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Isometry Groups of Chamfered Cube and Chamfered Octahedron Spaces

Özcan Gelişgen* and Serhat Yavuz

Abstract
Polyhedra have interesting symmetries. Therefore they have attracted the attention of scientists and artists from past to present. Thus polyhedra are discussed in a lot of scientific and artistic works. There are only five regular convex polyhedra known as the platonic solids. There are many relationships between metrics and polyhedra. Some of them are given in previous studies. In this study, we introduce two new metrics, and show that the spheres of the 3-dimensional analytical space furnished by these metrics are chamfered cube and chamfered octahedron. Also we give some properties about these metrics. We show that the group of isometries of the 3-dimesional space covered by $CC-$metric and $CO-$metric are the semi-direct product of O_h and $T(3)$, where octahedral group O_h is the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the 3-dimensional space.

Keywords: *Polyhedron; Metric; Isometry group; Octahedral Symmetry; Chamfered Cube; Chamfered Octahedron*

AMS Subject Classification (2010): *Primary: 51B20; 51N25; Secondary: 51F99; 51K05; 51K99.*

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1. Introduction

What is a polyhedron? This question is interestingly hard to answer simply! As it is stated in [2], a convex polyhedron in X which is a $d-$dimensional real affine space is a subset of X obtained a finite intersection of closed haf-spaces. A polytope is a compact convex polyhedron with non-empty interior. A flag of $d-$dimensional polytope P is a $d-$tuple $(F_0, F_1, ..., F_{d-1})$ consisting of $i-$faces F_i of P such that $F_i \subset F_{i+1}$ for $i = 0, 1, ..., d-2$. A polytope P is called regular its stabilizer $G(P) = Is_P(X)$ acts transitivelyonthe flags of P, where $Is_P(X)$ is isotropy group in the set of isometries of P. Therefore, A regular polyhedron is a polyhedron whose symmetry group acts transitively on its flags. (for more details see to [1, 2] and [5, 18]) There are many thinkers that worked on polyhedra among the ancient Greeks. Early civilizations worked out mathematics as problems and their solutions. Polyhedrons have been studied by mathematicians, scientists during many years, because of their symmetries.

Minkowski geometry is non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set. The points, lines and planes the same, and the angles are measured in the same way, but the distance function is different. (See [16] and [19]). Some mathematicians have been studied and improved metric space geometry. According to mentioned researches it is found that unit spheres of these metrics are associated with convex solids. For example, unit sphere of maximum metric is a cube which is a Platonic Solid. Taxicab metric's unit sphere is an octahedron, another Platonic Solid. And unit sphere of CC-metric is a deltoidal icositetrahedron, a Catalan solid. So there are some metrics which unit spheres are convex polyhedrons. That is, convex polyhedrons are associated with some metrics (See [3, 4, 6, 7, 9–15]). This influence us to the question "Are there some metrics of which unit spheres are the Catalan Solids?". For this goal, firstly, the related polyhedra are placed as fully symmetric such that symmetry center of it is origin in the 3-dimensional

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space. And then the coordinates of vertices are found. Later one can be obtained metric which always supply plane equation related with solid's surface.

In this study, two new metrics are introduced, and showed that the spheres of the 3-dimensional analytical space furnished by these metrics are chamfered cube and chamfered octahedron. Also some properties about these metrics are given. Morever, we show that the group of isometries of the 3-dimesional space covered by $CC-$metric and $CO-$metric are the semi-direct product of O_h and $T(3)$, where octahedral group O_h are the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the 3-dimensional space.

2. Chamfered Cube Metric and Some Properties

It has been stated in [18], there are many variations on the theme of the regular polyhedra. Firstly, one can meet the eleven solids which can be made by cutting off (truncating) the corners, and in some cases the edges, of the regular polyhedra so that all the faces of the faceted polyhedra obtained in this way are regular polygons. These polyhedra were first discovered by Archimedes (287-–212 B.C.E.) and so they are often called Archimedean solids (See [8] for brief history). Notice that vertices of the Archimedean polyhedra are all alike, but their faces, which are regular polygons, are of two or more different kinds. For this reason they are often called semiregular. Archimedes also showed that in addition to the eleven obtained by truncation, there are two more semiregular polyhedra: the snub cube and the snub dodecahedron.

The other operation about consctructing polyhedron from any polyhedra is chamfering. In geometry, chamfering or edge-truncation is a topological operator that modifies one polyhedron into another. It is similar to expansion, moving faces apart and outward, but also maintains the original vertices. For polyhedra, this operation adds a new hexagonal face in place of each original edge.

One of the solids which is obtained by chamfering from another solid is the chamfered cube. It has 12 hexagonal faces and 6 square faces, 32 vertices and 48 edges. The chamfered cube can be obtained by truncating operation from cube. Figure 1 shows the chamfered cube and the procress of chamfering operation applied to cube.

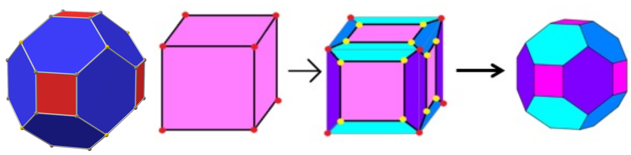

Figure 1: The cube, chamfering operation, chamfered cube

Before we give a description of the chamfered cube distance function, we must agree on some impressions. Therefore U, V, W denote the maximum, the middle and the minimum of quantities $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$. The metric that unit sphere is chamfered cube is described as following:

Definition 2.1. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{CC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ chamfered cube distance between P_1 and P_2 is defined by

$$d_{CC}(P_1, P_2) = \max\left\{U, \frac{5 - 2\sqrt{3}}{2}(U + V)\right\}.$$

According to chamfered cube distance, there are two different paths from P_1 to P_2. These paths are

$i)$ a line segment which is parallel to a coordinate axis.

$ii)$ union of two line segments each of which is parallel to a coordinate axis.

Thus chamfered cube distance between P_1 and P_2 is for (i) Euclidean lengths of line segment, and for (ii) $\frac{5 - 2\sqrt{3}}{2}$ times the sum of Euclidean lengths of mentioned two line segments.

Figure 2 illustrates some of chamfered cube way from P_1 to P_2

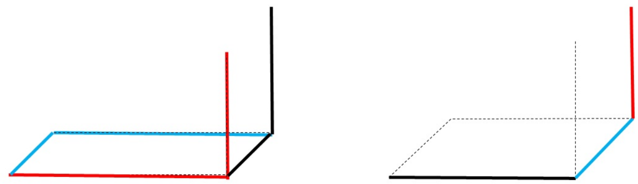

Figure 2: Some CC way from P_1 to P_2

It is well known that the sphere in the 3-dimensional analytical space with maximum metric is the cube. This metric for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is defined by $d_M(P_1, P_2) = U$.

Lemma 2.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. U_{12}, V_{12}, W_{12} denote the maximum, the middle and the minimum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. Then*

$$d_{CC}(P_1, P_2) \geq U_{12} \quad \text{and} \quad d_{CC}(P_1, P_2) \geq \frac{5 - 2\sqrt{3}}{2}(U_{12} + V_{12}).$$

Proof. Proof is trivial by the definition of maximum function. □

Theorem 2.1. *The distance function d_{CC} is a metric. Also according to d_{CC}, the unit sphere is a chamfered cube in $\mathbb{R}^3$.*

Proof. Let $d_{CC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ be the chamfered cube distance function and P_1=(x_1, y_1, z_1) , P_2=(x_2, y_2, z_2) and P_3=(x_3, y_3, z_3) are distinct three points in $\mathbb{R}^3$. U_{12}, V_{12}, W_{12} denote the maximum, the middle and the minimum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. To show that d_{CC} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

M1) $d_{CC}(P_1, P_2) \geq 0$ and $d_{CC}(P_1, P_2) = 0$ iff $P_1 = P_2$

M2) $d_{CC}(P_1, P_2) = d_{CC}(P_2, P_1)$

M3) $d_{CC}(P_1, P_3) \leq d_{CC}(P_1, P_2) + d_{CC}(P_2, P_3)$.

Since absolute values is always nonnegative value $d_{CC}(P_1, P_2) \geq 0$. If $d_{CC}(P_1, P_2) = 0$ then $d_{CC}(P_1, P_2) = \max\left\{U, \frac{5-2\sqrt{3}}{2}(U + V)\right\} = 0$, where U, V, W are the maximum, the middle and the minimum of quantities $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. Therefore, U=0, and $\frac{5-2\sqrt{3}}{2}(U + V)$=0. Hence, it is clearly obtained by $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$. That is, $P_1 = P_2$. Thus it is obtained that $d_{CC}(P_1, P_2) = 0$ iff $P_1 = P_2$.

Since $|x_1 - x_2| = |x_2 - x_1|$, $|y_1 - y_2|$=$|y_2 - y_1|$ and $|z_1 - z_2| = |z_2 - z_1|$, obviously $d_{CC}(P_1, P_2) = d_{CC}(P_2, P_1)$. That is, d_{CC} is symmetric.

U_{13}, V_{13}, W_{13} and U_{23}, V_{23}, W_{23} denote the maximum, the middle and the minimum of quantities of $\{|x_1 - x_3|, |y_1 - y_3|, |z_1 - z_3|\}$ and $\{|x_2 - x_3|, |y_2 - y_3|, |z_2 - z_3|\}$, respectively.

$$\begin{aligned}
&d_{CC}(P_1, P_3) = \max\left\{U_{13}, \tfrac{5-2\sqrt{3}}{2}(U_{13} + V_{13})\right\} \\
&\leq \max\left\{U_{12} + U_{23}, \tfrac{5-2\sqrt{3}}{2}(U_{13} + U_{23} + V_{13} + V_{23})\right\} \\
&= I.
\end{aligned}$$

Therefore one can easily find that $I \leq d_{CC}(P_1, P_2) + d_{CC}(P_2, P_3)$ from Lemma 2.1. So $d_{CC}(P_1, P_3) \leq d_{CC}(P_1, P_2) + d_{CC}(P_2, P_3)$. Consequently, chamfered cube distance is a metric in 3-dimensional analytical space. Finally, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that chamfered cube distance is 1 from $O = (0, 0, 0)$ is

$$S_{CC} = \left\{(x, y, z) : \max\left\{U, \frac{5 - 2\sqrt{3}}{2}(U + V)\right\} = 1\right\},$$

where U, V, W are the maximum, the middle and the minimum of quantities $\{|x|, |y|, |z|\}$, respectively. Thus the graph of S_{CC} is as in the figure 3:

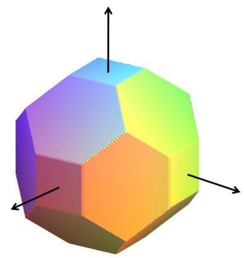

Figure 3 The unit sphere in terms of d_{CC}: Chamfered Cube

□

Corollary 2.1. *The equation of the chamfered cube with center* (x_0, y_0, z_0) *and radius* r *is*

$$\max\left\{U_0, \frac{5-2\sqrt{3}}{2}(U_0+V_0)\right\} = r,$$

which is a polyhedron which has 18 faces and 32 vertices, where U_0, V_0, W_0 *are the maximum, the middle and the minimum of quantities* $\{|x-x_0|, |y-y_0|, |z-z_0|\}$*, respectively. Coordinates of the vertices are translation to* (x_0, y_0, z_0) *all permutations of the three axis components and all possible +/- sign changes of each axis component of* $\left(\frac{4\sqrt{3}-3}{13}r, \frac{4\sqrt{3}-3}{13}r, r\right)$ *and* $\left(\frac{5+2\sqrt{3}}{13}r, \frac{5+2\sqrt{3}}{13}r, \frac{5+2\sqrt{3}}{13}r\right)$.

Lemma 2.2. *Let* l *be the line through the points* $P_1=(x_1, y_1, z_1)$ *and* $P_2=(x_2, y_2, z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector* (p, q, r)*, then*

$$d_{CC}(P_1, P_2) = \mu(P_1P_2)d_E(P_1, P_2)$$

where

$$\mu(P_1P_2) = \frac{\max\left\{U_d, \frac{5-2\sqrt{3}}{2}(U_d+V_d)\right\}}{\sqrt{p^2+q^2+r^2}},$$

U_d, V_d, W_d *are the maximum, the middle and the minimum of quantities* $\{|p|, |q|, |r|\}$*, respectively.*

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R}$. Thus,

$$d_{CC}(P_1, P_2) = |\lambda| \left(\max\left\{U_d, \frac{5-2\sqrt{3}}{2}(U_d+V_d)\right\}\right),$$

where U_d, V_d, W_d are the maximum, the middle and the minimum of quantities $\{|p|, |q|, |r|\}$, respectively, and $d_E(A, B) = |\lambda|\sqrt{p^2+q^2+r^2}$ which implies the required result. □

The above lemma says that d_{CC}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 2.2. *If* P_1, P_2 *and* X *are any three collinear points in* $\mathbb{R}^3$, *then* $d_E(P_1, X) = d_E(P_2, X)$ *if and only if* $d_{CC}(P_1, X) = d_{CC}(P_2, X)$.

Corollary 2.3. *If* P_1, P_2 *and* X *are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{CC}(X, P_1) \,/\, d_{CC}(X, P_2) = d_E(X, P_1) \,/\, d_E(X, P_2)\ .$$

That is, the ratios of the Euclidean and $d_{CC}-$*distances along a line are the same.*

3. Chamfered Octahedron Metric and Some Properties

The chamfered octahedron can be obtained by using chamfering operation from octahedron. The chamfered octahedron has 8 equilateral triangular faces and 12 bi-mirror-symmetric hexagonal faces, 30 vertices and 48 edges. Figure 4 shows the chamfered octahedron.

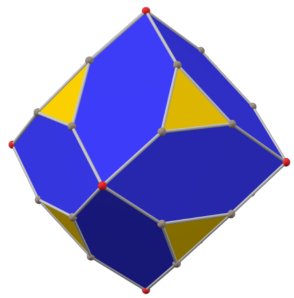

Figure 4: The chamfered octahedron

The notations U, V, W shall be used as defined in the previous section. The metric that unit sphere is the chamfered octahedron is described as following:

Definition 3.1. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{CO} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ chamfered octahedron distance between P_1 and P_2 is defined by

$$d_{CO}(P_1, P_2) = \max\left\{U + V, \frac{9 + \sqrt{6}}{15}(U + V + W)\right\}.$$

According to chamfered octahedron distance, there are three different paths from P_1 to P_2. These paths are
$i)$ union of two line segments each of which is parallel to a coordinate axis,
$ii)$ union of three line segments each of which is parallel to a coordinate axis.

Thus chamfered octahedron distance between P_1 and P_2 is for (i) the sum of Euclidean lengths of two line segments, for (ii) $\frac{9+\sqrt{6}}{15}$ times the sum of Euclidean lengths of mentioned above three line segments. In case of $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$, Figure 5 shows that some of the chamfered octahedron paths between P_1 and P_2.

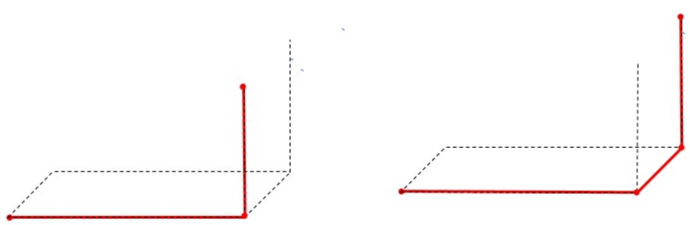

Figure 5: CO way from P_1 to P_2

It is well known that the sphere in the 3-dimensional analytical space with taxicab metric is the octahedron. This metrics for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ are defined as follows:

$$d_T(P_1, P_2) = U + V + W,$$

where U, V, W are denoted the maximum, the middle and the minimum of quantities $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively.

Lemma 3.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. U_{12}, V_{12}, W_{12} denote the maximum, the middle and the minimum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. Then*

$$d_{CO}(P_1, P_2) \geq U_{12} + V_{12},$$
$$d_{CO}(P_1, P_2) \geq \tfrac{9+\sqrt{6}}{15}(U_{12} + V_{12} + W_{12}).$$

Proof. Proof is trivial by the definition of maximum function. □

Theorem 3.1. *The distance function d_{CO} is a metric. Also according to d_{CO}, unit sphere is a chamfered octahedron in $\mathbb{R}^3$.*

Proof. One can easily show that the chamfered octahedron distance function satisfies the metric axioms by similar way in Theorem 2.1.

Consequently, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that chamfered octahedron distance is 1 from $O = (0, 0, 0)$ is

$$S_{CO} = \left\{ (x, y, z) : \max \left\{ U + V, \frac{9 + \sqrt{6}}{15} (U + V + W) \right\} = 1 \right\},$$

where U, V, W are the maximum, the middle and the minimum of quantities $\{|x|, |y|, |z|\}$, respectively. Thus the graph of S_{CO} is as in the figure 6:

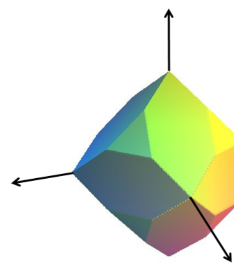

Figure 6 The unit sphere in terms of d_{CO}: Chamfered Octahedron

□

Corollary 3.1. *The equation of the chamfered octahedron with center (x_0, y_0, z_0) and radius r is*

$$\max \left\{ U_0 + V_0, \frac{9 + \sqrt{6}}{15} (U_0 + V_0 + W_0) \right\} = r,$$

which is a polyhedron which has 20 faces and 30 vertices, where U_0, V_0, W_0 are the maximum, the middle and the minimum of quantities $\{|x - x_0|, |y - y_0|, |z - z_0|\}$, respectively. Coordinates of the vertices are translation to (x_0, y_0, z_0) all permutations of the three axis components and all possible +/- sign changes of each axis component of $(0, 0, r)$ and $\left(\frac{4-\sqrt{6}}{5} r, \frac{4-\sqrt{6}}{5} r, \frac{3+\sqrt{6}}{5} r \right)$.

Lemma 3.2. *Let l be the line through the points P_1=(x_1, y_1, z_1) and P_2=(x_2, y_2, z_2) in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p, q, r), then*

$$d_{CO}(P_1, P_2) = \mu(P_1 P_2) d_E(P_1, P_2)$$

where

$$\mu(P_1 P_2) = \frac{\max \left\{ U_d + V_d, \frac{9+\sqrt{6}}{15} (U_d + V_d + W_d) \right\}}{\sqrt{p^2 + q^2 + r^2}},$$

U_d, V_d, W_d are the maximum, the middle and the minimum of quantities $\{|p|, |q|, |r|\}$, respectively.

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R}$. Thus,

$$d_{CO}(P_1, P_2) = |\lambda| \left(\max \left\{ U_d + V_d, \frac{9 + \sqrt{6}}{15} (U_d + V_d + W_d) \right\} \right)$$

where U_d, V_d, W_d are the maximum, the middle and the minimum of quantities $\{|p|, |q|, |r|\}$, respectively, and $d_E(A, B) = |\lambda| \sqrt{p^2 + q^2 + r^2}$ which implies the desired result. □

The above lemma says that d_{CO}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 3.2. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{CO}(P_1, X) = d_{CO}(P_2, X)$.*

Corollary 3.3. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{CO}(X, P_1) \;/\; d_{CO}(X, P_2) = d_E(X, P_1) \;/\; d_E(X, P_2) \;.$$

That is, the ratios of the Euclidean and $d_{CO}-$distances along a line are the same.

4. Isometry Group of Chamfered Octahedron and Chamfered Cube Spaces

Three essential methods geometric investigations; synthetic, metric and group approach. The group approach takes isometry groups of a geometry and convex sets plays an substantial role in indication of the group of isometries of geometries. Those properties are invariant under the group of motions and geometry studies those properties. There are a lot of studies about group of isometries of a space (See [7, 10, 11])

It is mentioned in introduction section that in a Minkowski geometry the linear structure is the same as the Euclidean one but distance is not uniform in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a certain symmetric closed convex set. In [17] the author give the following thereom:

Theorem 4.1. *If the unit ball C of $(V, \|\|)$ does not intersect a two-plane in an ellipse, then the group $I\,(3)$of isometries of $(V, \|\|)$ is isomorphic to the semi-direct product of the translation group $T(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant ± 1.*

After this theorem remains a single question. This question is that what is the relevant subgroup?

Now we show that the group of isometries of the 3-dimesional space covered by $CC-$metric and $CO-$metric are the semi-direct product of O_h and $T(3)$, where octahedral group O_h are the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the 3-dimensional space. In the rest of article we take $\triangle = CC$ or $\triangle = CO$. That is, $\triangle \in \{CC, CO\}$.

Definition 4.1. Let P, Q be two points in $\mathbb{R}^3_\triangle$. The minimum distance set of P, Q is defined by

$$\{X \mid d_\triangle(P, X) + d_\triangle(Q, X) = d_\triangle(P, Q)\}$$

and denoted by $[PQ]$.

In general, $[PQ]$ stand for a hexagonal dipyramid which is not necessary uniform in $\mathbb{R}^3_{CC}$ and $\mathbb{R}^3_{CO}$ as shown in Figure 7.

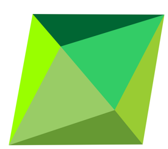

Figure 7

Proposition 4.1. *Let $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ be an isometry and let $[PQ]$ be the minimum distance set of P, Q. Then $\phi([PQ]) = [\phi(P)\phi(Q)]$.*

Proof. Let $Y \in \phi([PQ])$. Then,

$$\begin{array}{rcl} Y \in \phi([PQ]) & \Leftrightarrow & \exists X \in [PQ] \ni Y = \phi(X) \\ & \Leftrightarrow & d_\triangle(P, X) + d_\triangle(Q, X) = d_\triangle(P, Q) \\ & \Leftrightarrow & d_\triangle(\phi(P), \phi(X)) + d_\triangle(\phi(Q), \phi(X)) = d_\triangle(\phi(P), \phi(Q)) \\ & \Leftrightarrow & Y = \phi(X) \in [\phi(P)\phi(Q)] \,. \end{array}$$

□

Corollary 4.1. *Let $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ be an isometry and $[PQ]$ be the minimum distance set. Then ϕ maps vertices to vertices and preserves the lengths of the edges of $[PQ]$.*

Proposition 4.2. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle T}$ be an isometry such that $\phi(O) = O$. Then $\phi \in O_h$.*

Proof. Since $\triangle \in \{CC, CO\}$, there are two possibility for $\triangle$. Let $\triangle = CC$, $C_1 = \frac{4\sqrt{3}-3}{13}$, $C_2 = \frac{2\sqrt{3}+5}{3}$, $C_3 = 1$, and let $P_1 = (C_1, C_1, C_3)$, $P_2 = (-C_1, C_1, C_3)$, $P_3 = (C_1, C_3, C_1)$, $P_4 = (-C_1, C_3, C_1)$, $P_5 = (C_2, C_2, C_2)$, $P_6 = (-C_2, C_2, C_2)$ and $R = \left(0, \frac{10+4\sqrt{3}}{13}, \frac{10+4\sqrt{3}}{13}\right)$ be seven points in $\mathbb{R}^3_{CC}$. Consider $[OR]$ which is the hexagonal dipyramid (Figure 8(a)).

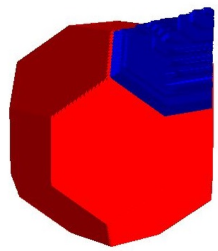

Figure 8(a)

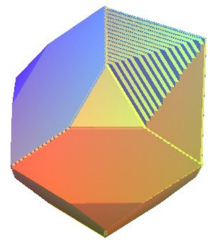

Figure 8(b)

Also points $P_1, P_2, P_3, P_4, P_5, P_6$ lie on minimum distance set $[OR]$ and unit sphere with center at origin. Moreover these six points are the corner points of a chamfered cube's hexagonal face. ϕ maps points P_i to the vertices of a chamfered cube by Corollary 4.1. Since ϕ preserve the lengths of the edges and chamfered cube have 12 hexagonal faces, there are 12 possibility to points which they can map, and also there are four possibility to points which they can map on the hexagonal face of chamfered cube. Therefore total number of possibility are forty eight. If these possibilities are handled one by one, it is seen that the elements of the desired subgroup are obtained.

Let $\triangle$=CO, $C_1 = \frac{4-\sqrt{6}}{5}$, $C_2 = \frac{1+\sqrt{6}}{5}$, $C_3 = 1$, and let $P_1 = (0, 0, 1)$, $P_2 = (0, 0, 1)$, $P_3 = (C_1, C_1, C_2)$, $P_4 = (-C_1, C_1, C_2)$, $P_5 = (C_1, C_2, C_1)$, $P_6 = (-P_1, C_2, C_1)\, \mathbb{R}^3_{CO}$. Consider $[OR]$ such that $R = (0, 1, 1)$. that is the hexagonal dipyramid with diagonal OR. (Figure 8(b)) Also points P_i lie on minimum distance set $[OR]$ and unit sphere with center at origin. Moreover these six points are the corner points of a chamfered octahedron's octagonal face. ϕ maps points P_i to the vertices of a chamfered octahedron by Corollary 4.1. Since ϕ preserve the lengths of the edges, and chamfeerd octahedron have 12 octagonal faces, there are 12 possibility to points which they can map, and also there are four possibility to points which they can map on the octagonal face of chamfered octagon. Therefore total number of possibility are forty eight. Similar way, If these possibilities are handled one by one, it is seen that the elements of the desired subgroup are obtained. □

Theorem 4.2. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry. Then there exists a unique $T_A \in T(3)$ and $\psi \in O_h$ where $\phi = T_A \circ \psi$*

Proof. Let $\phi(O) = A$ such that $A = (a_1, a_2, a_3)$. Define $\psi = T_{-A} \circ \phi$. We know that $\psi(O) = O$ and ψ is an isometry. Thereby, $\psi \in O_h$ and $\phi = T_A \circ \psi$ by Proposition 4.2. The proof of uniqueness is trivial. □

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