Relations Among Bell Polynomials, Central Factorial Numbers, and Central Bell Polynomials

Feng Qi and Bai-Ni Guo*

Abstract
In the note, by virtue of the Faà di Bruno formula and two identities for the Bell polynomials of the second kind, the authors derive three relations among the Bell polynomials, central factorial numbers of the second kind, and central Bell polynomials.

Keywords: Bell polynomial; central factorial number of the second kind; central Bell polynomial; Bell polynomial of the second kind; Faà di Bruno formula.

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1. Preliminaries

The Bell numbers $B_k$ for $k \geq 0$ can be generated \[4, 7, 12\] by

$$e^{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = 1 + t + t^2 + \frac{5}{6} t^3 + \frac{5}{8} t^4 + \frac{13}{30} t^5 + \frac{203}{720} t^6 + \frac{877}{5040} t^7 + \cdots$$

As a generalization of the Bell numbers $B_k$ for $k \geq 0$, the Bell polynomials $T_k(x)$ for $k \geq 0$ can be generated \[8–10, 15, 17\] by

$$e^{x(e^t-1)} = \sum_{k=0}^{\infty} T_k(x) \frac{t^k}{k!} = 1 + xt + \frac{1}{2} x(x+1)t^2 + \frac{1}{6} x(x^2+3x+1)t^3$$

$$+ \frac{1}{24} x(x^3+6x^2+7x+1)t^4 + \frac{1}{120} x(x^4+10x^3+25x^2+15x+1)t^5 + \cdots$$

The polynomials $T_k(x)$ for $k \geq 0$ are also called \[11, 18\] the Touchard polynomials or the exponential polynomials. It is clear that $T_k(1) = B_k$.

The central factorial numbers of the second kind $T(n, k)$ for $n \geq k \geq 0$ can be generated \[1, 6\] by

$$\frac{1}{k!} \left( 2 \sinh \frac{t}{2} \right)^k = \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!},$$

where

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

is the hyperbolic sine function.

The central Bell polynomials $B_k^{(c)}(x)$ for $k \geq 0$ can be generated \[5\] by

$$\exp \left( 2x \sinh \frac{t}{2} \right) = \sum_{k=0}^{\infty} B_k^{(c)}(x) \frac{t^k}{k!}.$$
In this note, by virtue of the Faà di Bruno formula and two identities for the Bell polynomials of the second kind, we will discuss relations among the Bell polynomials $T_k(x)$, central factorial numbers of the second kind $T(n, k)$, and central Bell polynomials $B_k^{(c)}(x)$.

2. Lemmas

The Bell polynomials of the second kind, denoted by $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ for $n \geq k \geq 0$, are defined [2, 3, 11, 13, 15–17] by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{1 \leq i \leq n-k+1} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{\ell_i} \right).$$

For proving our main results, we need the following lemmas.

Lemma 2.1 ([2, 3, 14, 19–21]). The Faà di Bruno formula can be described in terms of $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ by

$$\frac{d^n}{dx^n} f \circ h(x) = \sum_{k=0}^{n} f^{(k)}(h(x)) B_{n,k}(h'(x), h''(x), \ldots, h^{(n-k+1)}(x)).$$

For $n \geq k \geq 0$, the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ satisfy the identity

$$B_{n,k}(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$$

is valid, where $a, b \in \mathbb{C}$.

Lemma 2.2 ([14, 19, 21]). For $n \geq k \geq 0$, the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ satisfy the closed formula

$$B_{n,k}(1, 0, 1, \ldots, 1 - (-1)^{n-k+1} \frac{2}{2}) = \frac{1}{2^{k}k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} (k - 2\ell)^n.$$  (2.3)

3. Main results and their proofs

Now we are in a position to state and prove our main results.

Theorem 3.1. For $k \geq 0$, we have

$$B_k^{(c)}(x) = \frac{(-1)^k}{2^k} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} T_\ell(x) T_{k-\ell}(-x),$$

$$B_k^{(c)}(x) = \sum_{\ell=0}^{k} T(k, \ell) x^\ell,$$

and

$$\sum_{\ell=0}^{k} T(k, \ell) x^\ell = \left( -\frac{1}{2} \right)^k \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} T_\ell(x) T_{k-\ell}(-x).$$

Proof. Utilizing (1.2) and (1.1) in sequence, it follows that

$$\exp \left( 2x \sinh \frac{t}{2} \right) = e^{x(e^{t/2} - 1) - x(e^{-t/2} - 1)} = \sum_{k=0}^{\infty} \frac{T_k(x) t^k}{2^k k!} \left( \sum_{k=0}^{\infty} \frac{T_k(-x) t^k}{(-2)^k k!} \right)$$

Comparing this with (1.3) yields (3.1).
From (1.3), (2.1), (2.2), and (2.3) in sequence, it follows that

\[
B_{k}^{(c)}(x) = \lim_{t \to 0} \frac{d^k}{dt^k} \left[ \exp \left( 2x \sinh \frac{t}{2} \right) \right] \\
= \lim_{t \to 0} \sum_{\ell=0}^{k} \left( e^{2x u} \right)^{\ell} B_{k,\ell} \left( \frac{1}{2} \cosh \frac{t}{2}, \frac{1}{2} \frac{1}{2} \sinh \frac{t}{2}, \ldots, \frac{1}{2n-k+1} \sinh^{(n-k+1)} \frac{t}{2} \right) \\
= \sum_{\ell=0}^{k} \frac{2^\ell}{\ell!} B_{k,\ell} \left( 1, 0, 0, \ldots, \frac{1 - (-1)^{n-k+1}}{2} \right) \\
= \sum_{\ell=0}^{k} \frac{2^\ell}{\ell!} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} (\ell - 2m)^k \\
= \sum_{\ell=0}^{k} \frac{2^\ell}{\ell!} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} \left( \frac{\ell}{2} - m \right)^k \\
= \sum_{\ell=0}^{k} T(k, \ell) x^\ell,
\]

where \( u = u(t) = \sinh \frac{t}{2} \). Consequently, we derive the relation (3.2).

Combining the relation (3.1) with the equation (3.2) leads to the equality (3.3). The proof of Theorem 3.1 is complete.

References


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**Affiliations**

FENG QI  
**ADDRESS:** Inner Mongolia University for Nationalities, College of Mathematics, Tongliao 028043, Inner Mongolia, China; Tianjin Polytechnic University, School of Mathematical Sciences, Tianjin 300387, China.  
**E-MAIL:** qifeng618@gmail.com  
**ORCID ID:** https://orcid.org/0000-0001-6239-2968

BAI-NI GUO (CORRESPONDING AUTHOR)  
**ADDRESS:** Henan Polytechnic University, School of Mathematics and Informatics, Jiaozuo 454010, Henan, China.  
**E-MAIL:** bai.ni.guo@gmail.com  
**ORCID ID:** https://orcid.org/0000-0001-6156-2590