], the Fibonacci band matrix $\widehat{F}$ is defined by using Fibonacci numbers. Also, in [27] the Fibonacci difference sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ are introduced.

The Riesz matrix $R_q = (r_{nk})$ is defined by

$$r_{nk} = \begin{cases} \frac{q_k}{Q_n} & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$ and where (q_k) is the sequence of positive numbers and $Q_n = \sum_{k=0}^{n} q_k$ for all $n \in \mathbb{N}$. In [28], the paranormed Riesz sequence space is introduced.

In [29], the band matrix $T = (t_{nk})$ is defined by

$$t_{nk} = \begin{cases} t_n & , \quad k = n \\ -\frac{1}{t_n} & , \quad k < n \\ 0 & , \quad k > n \end{cases}$$

where $t_n > 0$ for all $n \in \mathbb{N}$ and $t = (t_n) \in c\backslash c_0$. Also in [29] the difference sequence spaces are introduced as follows:

$$\ell_p(T) = \left\{ u = (u_n) \in \omega : \sum_n \left| t_n u_n - \frac{1}{t_n} u_{n-1} \right|^p < \infty \right\} \quad (1 \le p < \infty)$$

and

$$\ell_\infty(T) = \left\{ u = (u_n) \in \omega : \sup_n \left| t_n u_n - \frac{1}{t_n} u_{n-1} \right| < \infty \right\}.$$

For more information on some new difference sequence spaces we refer to [30]-[37].

The paper is organized so that this section is followed by three sections. In Section 2 we give the definition of a new matrix and introduce the sequence spaces $\ell_p(\hat{T}^q)$ and $\ell_\infty(\hat{T}^q)$, where $1 \le p < \infty$. We prove that $\ell_p(\hat{T}^q)$ and $\ell_\infty(\hat{T}^q)$ are Banach spaces with respect to the norm defined on these spaces. Further, we establish some inclusion theorems related to the space $\ell_p(\hat{T}^q)$, where $1 \le p \le \infty$. In section 3 we determine the α-, β-, γ- duals of the space $\ell_p(\hat{T}^q)$ for $1 \le p \le \infty$. In the last section we characterize the classes $(\ell_p(\hat{T}^q), \lambda)$ and $(\lambda, \ell_p(\hat{T}^q))$ for $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$.

2. The difference sequence space $\ell_p(\hat{T}^q)$

In this section, we introduce a new matrix $\hat{T}^q$ by multiplying Riesz matrix and the band matrix T and introduce the difference sequence space $\ell_p(\hat{T}^q)$ derived by using this matrix, where $1 \le p \le \infty$. Also, we give some theorems which give inclusion relations corcerning this space. By multiplying these matrices we derive a new matrix $\hat{T}^q = (\hat{t}^q_{nk})$ as

$$\hat{t}^q_{nk} = \begin{cases} \frac{q_n}{Q_n} t_n & , \quad k = n \\ \frac{q_k}{Q_n} t_k - \frac{q_{k+1}}{Q_n} \frac{1}{t_{k+1}} & , \quad k < n \\ 0 & , \quad k > n. \end{cases}$$

$(\hat{T}^q)^{-1} = ((\hat{t}^q)^{-1}_{nk})$, the inverse of $\hat{T}^q$ can be easily computed as

$$(\hat{t}^q)^{-1}_{nk} = \begin{cases} \dfrac{Q_n}{q_n} \dfrac{1}{t_n} & , \quad k = n \\ Q_k \left[\dfrac{1}{q_k} \left(t_k \prod_{j=k}^{n} \dfrac{1}{t_j^2} \right) - \dfrac{1}{q_{k+1}} \left(t_{k+1} \prod_{j=k+1}^{n} \dfrac{1}{t_j^2} \right) \right] & , \quad k < n \\ 0 & , \quad k > n. \end{cases}$$

Now, let give the definitions of the difference sequence spaces $\ell_p(\hat{T}^q)$ and $\ell_\infty(\hat{T}^q)$ derived by this matrix

$$\ell_p(\hat{T}^q) = \left\{ u = (u_n) \in \omega : \sum_n \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k \left(t_k u_k - \frac{u_{k-1}}{t_k} \right) \right|^p < \infty \right\} \quad (1 \le p < \infty)$$

and

$$\ell_\infty(\hat{T}^q) = \left\{ u = (u_n) \in \omega : \sup_n \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k \left(t_k u_k - \frac{u_{k-1}}{t_k} \right) \right| < \infty \right\}.$$

For the $\hat{T}^q$-transform of a sequence $u = (u_n)$, we will use the sequence $\hat{u} = (\hat{u}_n)$ defined as

$$\hat{u}_n = \hat{T}^q_n(u) = \frac{1}{Q_n} \sum_{k=0}^{n} q_k \left(t_k u_k - \frac{u_{k-1}}{t_k} \right) \ (n \in \mathbb{N}). \tag{2.1}$$

Theorem 2.1. *For $1 \leq p \leq \infty$, $\ell_p(\hat{T}^q)$ is a Banach space with the norm $\|u\|_{\ell_p(\hat{T}^q)} = \|\hat{T}^q u\|_{\ell_p}$, defined as,*

$$\|u\|_{\ell_p(\hat{T}^q)} = \begin{cases} \left(\sum_n |\hat{T}^q_n(u)|^p \right)^{1/p} & , \quad 1 \leq p < \infty \\ \sup_n |\hat{T}^q_n(u)| & , \quad p = \infty. \end{cases}$$

Proof. If we assume that $\|u\|_{\ell_p(\hat{T}^q)} = 0$. Then, $\|\hat{T}^q u\|_{\ell_p} = 0$ and since $\|.\|_{\ell_p}$ is a norm we have $\hat{T}^q u = \theta$. Since it is known that $\hat{T}^q$ is invertible, we have $u = \theta$.

Let $\alpha \in \mathbb{C}$ and $u \in \ell_p(\hat{T}^q)$. Then,

$$\begin{aligned} \|\alpha u\|_{\ell_p(\hat{T}^q)} &= \|\hat{T}^q(\alpha u)\|_{\ell_p} = \|\alpha \hat{T}^q u\|_{\ell_p} \\ &= |\alpha| \|\hat{T}^q u\|_{\ell_p} = |\alpha| \|u\|_{\ell_p(\hat{T}^q)}. \end{aligned}$$

Finally, for $u, v \in \ell_p(\hat{T}^q)$ we have

$$\begin{aligned} \|u + v\|_{\ell_p(\hat{T}^q)} &= \|\hat{T}^q(u + v)\|_{\ell_p} = \|\hat{T}^q u + \hat{T}^q v\|_{\ell_p} \\ &\leq \|\hat{T}^q u\|_{\ell_p} + \|\hat{T}^q v\|_{\ell_p} = \|u\|_{\ell_p(\hat{T}^q)} + \|v\|_{\ell_p(\hat{T}^q)} \end{aligned}$$

and so the triangle inequality holds.

This means that, $(\ell_p(\hat{T}^q), \|.\|_{\ell_p(\hat{T}^q)})$ is a normed sequence space for $1 \leq p \leq \infty$. To show that $\ell_p(\hat{T}^q)$ is a Banach space, let (u_n) be a Cauchy sequece in $\ell_p(\hat{T}^q)$. Then, $(\hat{u}_n)$ is a sequence in ℓ_p. Obviously,

$$\begin{aligned} \|u_n - u_m\|_{\ell_p(\hat{T}^q)} &= \|\hat{T}^q(u_n - u_m)\|_{\ell_p} \\ &= \|\hat{T}^q u_n - \hat{T}^q u_m\|_{\ell_p} = \|\hat{u}_n - \hat{u}_m\|_{\ell_p}, \end{aligned}$$

hence, $(\hat{u}_n)$ is a Cauchy sequence in ℓ_p. Since $(\ell_p, \|.\|_{\ell_p})$ is a Banach space, there exists $\hat{u} \in \ell_p$ such that $\lim_{n\to\infty} \hat{u}_n = \hat{u}$ in ℓ_p. Since $u = (\hat{T}^q)^{-1}\hat{u}$, we have

$$\begin{aligned} \lim_{n\to\infty} \|u_n - u\|_{\ell_p(\hat{T}^q)} &= \lim_{n\to\infty} \|\hat{T}^q(u_n - u)\|_{\ell_p} \\ &= \lim_{n\to\infty} \|\hat{T}^q u_n - \hat{T}^q u\|_{\ell_p} = \lim_{n\to\infty} \|\hat{u}_n - \hat{u}\|_{\ell_p} = 0. \end{aligned}$$

Hence $\lim_{n\to\infty} u_n = u$ in $\ell_p(\hat{T}^q)$, where $u \in \ell_p(\hat{T}^q)$.

Remark 2.1. $\ell_p(\hat{T}^q)$ is a BK-space for $1 \leq p \leq \infty$.

Theorem 2.2. *The sequence spaces $\ell_p(\hat{T}^q)$ and ℓ_p are linearly isomorphic; that is, $\ell_p(\hat{T}^q) \cong \ell_p$ for $1 \leq p \leq \infty$.*

Proof. It must be shown that there exists a linear bijection between the spaces $\ell_p(\hat{T}^q)$ and ℓ_p for $1 \leq p \leq \infty$. Let $\hat{T}^q$ be the transformation defined from $\ell_p(\hat{T}^q)$ to ℓ_p by $u \to \hat{u} = \hat{T}^q u = (\hat{T}^q_n(u))$. Then, we have $\hat{T}^q u = \hat{u} \in \ell_p$ for every $u \in \ell_p(\hat{T}^q)$. Hence, $\hat{T}^q$ is a linear transformation. Also, $\hat{T}^q$ is injective since $u = \theta$ whenever $\hat{T}^q u = \theta$.

Moreover, let $v = (v_n) \in \ell_p$ be given for $1 \leq p \leq \infty$ and define the sequence $u = (u_n)$ as follows:

$$u_n = \sum_{k=0}^{n} Q_k \left[\frac{1}{q_k} \left(t_k \prod_{j=k}^{n} \frac{1}{t_j^2} \right) - \frac{1}{q_{k+1}} \left(t_{k+1} \prod_{j=k+1}^{n} \frac{1}{t_j^2} \right) \right] v_k \quad (n \in \mathbb{N}). \tag{2.2}$$

Then, by combining (2.1) and (2.2), we get for every $n \in \mathbb{N}$

$$\begin{aligned}\hat{T}^q_n(u) &= \frac{1}{Q_n}\sum_{k=0}^{n} q_k \left(t_k \sum_{r=0}^{k} Q_r \left[\frac{1}{q_r}\left(t_r \prod_{j=r}^{k} \frac{1}{t_j^2}\right) - \frac{1}{q_{r+1}}\left(t_{r+1} \prod_{j=r+1}^{k} \frac{1}{t_j^2}\right)\right] v_r \right) \\ &- \frac{1}{Q_n}\sum_{k=0}^{n} q_k \left(\frac{1}{t_k} \sum_{r=0}^{k-1} Q_r \left[\frac{1}{q_r}\left(t_r \prod_{j=r}^{k-1} \frac{1}{t_j^2}\right) - \frac{1}{q_{r+1}}\left(t_{r+1} \prod_{j=r+1}^{k-1} \frac{1}{t_j^2}\right)\right] v_r \right) \\ &= v_n\end{aligned}$$

This means that $\hat{T}^q u = v$. Since $v \in \ell_p$, we have $\hat{T}^q u \in \ell_p$. Thus, we conclude that $u \in \ell_p(\hat{T}^q)$ for any $v \in \ell_p$. Hence $\hat{T}^q$ is surjective.

Since $\|u\|_{\ell_p(\hat{T}^q)} = \|\hat{T}^q u\|_{\ell_p}$ for any $u \in \ell_p(\hat{T}^q)$, we have

$$\|v\|_{\ell_p} = \|\hat{T}^q u\|_{\ell_p} = \|u\|_{\ell_p(\hat{T}^q)}$$

which shows that $\hat{T}^q$ preserves the norm, where $1 \leq p \leq \infty$. Hence, $\hat{T}^q$ is an isometry. As a result, the space $\ell_p(\hat{T}^q)$ is isometrically isomorphic to ℓ_p for $1 \leq p \leq \infty$.

It is known that the space ℓ_p is not a Hilbert space with $p \neq 2$. The similar result is valid for the space $\ell_p(\hat{T}^q)$ and the following theorem gives this result.

Theorem 2.3. *The space $\ell_p(\hat{T}^q)$ is not an inner product space in the case $p \neq 2$. Hence, $\ell_p(\hat{T}^q)$ is not a Hilbert space for $1 \leq p < \infty$ and $p \neq 2$.*

Proof. We must show that the space $\ell_2(\hat{T}^q)$ is a Hilbert space while $\ell_p(\hat{T}^q)$ is not a Hilbert space for $p \neq 2$. By Theorem 2.1, we know that $\ell_2(\hat{T}^q)$ is a Banach space with the norm $\|u\|_{\ell_2(\hat{T}^q)} = \|\hat{T}^q u\|_{\ell_2}$ and its norm can be obtained as follows:

$$\|u\|_{\ell_2(\hat{T}^q)} = \langle u, u\rangle^{1/2}_{\ell_2(\hat{T}^q)} = \langle \hat{T}^q u, \hat{T}^q u\rangle^{1/2}_{\ell_2} = \|\hat{T}^q u\|_{\ell_2}$$

for every $u \in \ell_2(\hat{T}^q)$. Hence $\ell_2(\hat{T}^q)$ is a Hilbert space.

Consider the sequences

$$s = (s_n) = \begin{cases} \frac{1}{t_0} & , \quad n = 0 \\ \frac{1}{t_0 t_1^2} + \frac{1}{t_1} & , \quad n = 1 \\ t_0 \prod_{i=0}^{n} \frac{1}{t_i^2} + t_1 \prod_{i=1}^{n} \frac{1}{t_i^2} - \frac{Q_1 t_2}{q_2} \prod_{i=2}^{n} \frac{1}{t_i^2} & , \quad n \geq 2 \ \ (n \in \mathbb{N}) \end{cases}$$

and

$$t = (t_n) = \begin{cases} \frac{1}{t_0} & , \quad n = 0 \\ \frac{1}{t_0 t_1^2} - \frac{(Q_0+Q_1)}{q_1 t_1} & , \quad n = 1 \\ t_0 \prod_{i=0}^{n} \frac{1}{t_i^2} - \frac{(Q_0+Q_1)}{q_1} t_1 \prod_{i=1}^{n} \frac{1}{t_i^2} + \frac{Q_1 t_2}{q_2} \prod_{i=2}^{n} \frac{1}{t_i^2} & , \quad n \geq 2 \ \ (n \in \mathbb{N}) \end{cases}$$

With the $\hat{T}^q$-transforms of s and t, we have the following sequences

$$\hat{T}^q s = (1, 1, 0, 0, ...) \text{ and } \hat{T}^q t = (1, -1, 0, 0, ...).$$

Also, it can be easily seen that

$$\|s+t\|^2_{\ell_p(\hat{T}^q)} + \|s-t\|^2_{\ell_p(\hat{T}^q)} = 8 \neq 4(2^{2/p}) = 2(\|s\|^2_{\ell_p(\hat{T}^q)} + \|t\|^2_{\ell_p(\hat{T}^q)})$$

for $p \neq 2$. This means that the parallelogram equality cannot be satisfied by the norm of the space $\ell_p(\hat{T}^q)$ for $p \neq 2$. Therefore, this norm cannot be gained from an inner product. Therefore, the space $\ell_p(\hat{T}^q)$ with $p \neq 2$ is a Banach space but it is not a Hilbert space, where $1 \leq p < \infty$. The proof is completed.

Remark 2.2. Obviously, the space $\ell_\infty(\hat{T}^q)$ is also a Banach space but it is not a Hilbert space.

Now, we give some theorems on inclusion relations associated with the space $\ell_p(\hat{T}^q)$.

Theorem 2.4. *For $1 \leq p < q < \infty$ the inclusion relation $\ell_p(\hat{T}^q) \subset \ell_q(\hat{T}^q)$ strictly holds.*

Proof. Let $1 \leq p < q < \infty$. If u is any sequence in $\ell_p(\hat{T}^q)$, then its $\hat{T}^q$-transform $\hat{T}^q u$ is in ℓ_p. Since the inclusion $\ell_p \subset \ell_q$ holds, $\hat{T}^q u$ is also in ℓ_q. Hence $u \in \ell_q(\hat{T}^q)$ which means that $\ell_p(\hat{T}^q) \subset \ell_q(\hat{T}^q)$. Now, we must prove that the inclusion holds strictly. For this, there should be a sequence $\hat{v} = (\hat{v}_n) \in \ell_q$ but not in ℓ_p, i.e., $\hat{v} \in \ell_q \backslash \ell_p$. The existence of $\hat{v} \in \ell_q \backslash \ell_p$ is clear since, as a well known fact, $\ell_p \subset \ell_q$ is a strict inclusion. Let define the sequence $v = (v_n)$ in terms of the sequence $\hat{v}$ as follows:

$$v_n = \sum_{k=0}^{n} Q_k \left[\frac{1}{q_k} \left(t_k \prod_{j=k}^{n} \frac{1}{t_j^2} \right) - \frac{1}{q_{k+1}} \left(t_{k+1} \prod_{j=k+1}^{n} \frac{1}{t_j^2} \right) \right] \hat{v}_k \quad (n \in \mathbb{N}).$$

Then, it is clear that

$$\hat{T}_n^q(v) = \hat{v}_n$$

for every $n \in \mathbb{N}$. This shows that $\hat{T}^q v = \hat{v}$ and since $\hat{v} \in \ell_q \backslash \ell_p$, we have $\hat{T}^q v \in \ell_q \backslash \ell_p$. Hence, the sequence v must be in $\ell_q(\hat{T}^q)$ but cannot be in $\ell_p(\hat{T}^q)$, that is, the inclusion $\ell_p(\hat{T}^q) \subset \ell_q(\hat{T}^q)$ is strict. The proof is completed.

Theorem 2.5. *For $1 \leq p < \infty$ the inclusion $\ell_p(\hat{T}^q) \subset \ell_\infty(\hat{T}^q)$ is strict.*

Proof. If $u \in \ell_p(\hat{T}^q)$, then $\hat{T}^q u \in \ell_p$. Since $\ell_p \subset \ell_\infty$, $\hat{T}^q u \in \ell_\infty$. Hence, $u \in \ell_\infty(\hat{T}^q)$ which shows that $\ell_p(\hat{T}^q) \subset \ell_\infty(\hat{T}^q)$. To show that this inclusion is strict, we define the sequence $v = (v_n)$ by

$$v_n = t_0 \prod_{i=0}^{n} \frac{1}{t_i^2} + \sum_{i=2}^{n} (-1)^{i-1} \frac{(Q_{i-2} + Q_{i-1})}{q_{i-1}} \left(t_{i-1} \prod_{k=i-1}^{n} \frac{1}{t_k^2} \right) + (-1)^n \frac{(Q_{n-1} + Q_n)}{q_n t_n} \quad (n \in \mathbb{N}).$$

Then, we have for every $n \in \mathbb{N}$ that

$$\begin{aligned} \hat{T}_n^q(v) &= \frac{1}{Q_n} \sum_{k=0}^{n} q_k \left(t_k v_k - \frac{v_{k-1}}{t_k} \right) \\ &= (-1)^n. \end{aligned}$$

Then, $\hat{T}^q v \in \ell_\infty \backslash \ell_p$ since $((-1)^n) \in \ell_\infty$ but not in ℓ_p. Thus, v is in $\ell_\infty(\hat{T}^q)$ but not in $\ell_p(\hat{T}^q)$ which means that the inclusion $\ell_p(\hat{T}^q) \subset \ell_\infty(\hat{T}^q)$ strictly holds. The proof is completed.

3. The α-, β- and γ-duals of the space $\ell_p(\hat{T}^q)$

In this section, we determine the α-, β- and γ-duals of the sequence space $\ell_p(\hat{T}^q)$, where $1 \leq p \leq \infty$. Also, we give a sequence of the points of the space $\ell_p(\hat{T}^q)$ which forms a basis for this space.

The following known results in [38] and [39] are fundamental for our investigation.

$$\sup_n \sum_k |b_{nk}|^q < \infty. \tag{3.1}$$

$$\lim_{n\to\infty} b_{nk} \ \ exists \ for \ all \ k \in \mathbb{N}. \tag{3.2}$$

$$\lim_{n\to\infty} b_{nk} = 0 \ for \ all \ k \in \mathbb{N}. \tag{3.3}$$

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} b_{nk} \right|^q < \infty. \tag{3.4}$$

$$\sup_{n,k} |b_{nk}| < \infty. \tag{3.5}$$

$$\sup_k \sum_n |b_{nk}| < \infty. \tag{3.6}$$

$$\lim_{n\to\infty}\sum_k |b_{nk}| = \sum_k \left|\lim_{n\to\infty} b_{nk}\right|. \tag{3.7}$$

$$\lim_{n\to\infty}\sum_k |b_{nk}| = 0. \tag{3.8}$$

Lemma 3.1. *Let $B=(b_{nk})$ be an infinite matrix. The following statements hold:*

1.$B\in(\ell_p,\ell_\infty)\Leftrightarrow$ *(3.1).*
2. $B\in(\ell_1,\ell_\infty)\Leftrightarrow$ *(3.5).*
3. $B\in(\ell_\infty,\ell_\infty)\Leftrightarrow$ *(3.1) with q=1.*
4. $B\in(\ell_p,c)\Leftrightarrow$ *(3.1) and (3.2).*
5. $B\in(\ell_1,c)\Leftrightarrow$ *(3.2) and (3.5).*
6. $B\in(\ell_\infty,c)\Leftrightarrow$ *(3.2) and (3.7).*
7. $B\in(\ell_p,c_0)\Leftrightarrow$ *(3.1) and (3.3).*
8. $B\in(\ell_1,c_0)\Leftrightarrow$ *(3.3) and (3.5).*
9. $B\in(\ell_\infty,c_0)\Leftrightarrow$ *(3.3) and (3.8).*
10. $B\in(\ell_p,\ell_1)\Leftrightarrow$ *(3.4).*
11. $B\in(\ell_1,\ell_1)\Leftrightarrow$ *(3.6).*
12. $B\in(\ell_\infty,\ell_1)\Leftrightarrow$ *(3.4) with q=1.*

Now, let give two lemmas which are needed to determine the $\alpha-,\beta$- and γ-duals of the space $\ell_p(\hat{T}^q)$, where $1\le p\le\infty$.

Lemma 3.2. *Let $a=(a_n)\in\omega$ and the matrix $\hat{B}=(\hat{b}_{nk})$ be defined by $\hat{B}_n=a_n(\hat{T}^q_n)^{-1}$, that is,*

$$\hat{b}_{nk}=\begin{cases} 0 & , \quad k>n \\ a_n(\hat{t}^q)^{-1}_{nk} & , \quad 0\le k\le n \end{cases}$$

for all $k,n\in\mathbb{N}$. Then, $a\in(\ell_p(\hat{T}^q))^\alpha$ if and only if $\hat{B}\in(\ell_p,\ell_1)$, where $1\le p\le\infty$.

Proof. Let $\hat{u}$ be the $\hat{T}^q$-transform of a sequence $u=(u_n)\in\omega$. Then, we have

$$a_nu_n=a_n(\hat{T}^q)^{-1}_n(\hat{u})=\hat{B}_n(\hat{u})$$

for all $n\in\mathbb{N}$. So, from this equality it can be easily seen that $au=(a_nu_n)\in\ell_1$ with $u\in\ell_p(\hat{T}^q)$ if and only if $\hat{B}\hat{u}\in\ell_1$ with $\hat{u}\in\ell_p$. This implies that $a\in(\ell_p(\hat{T}^q))^\alpha$ if and only if $\hat{B}\in(\ell_p,\ell_1)$. The proof is completed.

Lemma 3.3. *[40, Theorem 3.1] Let $C=(c_{nk})$ be defined via a sequence $a=(a_k)\in\omega$ and the inverse matrix $V=(v_{nk})$ of the triangle matrix $U=(u_{nk})$ by*

$$c_{nk}=\begin{cases} 0 & , \quad k>n \\ \sum_{j=k}^n a_jv_{jk} & , \quad 0\le k\le n \end{cases}$$

for all $k,n\in\mathbb{N}$. Then,

$$(\ell_p(U))^\gamma=\{a=(a_k)\in\omega : C\in(\ell_p,\ell_\infty)\},$$
$$(\ell_p(U))^\beta=\{a=(a_k)\in\omega : C\in(\ell_p,c)\},$$

where $1\le p\le\infty$.

Combining Lemmas 3.1-3.3 we have;

Corollary 3.1. *Let the sets $\hat{d}_1,\hat{d}_2,\hat{d}_3,\hat{d}_4,\hat{d}_5$ and $\hat{d}_6$ be defined as follows:*

$$\hat{d}_1=\left\{a=(a_k)\in\omega : \sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}\left(Q_k\left[\frac{1}{q_k}\left(t_k\prod_{j=k}^n\frac{1}{t_j^2}\right)-\frac{1}{q_{k+1}}\left(t_{k+1}\prod_{j=k+1}^n\frac{1}{t_j^2}\right)\right]\right)a_n\right|^q<\infty\right\},$$

$$\hat{d}_2=\left\{a=(a_k)\in\omega : \sum_{j=k}^\infty\left(Q_k\left[\frac{1}{q_k}\left(t_k\prod_{i=k}^j\frac{1}{t_i^2}\right)-\frac{1}{q_{k+1}}\left(t_{k+1}\prod_{i=k+1}^j\frac{1}{t_i^2}\right)\right]\right)a_j \text{ exists for each } k\in\mathbb{N}\right\},$$

$$\hat{d_3} = \left\{ a = (a_k) \in \omega : \sup_n \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \left(Q_k \left[\frac{1}{q_k} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) - \frac{1}{q_{k+1}} \left(t_{k+1} \prod_{i=k+1}^{j} \frac{1}{t_i^2} \right) \right] \right) a_j \right|^q < \infty \right\},$$

$$\hat{d_4} = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \left(Q_k \left[\frac{1}{q_k} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) - \frac{1}{q_{k+1}} \left(t_{k+1} \prod_{i=k+1}^{j} \frac{1}{t_i^2} \right) \right] \right) a_j \right| \right\},$$

$$\hat{d_5} = \left\{ a = (a_k) \in \omega : \sup_k \sum_{n} \left| \left(Q_k \left[\frac{1}{q_k} \left(t_k \prod_{j=k}^{n} \frac{1}{t_j^2} \right) - \frac{1}{q_{k+1}} \left(t_{k+1} \prod_{j=k+1}^{n} \frac{1}{t_j^2} \right) \right] \right) a_n \right| < \infty \right\}$$

and

$$\hat{d_6} = \left\{ a = (a_k) \in \omega : \sup_{n,k} \left| \sum_{j=k}^{n} \left(Q_k \left[\frac{1}{q_k} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) - \frac{1}{q_{k+1}} \left(t_{k+1} \prod_{i=k+1}^{j} \frac{1}{t_i^2} \right) \right] \right) a_j \right| < \infty \right\}.$$

Then, the following statements hold:
(a) $(\ell_p(\hat{T}^q))^\alpha = \hat{d_1}$ *and* $(\ell_1(\hat{T}^q))^\alpha = \hat{d_5}$, *where* $1 < p \leq \infty$.
(b) $(\ell_p(\hat{T}^q))^\beta = \hat{d_2} \cap \hat{d_3}$, $(\ell_\infty(\hat{T}^q))^\beta = \hat{d_2} \cap \hat{d_4}$ *and* $(\ell_1(\hat{T}^q))^\beta = \hat{d_2} \cap \hat{d_6}$, *where* $1 < p < \infty$.
(c) $(\ell_p(\hat{T}^q))^\gamma = \hat{d_3}$ *and* $(\ell_1(\hat{T}^q))^\gamma = \hat{d_6}$, *where* $1 < p \leq \infty$.

Now, we give the Schauder basis of the space $\ell_p(\hat{T}^q)$ $(1 \leq p < \infty)$.

Theorem 3.1. *Let* $1 \leq p < \infty$ *and define the sequence* $c^{(k)} \in \ell_p(\hat{T}^q)$ *for every fixed* $k \in \mathbb{N}$ *by*

$$(c^{(k)})_n = \begin{cases} 0 & , \quad n < k \\ \left(Q_k \left[\frac{1}{q_k} \left(t_k \prod_{j=k}^{n} \frac{1}{t_j^2} \right) - \frac{1}{q_{k+1}} \left(t_{k+1} \prod_{j=k+1}^{n} \frac{1}{t_j^2} \right) \right] \right) & , \quad n \geq k \end{cases} \quad (n \in \mathbb{N}). \tag{3.9}$$

Then the sequence $(c^{(k)})$ *is a basis for the space* $\ell_p(\hat{T}^q)$, *and every* $u \in \ell_p(\hat{T}^q)$ *has a unique representation of the form*

$$u = \sum_k \hat{T}^q_k(u) c^{(k)}. \tag{3.10}$$

Proof. Let $1 \leq p < \infty$. By (3.9), it is clear that $\hat{T}^q(c^{(k)}) = e^{(k)} \in \ell_p$ and $c^{(k)} \in \ell_p(\hat{T}^q)$ for all $k \in \mathbb{N}$.
Also, let $u \in \ell_p(\hat{T}^q)$ given. For every non-negative integer m, we put

$$u^{(m)} = \sum_{k=0}^{m} \hat{T}^q_k(u) c^{(k)}.$$

Then, we obtain

$$\hat{T}^q(u^{(m)}) = \sum_{k=0}^{m} \hat{T}^q_k(u) \hat{T}^q(c^{(k)}) = \sum_{k=0}^{m} \hat{T}^q_k(u) e^{(k)}$$

and so

$$\hat{T}^q_n(u - u^{(m)}) = \begin{cases} 0 & (0 \leq n \leq m) \\ \hat{T}^q_n(u) & (n > m). \end{cases}$$

Let $\epsilon > 0$ be given. Then, there exists a non-negative integer m_0 which satisfies

$$\sum_{n=m_0+1}^{\infty} |\hat{T}^q_n(u)|^p \leq \left(\frac{\epsilon}{2} \right)^p.$$

So, we obtain for every $m \geq m_0$ that

$$\|u - u^{(m)}\|_{\ell_p(\hat{T}^q)} = \left(\sum_{n=m+1}^{\infty} |\hat{T}^q_n(u)|^p \right)^{1/p} \leq \left(\sum_{n=m_0+1}^{\infty} |\hat{T}^q_n(u)|^p \right)^{1/p} \leq \frac{\epsilon}{2} < \epsilon$$

which indicates that $\lim_{m\to\infty} \|u - u^{(m)}\|_{\ell_p(\hat{T}^q)} = 0$ and hence u is shown as in (3.10).

Finally, we must prove that the representation (3.10) of $u \in \ell_p(\hat{T}^q)$ is unique. Assume that $u = \sum_k \mu_k(u)c^{(k)}$. The continuity of the linear transformation $\hat{T}^q : \ell_p(\hat{T}^q) \to \ell_p$ which is defined in the proof of Theorem 2.2 is clear, we have

$$\hat{T}^q_n(u) = \sum_k \mu_k(u)\hat{T}^q_n(c^{(k)}) = \sum_k \mu_k(u)\delta_{nk} = \mu_n(u) \quad (n \in \mathbb{N}).$$

Hence, the representation (3.10) of $u \in \ell_p(\hat{T}^q)$ is unique. The proof is completed.

4. Characterization of some matrix transformations on $\ell_p(\hat{T}^q)$

In this section of the study, we obtain the characterization of the classes $(\ell_p(\hat{T}^q), \lambda)$, $(\lambda, \ell_p(\hat{T}^q))$, where $1 \le p \le \infty$, $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ and $\mu \in \{\ell_1, \ell_\infty\}$.

Throughout this section, we write $b(n,k) = \sum_{j=0}^n b_{jk}$ for given an infinite matrix $B = (b_{nk})$, where $n, k \in \mathbb{N}$.

Firstly, we give a theorem which is essential for our results.

Theorem 4.1. *Let $1 \le p \le \infty$. Then, we have $B = (b_{nk}) \in (\ell_p(\hat{T}^q), \lambda)$ if and only if*

$$E^{(m)} = \left(e^{(m)}_{nk}\right) \in (\ell_p, c) \; for \; all \; n \in \mathbb{N}, \tag{4.1}$$

$$E = (e_{nk}) \in (\ell_p, \lambda), \tag{4.2}$$

where $$e^{(m)}_{nk} = \begin{cases} 0 & , \quad k > m \\ \sum_{j=k}^m Q_k \left[\frac{1}{q_k}\left(t_k \prod_{i=k}^j \frac{1}{t_i^2}\right) - \frac{1}{q_{k+1}}\left(t_{k+1} \prod_{i=k+1}^j \frac{1}{t_i^2}\right)\right] b_{nj} & , \quad 0 \le k \le m \end{cases}$$

and $$e_{nk} = \sum_{j=k}^\infty Q_k \left[\frac{1}{q_k}\left(t_k \prod_{i=k}^j \frac{1}{t_i^2}\right) - \frac{1}{q_{k+1}}\left(t_{k+1} \prod_{i=k+1}^j \frac{1}{t_i^2}\right)\right] b_{nj} \; for \; all \; k, m, n \in \mathbb{N}.$$

Proof. For the proof, we follow the similar tecnique due to Kirişçi and Başar [4]. Let $B = (b_{nk}) \in (\ell_p(T), \lambda)$ and $u = (u_k) \in \ell_p(\hat{T}^q)$. By (2.2), we have

$$\begin{aligned} \sum_{k=0}^m b_{nk}u_k &= \sum_{k=0}^m b_{nk}\left(\sum_{j=0}^k Q_j \left[\frac{1}{q_j}\left(t_j \prod_{i=j}^k \frac{1}{t_i^2}\right) - \frac{1}{q_{j+1}}\left(t_{j+1} \prod_{i=j+1}^j \frac{1}{t_i^2}\right)\right]\right)\hat{u}_j \\ &= \sum_{k=0}^m \left(\sum_{j=k}^m Q_k \left[\frac{1}{q_k}\left(t_k \prod_{i=k}^j \frac{1}{t_i^2}\right) - \frac{1}{q_{k+1}}\left(t_{k+1} \prod_{i=k+1}^j \frac{1}{t_i^2}\right)\right] b_{nj}\right)\hat{u}_k \\ &= \sum_{k=0}^m e^{(m)}_{nk}\hat{u}_k \\ &= E^{(m)}_n(\hat{u}) \end{aligned}$$

for all $m, n \in \mathbb{N}$. Since Bu exists, $E^{(m)}$ belongs to the class (ℓ_p, c). Letting $m \to \infty$ in the last equality, we obtain $Bu = U\hat{u}$ which gives the result $E \in (\ell_p, \lambda)$.

Conversely, suppose the conditions (4.1), (4.2) hold and take any $u \in \ell_p(T)$. Then, we have $(e_{nk})_{k\in\mathbb{N}} \in \ell_p^\beta$ which gives together with (4.1) that $B_n = (b_{nk})_{k\in\mathbb{N}} \in (\ell_p(\hat{T}^q))^\beta$ for all $n \in \mathbb{N}$. Thus, Bu exists. Therefore, we derive by the above equality as $m \to \infty$ that $Bu = E\hat{u}$, and this shows that $B \in (\ell_p(\hat{T}^q), \lambda)$.

The following conditions are necessary for our study:

$$\sup_n \sum_k |e_{nk}|^q < \infty. \tag{4.3}$$

$$\lim_{n\to\infty} e_{nk} \; exists \; for \; all \; k \in \mathbb{N}. \tag{4.4}$$

$$\lim_{n\to\infty} e_{nk} = 0 \; for \; all \; k \in \mathbb{N}. \tag{4.5}$$

$$\sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}e_{nk}\right|^q<\infty. \tag{4.6}$$

$$\sup_{n,k}|e_{nk}|<\infty. \tag{4.7}$$

$$\sup_k\sum_n|e_{nk}|<\infty. \tag{4.8}$$

$$\lim_{n\to\infty}\sum_k|e_{nk}|=\sum_k\left|\lim_{n\to\infty}e_{nk}\right|. \tag{4.9}$$

$$\lim_{n\to\infty}\sum_k|e_{nk}|=0. \tag{4.10}$$

$$\lim_{m\to\infty}e_{nk}^{(m)}\ \ exists\ \ (\forall n,k\in\mathbb{N}), \tag{4.11}$$

$$\sup_{m,k}\left|e_{nk}^{(m)}\right|<\infty\ \ (\forall n\in\mathbb{N}) \tag{4.12}$$

$$\sup_m\sum_{k=0}^m\left|e_{nk}^{(m)}\right|^q<\infty \tag{4.13}$$

$$\lim_{m\to\infty}\sum_{k=0}^m\left|e_{nk}^{(m)}\right|=\sum_k|e_{nk}|\ \ for\ each\ n\in\mathbb{N} \tag{4.14}$$

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}e_{nk}\right|<\infty \tag{4.15}$$

We obtain the following results by combining Theorem 4.1 and previous conditions.

Theorem 4.2. *The following statements hold:*

1. $B=(b_{nk})\in(\ell_1(\hat{T}^q),\ell_\infty)\Leftrightarrow$ *(4.7), (4.11) and (4.12).*
2. $B=(b_{nk})\in(\ell_p(\hat{T}^q),\ell_\infty)\Leftrightarrow$ *(4.3), (4.11) and (4.13).*
3. $B=(b_{nk})\in(\ell_\infty(\hat{T}^q),\ell_\infty)\Leftrightarrow$ *(4.11), (4.14) and (4.3) with q=1.*
4. $B=(b_{nk})\in(\ell_1(\hat{T}^q),c)\Leftrightarrow$ *(4.4), (4.7), (4.11) and (4.12).*
5. $B=(b_{nk})\in(\ell_p(\hat{T}^q),c)\Leftrightarrow$ *(4.3), (4.4), (4.11) and (4.13).*
6. $B=(b_{nk})\in(\ell_\infty(\hat{T}^q),c)\Leftrightarrow$ *(4.4), (4.9), (4.11) and (4.14).*
7. $B=(b_{nk})\in(\ell_1(\hat{T}^q),c_0)\Leftrightarrow$ *(4.5), (4.7), (4.11) and (4.12).*
8. $B=(b_{nk})\in(\ell_p(\hat{T}^q),c_0)\Leftrightarrow$ *(4.3), (4.5), (4.11) and (4.13).*
9. $B=(b_{nk})\in(\ell_\infty(\hat{T}^q),c_0)\Leftrightarrow$ *(4.5), (4.10), (4.11) and (4.14).*
10. $B=(b_{nk})\in(\ell_1(\hat{T}^q),\ell_1)\Leftrightarrow$ *(4.8), (4.11) and (4.12).*
11. $B=(b_{nk})\in(\ell_p(\hat{T}^q),\ell_1)\Leftrightarrow$ *(4.6) , (4.11) and (4.13).*
12.$B=(b_{nk})\in(\ell_\infty(\hat{T}^q),\ell_1)\Leftrightarrow$ *(4.7), (4.11) and(4.14).*

By using Theorem 4.2, we derive the following result:

Corollary 4.1. *The following statements hold:*

1. $B=(b_{nk})\in(\ell_1(\hat{T}^q),cs_0)\Leftrightarrow$ (4.5), (4.7) *and* (4.11), (4.12).
2. $B=(b_{nk})\in(\ell_p(\hat{T}^q),cs_0)\Leftrightarrow$ (4.3), (4.5) *and* (4.11), (4.13).
3. $B=(b_{nk})\in(\ell_\infty(\hat{T}^q),cs_0)\Leftrightarrow$ (4.5), (4.10) *and* (4.11), (4.14).
4. $B=(b_{nk})\in(\ell_1(\hat{T}^q),cs)\Leftrightarrow$ (4.4), (4.7) *and* (4.11), (4.12).
5. $B=(b_{nk})\in(\ell_p(\hat{T}^q),cs)\Leftrightarrow$ (4.3), (4.4) *and* (4.11), (4.13).
6. $B=(b_{nk})\in(\ell_\infty(\hat{T}^q),cs)\Leftrightarrow$ (4.4), (4.9) *and* (4.11), (4.14).
7. $B=(b_{nk})\in(\ell_1(\hat{T}^q),bs)\Leftrightarrow$ (4.7) *and* (4.11), (4.12).
8. $B=(b_{nk})\in(\ell_p(\hat{T}^q),bs)\Leftrightarrow$ (4.3) *and* (4.11), (4.13).
9. $B=(b_{nk})\in(\ell_\infty(\hat{T}^q),bs)\Leftrightarrow$ (4.3) *with* $q=1$ *and* (4.11), (4.14)
hold with $d(n,k)$ *instead of* d_{nk}.

Now, we introduce the matrix transformations from the space $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ to $\ell_p(\hat{T}^q)$, where $1 \leq p \leq \infty$. Before this, we give the necessary and sufficient conditions for the matrix transformation B is in (λ, ℓ_p).

Lemma 4.1. *The following statements hold:*

(a) $B \in (\ell_\infty, \ell_p) = (c, \ell_p) = (c_0, \ell_p)$ *if and only if*

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} b_{nk} \right|^p < \infty, \ where \ 1 \leq p < \infty. \tag{4.16}$$

(b) $B \in (\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty)$ *if and only if*

$$\sup_n \sum_k |b_{nk}| < \infty. \tag{4.17}$$

(c) $B \in (\ell_1, \ell_p)$ *if and only if*

$$\sup_k \sum_n |b_{nk}|^p < \infty, \ where \ 1 \leq p < \infty. \tag{4.18}$$

When we change the roles of the spaces $\ell_p(\hat{T}^q)$ and ℓ_p with λ in Theorem 4.1, we obtain the following theorem.

Theorem 4.3. *Assume that the terms of the infinite matrices $B = (b_{nk})$ and $\tilde{B} = (\tilde{b}_{nk})$ satisfies the following relation*

$$\tilde{b}_{nk} = \sum_{r=0}^{n-1} \left(\frac{q_r}{Q_n} t_r - \frac{q_{r+1}}{Q_n} \frac{1}{t_{r+1}} \right) b_{rk} + \frac{q_n}{Q_n} t_n b_{nk} \tag{4.19}$$

for all $k, n \in \mathbb{N}$ and λ be any given sequence space. Then, $B \in (\lambda, \ell_p(\hat{T}^q))$ if and only if $\tilde{B} \in (\lambda, \ell_p)$, where $1 \leq p \leq \infty$.

Proof. Let $u = (u_k) \in \lambda$. Then, by using the relation (4.19) one can easily obtain the following equality

$$\sum_{k=0}^{m} \tilde{b}_{nk} u_k = \sum_{k=0}^{m} \left(\sum_{r=0}^{n-1} \left(\frac{q_r}{Q_n} t_r - \frac{q_{r+1}}{Q_n} \frac{1}{t_{r+1}} \right) b_{rk} + \frac{q_n}{Q_n} t_n b_{nk} \right) u_k \text{ for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(\tilde{B}_n(u)) = (\hat{T}^q_n(Bu))$. Therefore, we conclude that $Bu \in \ell_p(\hat{T}^q)$ for $u \in \lambda$ if and only if $\tilde{B}u \in \ell_p$ for $u \in \lambda$, where $1 \leq p \leq \infty$. The proof is completed.

By combining Lemma 4.1 and Theorem 4.3, we obtain the following results:

Corollary 4.2. *Let the matrices $B = (b_{nk})$ and $\tilde{B} = (\tilde{b}_{nk})$ be connected by (4.19). Then, we obtain:*

(a) $B = (b_{nk}) \in (\ell_\infty, \ell_1(\hat{T}^q)) = (c, \ell_1(\hat{T}^q)) = (c_0, \ell_1(\hat{T}^q))$ *if and only if (4.16) holds with $p = 1$ and $\tilde{b}_{nk}$ instead of b_{nk}.*

(b) $B = (b_{nk}) \in (\ell_1, \ell_1(\hat{T}^q))$ *if and only if (4.18) holds with $p = 1$ and $\tilde{b}_{nk}$ instead of b_{nk}.*

Corollary 4.3. *Let the matrices $B = (b_{nk})$ and $\tilde{B} = (\tilde{b}_{nk})$ be connected by (4.19). For $1 < p < \infty$, we obtain:*

(a) $B = (b_{nk}) \in (\ell_\infty, \ell_p(\hat{T}^q)) = (c, \ell_p(\hat{T}^q)) = (c_0, \ell_p(\hat{T}^q))$ *if and only if (4.16) holds with $\tilde{b}_{nk}$ instead of b_{nk}.*

(b) $B = (b_{nk}) \in (\ell_1, \ell_p(\hat{T}^q))$ *if and only if (4.18) holds with $\tilde{b}_{nk}$ instead of b_{nk}.*

Corollary 4.4. *Let the matrices $B = (b_{nk})$ and $\tilde{B} = (\tilde{b}_{nk})$ be connected by (4.19). Then, we obtain:*

(a) $B = (b_{nk}) \in (\ell_\infty, \ell_\infty(\hat{T}^q)) = (c, \ell_\infty(\hat{T}^q)) = (c_0, \ell_\infty(\hat{T}^q))$ *if and only if (4.17) holds with $\tilde{b}_{nk}$ instead of b_{nk}.*

(b) $B = (b_{nk}) \in (\ell_1, \ell_\infty(\hat{T}^q))$ *if and only if (3.5) holds with $\tilde{b}_{nk}$ instead of b_{nk}.*

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Affiliations

PINAR ZENGIN ALP
ADDRESS: Düzce University, Dept. of Mathematics, 81620, Düzce-Turkey.
E-MAIL: pinarzenginalp@gmail.com
ORCID ID:0000-0001-9699-7199

MERVE İLKHAN
ADDRESS: Düzce University, Dept. of Mathematics, 81620, Düzce-Turkey.
E-MAIL: merveilkhan@duzce.edu.tr
ORCID ID:0000-0002-0831-1474