Available online: October 18, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 193–212 (2020)
DOI: 10.31801/cfsuasmas.567501
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

A TYCHONOFF THEOREM FOR GRADED DITOPOLOGICAL TEXTURE SPACES

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ABSTRACT. In this paper, initial and product graded ditopologies are formulated and accordingly it is shown that **dfGDitop** is a topological structure over **dfTex** $\times$ **dfTex**. By means of spectrum idea, (di)compactness in graded ditological texture spaces is defined as a generalization of (di)compactness in ditopological case and its relation with the ditopological case is investigated. Moreover, the relations between graded difilters and dicompactness of graded ditological texture spaces are studied.

1. INTRODUCTION

The idea "graded ditopology" has been introduced in [7] by Brown and Šostak. This new structure is more comprehensive than ditopologies basically given in [2, 3] and fuzzy topologies given independently by Šostak in [11] and Kubiak in [10]. Unlike ditopological case, in graded ditopologies, openness and closedness are given by means of independent grading functions.

In this work, we formulate the initial and product graded ditopologies and then we show that **dfGDitop** (given in Theorem 1.15) is a topological structure over **dfTex** $\times$ **dfTex**. Note that **dfTex** is the category of textures and difunctions between them [4]. Also **dfTex** $\times$ **dfTex** is the product category whose objects are all pairs of textures $((S,\mathscr{S}),(V,\mathscr{V}))$ and morphisms are all pairs of difunctions $((f,F),\ (h,H))$ from $((S,\mathscr{S}),\ (V,\mathscr{V}))$ to $((S',\mathscr{S}'),\ (V',\mathscr{V}'))$ with $(f,F):(S,\mathscr{S})\to(S',\mathscr{S}')$, $(h,H):(V,\mathscr{V})\to(V',\mathscr{V}')$. By using spectral theory as in [12, 13], we define (di)compactness in graded ditopological texture spaces as a generalization of (di)compactness in ditopological case and then a Tychonoff Theorem for that spaces is proved. The relationship between dicompactness spectrum and diconvergence (diclustering) spectrum is also studied.

Textures: [2] For a set S, a subset $\mathscr{S}\subseteq\mathscr{P}(S)$ is called a texturing on S if it is a point separating (i.e. for all $s,t\in S,\ s\neq t$ there exists a set $A\in\mathscr{S}$ such that $s\in A,\ t\notin A$ or $s\notin A,\ t\in A$), completely distributive, complete lattice with respect to inclusion which

Received by the editors: May 19, 2019; Accepted: October 04, 2019.
2010 *Mathematics Subject Classification.* Primary 54A05, 54D30; Secondary 54B10, 54A20.
Key words and phrases. Dicompactness spectrum, product graded ditopology, initial graded ditopology, compactness, cocompactness.

contains $\emptyset$, S and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. In this case $(S,\mathscr{S})$ is called a texture space or simply a texture.

For any texture $(S,\mathscr{S})$, many properties are conveniently defined in terms of the $p-sets$

$$P_s=\bigcap\{A\in\mathscr{S}\mid s\in A\}$$

and the $q-sets$

$$Q_s=\bigvee\{A\in\mathscr{S}\mid s\notin A\}=\bigvee\{P_u\mid u\in S, s\notin P_u\}.$$

A texture $(S,\mathscr{S})$ is called plain if $P_s\not\subseteq Q_s$ for all $s\in S$ or equivalently $A=\bigvee_{i\in I}A_i=\bigcup_{i\in I}A_i$ for all $A_i\in\mathscr{S}, i\in I$.

In general, a texturing of S need not be closed under set complementation, but there may exist a mapping $\sigma:\mathscr{S}\to\mathscr{S}$ satisfying $\sigma(\sigma(A))=A$ and $A\subseteq B\Rightarrow\sigma(B)\subseteq\sigma(A)$ for all $A,B\in\mathscr{S}$. In this case σ is called a complementation on $(S,\mathscr{S})$ and $(S,\mathscr{S},\sigma)$ is said to be a complemented texture.

For any set $A\in\mathscr{S}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat=\bigcap\big\{\bigcup\{A_i\mid i\in I\}\mid\{A_i\mid i\in I\}\subseteq\mathscr{S},\ A=\bigvee\{A_i\mid i\in I\}\big\}.$$

Product of textures: [3, 4, 5] Let $(S_j,\mathscr{S}_j)$, $j\in J$ be textures, $S=\prod_{j\in J}S_j$ and $A_k\in\mathscr{S}_k$ for some $k\in J$. If we write

$$E(k,A_k)=\prod_{j\in J}Y_j \text{ where } Y_j=\begin{cases} A_j & \text{if } j=k\\ S_j & \text{otherwise}\end{cases}$$

then the product texturing $\mathscr{S}=\bigotimes_{j\in J}\mathscr{S}_j$ of S consists of arbitrary intersections of elements of the set

$$\varepsilon=\{\bigcup_{j\in J_1}E(j,A_j)\mid J_1\subseteq J,\ A_j\in\mathscr{S}_j \text{ for } j\in J_1\}.$$

Consider two textures $(S,\mathscr{S})$ and $(V,\mathscr{V})$. The p-sets and q-sets of the product texture $(S\times V,\mathscr{P}(S)\otimes\mathscr{V})$ will be denoted by $\overline{P}_{(s,v)}$, $\overline{Q}_{(s,v)}$ respectively.

Definition 1.1. [4] *Let $(S,\mathscr{S})$ and $(V,\mathscr{V})$ be textures. Then*

(1) *$r\in\mathscr{P}(S)\otimes\mathscr{V}$ is called a relation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ if it satisfies*
  R1 *$r\not\subseteq\overline{Q}(s,v), P_{s'}\not\subseteq Q_s\Rightarrow r\not\subseteq\overline{Q}(s',v)$.*
  R2 *$r\not\subseteq\overline{Q}(s,v)\Rightarrow\exists s'\in S$ such that $P_s\not\subseteq Q_{s'}$ and $r\not\subseteq\overline{Q}(s',v)$.*

(2) *$R\in\mathscr{P}(S)\otimes\mathscr{V}$ is called a co-relation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ if it satisfies*
  CR1 *$\overline{P}(s,v)\not\subseteq R, P_s\not\subseteq Q_{s'}\Rightarrow\overline{P}(s',v)\not\subseteq R$.*
  CR2 *$\overline{P}(s,v)\not\subseteq R\Rightarrow\exists s'\in S$ such that $P_{s'}\not\subseteq Q_s$ and $\overline{P}(s',v)\not\subseteq R$.*

(3) *A pair (r,R), where r is a relation and R a co-relation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ is called a direlation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$.*

For a texture $(S,\mathscr{S})$ the identity direlation $(i_{(S,\mathscr{S})},I_{(S,\mathscr{S})})$ is defined by

$$i_{(S,\mathscr{S})}=\bigvee\{\overline{P}(s,s)\mid s\in S\} \text{ and } I_{(S,\mathscr{S})}=\bigcap\{\overline{Q}(s,s)\mid s\in S^\flat\}.$$

For $A \subseteq S$, $r^{\to}A = \bigcap\{Q_v \mid \forall s, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow A \subseteq Q_s\}$ is called the A-section of r and $R^{\to}A = \bigvee\{P_v \mid \forall s, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow P_s \subseteq A\}$ is called the A-section of R.

For $B \subseteq V$, $r^{\leftarrow}B = \bigvee\{P_s \mid \forall v, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow P_v \subseteq B\}$ is called the B-presection of r and $R^{\leftarrow}B = \bigcap\{Q_s \mid \forall v, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$ is called the B-presection of R.

Lemma 1.2. [4] *Let r, r_1, r_2 be relations, R, R_1, R_2 co-relations on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ with $r_1 \subseteq r_2$, $R_1 \subseteq R_2$, $A, A_1, A_2 \in \mathscr{S}$, $A_1 \subseteq A_2$, $B, B_1, B_2 \in \mathscr{V}$, $B_1 \subseteq B_2$.*

(1) $r \not\subseteq \overline{Q}_{(s,v)} \Leftrightarrow \overline{P}_{(v,s)} \not\subseteq r^{\leftarrow}$ *and* $\overline{P}_{(s,v)} \not\subseteq R \Leftrightarrow R^{\leftarrow} \not\subseteq \overline{Q}_{(v,s)}$ *for all* $s \in S$, $v \in V$.
(2) $(r^{\leftarrow})^{\leftarrow} = r$ *and* $(R^{\leftarrow})^{\leftarrow} = R$
(3) *For a second direlation* (m,M) *from* $(S,\mathscr{S})$ *to* $(V,\mathscr{V})$, $(r,R) \sqsubseteq (m,M) \Leftrightarrow (r,R)^{\leftarrow} \sqsubseteq (m,M)^{\leftarrow}$
(4) $r^{\to}\emptyset = \emptyset$, $A \subseteq r^{\leftarrow}(r^{\to}A)$, $r^{\to}(r^{\leftarrow}B) \subseteq B$
(5) $R^{\to}S = V$, $R^{\leftarrow}(R^{\to}A) \subseteq A$, $B \subseteq R^{\to}(R^{\leftarrow}B)$
(6) $r_1^{\to}A_1 \subseteq r_2^{\to}A_2$, $R_1^{\to}A_1 \subseteq R_2^{\to}A_2$, $r_2^{\leftarrow}B_1 \subseteq r_1^{\leftarrow}B_2$, $R_2^{\leftarrow}B_1 \subseteq R_1^{\leftarrow}B_2$.

Proposition 1.3. [4] *If (r,R) is a direlation on $(S,\mathscr{S})$ to $(V,\mathscr{V})$ then $r^{\to}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} r^{\to}A_i$, $R^{\to}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} R^{\to}A_i$, $r^{\leftarrow}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} r^{\leftarrow}B_j$ and $R^{\leftarrow}(\bigvee_{j\in J} B_j) = \bigvee_{j\in J} R^{\leftarrow}B_j$ for any $A_i \in \mathscr{S}$, $B_j \in \mathscr{V}$, $i \in I$, $j \in J$.*

Definition 1.4. [4] *Let (f,F) be a direlation from $(S,\mathscr{S})$ to $(V,\mathscr{V})$. Then (f,F) is called a difunction from $(S,\mathscr{S})$ to $(V,\mathscr{V})$ if it satisfies the following two conditions:*

(DF1) *For* $s, s' \in S$, $P_s \not\subseteq Q_{s'} \Rightarrow \exists v \in V$ *with* $f \not\subseteq \overline{Q}_{(s,v)}$ *and* $\overline{P}_{(s',v)} \not\subseteq F$.
(DF2) *For* $v, v' \in V$ *and* $s \in S$, $f \not\subseteq \overline{Q}_{(s,v)}$ *and* $\overline{P}_{(s,v')} \not\subseteq F \Rightarrow P_{v'} \not\subseteq Q_v$.

It is clear that (i_S, I_S) is a difunction on $(S,\mathscr{S})$ and we call it the identity difunction on $(S,\mathscr{S})$. (f,F) is called surjective if

$$\forall v, v' \in V\ P_v \not\subseteq Q_{v'} \Rightarrow \exists s \in S \text{ with } f \not\subseteq \overline{Q}_{(s,v')} \text{ and } \overline{P}_{(s,v)} \not\subseteq F.$$

Proposition 1.5. [4] *Let (f,F) be a difunction on $(S,\mathscr{S})$ to $(V,\mathscr{V})$.*

(1) $f^{\leftarrow}B = F^{\leftarrow}B$ *for each* $B \in \mathscr{V}$.
(2) $f^{\leftarrow}\emptyset = F^{\leftarrow}\emptyset = \emptyset$ *and* $f^{\leftarrow}V = F^{\leftarrow}V = S$.
(3) $A \subseteq F^{\leftarrow}(f^{\to}A)$ *and* $f^{\to}(F^{\leftarrow}B) \subseteq B$ *for all* $A \in \mathscr{S}$, $B \in \mathscr{V}$.
(4) *If* (f,F) *is surjective then* $F^{\to}(f^{\leftarrow}B) = B = f^{\to}(F^{\leftarrow}B)$ *for all* $B \in \mathscr{V}$.

Definition 1.6. [2] *A dichotomous topology, or ditopology for short, on a texture $(S,\mathscr{S})$ is a pair (τ,κ) of subsets of $\mathscr{S}$, where the set of open sets τ satisfies*

(T_1) $S, \emptyset \in \tau$
(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$
(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$

and the set of closed sets κ satisfies

(CT_1) $S, \emptyset \in \kappa$
(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$
(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no a priori relation between the open and closed sets.

Definition 1.7. [5] *Let $(S_k, \mathscr{S}_k, \tau_k, \kappa_k)$, $k = 1,2$ be ditopological texture spaces and $(f,F) : (S_1, \mathscr{S}_1) \to (S_2, \mathscr{S}_2)$ a difunction. (f,F) is called continuous if*

$$F^{\leftarrow} A \in \tau_1, \ \ \forall A \in \tau_2$$

and cocontinuous if

$$f^{\leftarrow} A \in \kappa_1, \ \ \forall A \in \kappa_2.$$

The difunction (f,F) is called bicontinuous if it is both continuous and cocontinuous.

Theorem 1.8. [5] *Ditopological texture spaces and bicontinuous difunctions form a category denoted by* **dfDiTop**.

For $s = (s_j) \in S$, $P_s = \bigcap_{j\in J} E(j, P_{s_j}) = \prod_{j\in J} P_{s_j}$. The jth-projection difunction $(\pi_j, \Pi_j) : (S, \mathscr{S}) \to (S_j, \mathscr{S}_j)$ is defined by

$$\pi_j = \bigvee \{\overline{P}_{(s,s_j)} \mid s = (s_j) \in S\},\ \Pi_j = \bigcap \{\overline{Q}_{(s,s_j)} \mid s = (s_j) \in S^\flat\}$$

and it is surjective by [6].

For ditopological texture spaces $(S_j, \mathscr{S}_j, \tau_j, \kappa_j)$, $j \in J$, the product ditopology on the product texture $(S, \mathscr{S})$ has subbase $\{E(j,G) \mid G \in \tau_j,\ j \in J\}$, cosubbase $\{E(j,K) \mid K \in \kappa_j,\ j \in J\}$.

Proposition 1.9. [5] *Let (π_j, Π_j) be the jth-projection difunction of the product texture $(S, \mathscr{S})$ of the textures $(S_j, \mathscr{S}_j)$, $j \in J$. Then*

(1) *If $A_i \in \mathscr{S}_i$, $i \in J$ and $A_i \neq \emptyset$, $i \neq j$ then $\pi_j^{\rightarrow} \bigcap_{j\in J} E(i, A_i) = A_j$.*
(2) *If $A_i \in \mathscr{S}_i$, $i \in J$ and $A_i \neq S_i$, $i \neq j$ then $\Pi_j^{\rightarrow} \bigcup_{j\in J} E(i, A_i) = A_j$.*

Proposition 1.10. [15] *Let $(S, \mathscr{S})$ be the product texturing of the textures $(S_j, \mathscr{S}_j)$, $j \in J$. $(S, \mathscr{S})$ is plain if and only if $(S_j, \mathscr{S}_j)$ is plain for all $j \in J$.*

Definition 1.11. [6] *Let $(S, \mathscr{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathscr{S}$. Then*

(1) *A is called compact if whenever $\{G_i \mid i \in I\}$ is an open cover of A (i.e. $\forall i \in I$ $G_i \in \tau$ and $A \subseteq \bigvee_{i\in I} G_i$) then there is a finite subset J of I with $A \subseteq \bigvee_{i\in J} G_i$. In particular $(S, \mathscr{S}, \tau, \kappa)$ is called compact if S is compact.*
(2) *A is called cocompact if whenever $\{K_i \mid i \in I\}$ is a closed cocover of A (i.e. $\forall i \in I$ $K_i \in \kappa$ and $\bigcap_{i\in I} K_i \subseteq A$) then there is a finite subset J of I with $\bigcap_{i\in J} K_i \subseteq A$. In particular $(S, \mathscr{S}, \tau, \kappa)$ is called cocompact if $\emptyset$ is compact.*
(3) *$(S, \mathscr{S}, \tau, \kappa)$ is called stable if every $K \in \kappa$ with $K \neq S$ is compact.*
(4) *$(S, \mathscr{S}, \tau, \kappa)$ is called costable if every $G \in \kappa$ with $G \neq \emptyset$ is cocompact.*
(5) *$(S, \mathscr{S}, \tau, \kappa)$ is called dicompact if it is compact, cocompact, stable and costable.*

Theorem 1.12. [6] *A product of non-empty ditopological texture space is dicompact if and only if each component space is dicompact.*

Graded Ditopological Texture Spaces: [7] Let $(S,\mathscr{S})$, $(V,\mathscr{V})$ be textures and consider $\mathscr{T},\mathscr{K}:\mathscr{S}\to\mathscr{V}$ satisfying

(GT_1) $\mathscr{T}(S)=\mathscr{T}(\emptyset)=V$
(GT_2) $\mathscr{T}(A_1)\cap\mathscr{T}(A_2)\subseteq\mathscr{T}(A_1\cap A_2)\ \forall A_1,A_2\in\mathscr{S}$
(GT_3) $\bigcap_{j\in J}\mathscr{T}(A_j)\subseteq\mathscr{T}(\bigvee_{j\in J}A_j)\ \forall A_j\in\mathscr{S},\ j\in J$

and

(GCT_1) $\mathscr{K}(S)=\mathscr{K}(\emptyset)=V$
(GCT_2) $\mathscr{K}(A_1)\cap\mathscr{K}(A_2)\subseteq\mathscr{K}(A_1\cup A_2)\ \forall A_1,A_2\in\mathscr{S}$
(GCT_3) $\bigcap_{j\in J}\mathscr{K}(A_j)\subseteq\mathscr{K}(\bigcap_{j\in J}A_j)\ \forall A_j\in\mathscr{S},\ j\in J$

Then $\mathscr{T}$ is called a $(V,\mathscr{V})$-graded topology, $\mathscr{K}$ a $(V,\mathscr{V})$-graded cotopology and $(\mathscr{T},\mathscr{K})$ a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$ and for any graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ and for each $v\in V$ it is defined that

$$\mathscr{T}^{v}=\{A\in\mathscr{S}\mid P_v\subseteq\mathscr{T}(A)\},\ \mathscr{K}^{v}=\{A\in\mathscr{S}\mid P_v\subseteq\mathscr{K}(A)\}.$$

Then $(\mathscr{T}^{v},\mathscr{K}^{v})$ is a ditopology on $(S,\mathscr{S})$ for each $v\in V$. That is, if $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is any graded ditopological texture space, then there exists a ditopology $(\mathscr{T}^{v},\mathscr{K}^{v})$ on the texture space $(S,\mathscr{S})$ for each $v\in V$.

If $(S,\mathscr{S},\sigma)$ is a complemented texture and $(\mathscr{T},\mathscr{K})$ a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$, then $(\mathscr{K}\circ\sigma,\mathscr{T}\circ\sigma)$ is also a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$. $(\mathscr{T},\mathscr{K})$ is called complemented if $(\mathscr{T},\mathscr{K})=(\mathscr{K}\circ\sigma,\mathscr{T}\circ\sigma)$.

Let $(\mathscr{T}_k,\mathscr{K}_k)$, $k=1,2$ be $(V,\mathscr{V})$-graded ditopologies on $(S,\mathscr{S})$. $(\mathscr{T}_1,\mathscr{K}_1)$ said to be coarser than $(\mathscr{T}_2,\mathscr{K}_2)$ and $(\mathscr{T}_2,\mathscr{K}_2)$ said to be finer than $(\mathscr{T}_1,\mathscr{K}_1)$ if $\mathscr{T}_1(A)\subseteq\mathscr{T}_2(A)$, $\mathscr{K}_1(A)\subseteq\mathscr{K}_2(A)$ for all $A\in\mathscr{S}$ [8].

Example 1.13. [7] *Let $(S,\mathscr{S},\tau,\kappa)$ be a ditopological texture space and $(V,\mathscr{V})$ the discrete texture on a singleton. Take $(V,\mathscr{V})=(1,\mathscr{P}(1))$ (The notation 1 denotes the set $\{0\}$) and define $\tau^g:\mathscr{S}\to\mathscr{P}(1)$ by $\tau^g(A)=1\Leftrightarrow A\in\tau$. Then τ^g is a $(V,\mathscr{V})$-graded topology on $(S,\mathscr{S})$. Likewise, κ^g defined by $\kappa^g(A)=1\Leftrightarrow A\in\kappa$ is a $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$ and (τ^g,κ^g) is called the graded ditopology on $(S,\mathscr{S})$ corresponding to ditopology (τ,κ).*

Definition 1.14. [7] *Let $(S_k,\mathscr{S}_k,\mathscr{T}_k,\mathscr{K}_k,V_k,\mathscr{V}_k)$, $k=1,2$ be graded ditopological texture spaces, $(f,F):(S_1,\mathscr{S}_1)\to(S_2,\mathscr{S}_2)$, $(h,H):(V_1,\mathscr{V}_1)\to(V_2,\mathscr{V}_2)$ difunctions. For the pair $((f,F),(h,H))$, (f,F) is called continuous with respect to (h,H) if*

$$H^{\leftarrow}\mathscr{T}_2(A)\subseteq\mathscr{T}_1(F^{\leftarrow}A)\ \textit{for all}\ A\in\mathscr{S}_2$$

and cocontinuous with respect to (h,H) if

$$h^{\leftarrow}\mathscr{K}_2(A)\subseteq\mathscr{K}_1(f^{\leftarrow}A)\ \textit{for all}\ A\in\mathscr{S}_2.$$

The difunction (f,F) is called bicontinuous with respect to (h,H) if it is both continuous and cocontinuous with respect to (h,H).

Theorem 1.15. [7] *The class of graded ditopological texture spaces and relatively bicontinuous difunction pairs between them form a category denoted by* **dfGDitop**.

2. PRODUCT GRADED DITOPOLOGY

Throughout the paper we denote the finite subset of a index set J by J_0 and the finite subfamily of a family $\mathscr{U}$ by $\mathscr{U}_0$.

Theorem 2.1. *Let $(S,\mathscr{S})$, $(V,\mathscr{V})$ be textures, $(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)_{j\in J}$ be graded ditopological texture spaces and $(f_j,F_j):(S,\mathscr{S})\to(S_j,\mathscr{S}_j)$, $(h_j,H_j):(V,\mathscr{V})\to(V_j,\mathscr{V}_j)$, $(j\in J)$ be difunctions. Then the mappings $\mathscr{T},\mathscr{K}:\mathscr{S}\to\mathscr{V}$ defined by*

$$\mathscr{T}(A)=\bigvee\{\bigcap_{j\in J_0}H_j^{\leftarrow}\mathscr{T}_j(G_j)\mid A=\bigcap_{j\in J_0}F_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}$$

$$\mathscr{K}(A)=\bigvee\{\bigcap_{j\in J_0}h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid A=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}$$

for all $A\in\mathscr{S}$ form a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$. $(\mathscr{T},\mathscr{K})$ is the coarsest $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$ that makes (f_j,F_j) bicontinuous with respect to (h_j,H_j) for each $j\in J$.

Proof. Firstly, we show that $\mathscr{K}$ is a $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$:
(i) Since $S=f_j^{\leftarrow}S_j$ and $h_j^{\leftarrow}V_j=V$ for all $j\in J$ by Proposition 1.5 (2), if we take a $j_0\in J$ then we have $\mathscr{K}(S)=\bigvee\{\bigcap_{j\in J_0}h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid S=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}\supseteq h^{\leftarrow}\mathscr{K}_{j_0}(S_{j_0})=h^{\leftarrow}V_{j_0}=V$ and so $\mathscr{K}(S)=V$.

On the other hand, since $\emptyset=f_j^{\leftarrow}\emptyset$ and $h_j^{\leftarrow}\emptyset=\emptyset$ for all $j\in J$ by Proposition 1.5 (2), if we take a $j_0\in J$ then we have $\mathscr{K}(\emptyset)=\bigvee\{\bigcap_{j\in J_0}h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid\emptyset=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}\supseteq h^{\leftarrow}\mathscr{K}_{j_0}(\emptyset)=h^{\leftarrow}V_{j_0}=V$ (by (GCT_1)) and so $\mathscr{K}(\emptyset)=V$.
(ii) Let $A,B\in\mathscr{S}$ be given. If there is no $G_j\in\mathscr{S}_j$ such that $A=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j$ or $B=\bigcup_{j\in J_0}f_j^{\leftarrow}G_j$ for a finite $J_0\subseteq J$ then $\mathscr{K}(A)\cap\mathscr{K}(B)=\emptyset\subseteq\mathscr{K}(A\cup B)$. So, let $A=\bigcup_{j\in J_1}f_j^{\leftarrow}G_j$ and $B=\bigcup_{j\in J_2}f_j^{\leftarrow}L_j$ for some finite subsets $J_1,J_2\subseteq J$ and for some $G_j,L_j\in\mathscr{S}_j$. If we redefine

$$G_j=\begin{cases}G_j, & j\in J_1\\ \emptyset, & j\in J_2\end{cases}$$

$$L_j=\begin{cases}L_j, & j\in J_2\\ \emptyset, & j\in J_1\end{cases}$$

then we have $\bigcup_{j\in J_1}f_j^{\leftarrow}G_j=\bigcup_{j\in J_1\cup J_2}f_j^{\leftarrow}G_j$ and $\bigcup_{j\in J_2}f_j^{\leftarrow}L_j=\bigcup_{j\in J_1\cup J_2}f_j^{\leftarrow}L_j$ by the fact that $f_j^{\leftarrow}\emptyset=\emptyset$ for all $j\in J$. Similarly, since $\mathscr{K}_j(\emptyset)=V_j$ and $h_j^{\leftarrow}V_j=V$ for all $j\in J$, we have $\bigcap_{j\in J_1}h_j^{\leftarrow}\mathscr{K}_j(G_j)=\bigcap_{j\in J_1\cup J_2}h_j^{\leftarrow}\mathscr{K}_j(G_j)$ and $\bigcap_{j\in J_2}h_j^{\leftarrow}\mathscr{K}_j(L_j)=\bigcap_{j\in J_1\cup J_2}h_j^{\leftarrow}\mathscr{K}_j(L_j)$. Thus we get

$$\begin{aligned}&\mathscr{K}(A)\cap\mathscr{K}(B)\\ &=\bigvee\{\bigcap_{j\in J_1}h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid A=\bigcup_{j\in J_1}f_j^{\leftarrow}G_j\}\cap\bigvee\{\bigcap_{j\in J_2}h_j^{\leftarrow}\mathscr{K}_j(L_j)\mid B=\bigcup_{j\in J_2}f_j^{\leftarrow}L_j\}\\ &=\bigvee\{\bigcap_{j\in J_1}h_j^{\leftarrow}\mathscr{K}_j(G_j)\cap\bigcap_{j\in J_2}h_j^{\leftarrow}\mathscr{K}_j(L_j)\mid A=\bigcup_{j\in J_1}f_j^{\leftarrow}G_j,\ B=\bigcup_{j\in J_2}f_j^{\leftarrow}L_j\}\end{aligned}$$

$$= \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}\mathscr{K}_j(G_j)\cap \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}\mathscr{K}_j(L_j) \mid A=\bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}G_j,\ B=\bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}L_j\}$$

$$= \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}\mathscr{K}_j(G_j)\cap h_j^{\leftarrow}\mathscr{K}_j(L_j) \mid A=\bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}G_j,\ B=\bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}L_j\}$$

$$= \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}(\mathscr{K}_j(G_j)\cap \mathscr{K}_j(L_j)) \mid A=\bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}G_j,\ B=\bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}L_j\}$$

$$\subseteq \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}(\mathscr{K}_j(G_j\cup L_j)) \mid A\cup B=\bigcup_{j\in J_1\cup J_2} (f_j^{\leftarrow}G_j\cup f_j^{\leftarrow}L_j)=\bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}(G_j\cup L_j)\}$$

$$= \bigvee\{ \bigcap_{j\in J_1\cup J_2} h_j^{\leftarrow}\mathscr{K}_j(M_j) \mid A\cup B=\bigcup_{j\in J_1\cup J_2} f_j^{\leftarrow}M_j\} = \mathscr{K}(A\cup B)$$

(iii) Let $A_i\in\mathscr{S}$ for all $i\in I$ where I is a nonempty index set. If for some $i\in I$, A_i can not be written as $A_i=\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i$ where J_i is a finite subset of J then $\bigcap_{i\in I}\mathscr{K}(A_i)=\emptyset\subseteq\mathscr{K}(\bigcap_{i\in I}A_i)$. So, for each $i\in I$ let $A_i=\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i$ for some $G_j^i\in\mathscr{S}_j$, $j\in J_i$. If we redefine

$$G_j^i=\begin{cases} G_j^i, & j\in J_i\\ \emptyset, & j\notin J_i\end{cases}$$

then considering $f_j^{\leftarrow}\emptyset=\emptyset$, $\mathscr{K}_j(\emptyset)=V_j$ (by (GCT_1)), $h_j^{\leftarrow}V_j=V$ for all $j\in J$ and "$j\notin\bigcap_{i\in I}J_i\Rightarrow\bigcap_{i\in I}G_j^i=\emptyset$" we have

$$\bigcap_{i\in I}A_i=\bigcap_{i\in I}\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i=\bigcap_{i\in I}\bigcup_{j\in J} f_j^{\leftarrow}G_j^i=\bigcup_{j\in J}\bigcap_{i\in I} f_j^{\leftarrow}G_j^i$$

$$=\bigcup_{j\in J} f_j^{\leftarrow}(\bigcap_{i\in I}G_j^i)=\bigcup_{j\in\bigcap_{i\in I}J_i} f_j^{\leftarrow}(\bigcap_{i\in I}G_j^i)$$

and

$$\bigcap_{i\in I}(\bigcap_{j\in J_i} h_j^{\leftarrow}\mathscr{K}_j(G_j^i))=\bigcap_{i\in I}(\bigcap_{j\in J} h_j^{\leftarrow}\mathscr{K}_j(G_j^i))=\bigcap_{j\in J} h_j^{\leftarrow}\bigcap_{i\in I}\mathscr{K}_j(G_j^i)$$

$$\subseteq\bigcap_{j\in J} h_j^{\leftarrow}\mathscr{K}_j(\bigcap_{i\in I}G_j^i)=\bigcap_{j\in\bigcap_{i\in I}J_i} h_j^{\leftarrow}\mathscr{K}_j(\bigcap_{i\in I}G_j^i)$$

Therefore we get

$$\bigcap_{i\in I}\mathscr{K}(A_i)=\bigcap_{(i\in I)}(\bigvee_{(A_i=\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i)}(\bigcap_{(j\in J_i)} h_j^{\leftarrow}\mathscr{K}_j(G_j^i)))$$

$$=\bigvee_{(A_i=\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i,\ i\in I)}\bigcap_{(i\in I)}(\bigcap_{(j\in J_i)} h_j^{\leftarrow}\mathscr{K}_j(G_j^i))$$

$$\subseteq\bigvee_{(\bigcap_{i\in I}A_i=\bigcap_{i\in I}\bigcup_{j\in J_i} f_j^{\leftarrow}G_j^i)}(\bigcap_{(j\in\bigcap_{i\in I}J_i)} h_j^{\leftarrow}\mathscr{K}_j(\bigcap_{i\in I}G_j^i))$$

$$= \bigvee_{(\bigcap_{i\in I} A_i = \bigcup_{j\in(\bigcap_{i\in I} J_i)} f_j^{\leftarrow}(\bigcap_{i\in I} G_j^i))} \Big(\bigcap_{(j\in\bigcap_{i\in I} J_i)} h_j^{\leftarrow}\mathscr{K}_j(\bigcap_{i\in I} G_j^i)\Big)$$

$$= \bigvee_{(\bigcap_{i\in I} A_i = \bigcup_{j\in(\bigcap_{i\in I} J_i)} f_j^{\leftarrow} B_j)} \Big(\bigcap_{(j\in\bigcap_{i\in I} J_i)} h_j^{\leftarrow}\mathscr{K}_j(B_j)\Big) = \mathscr{K}(\bigcap_{i\in I} A_i)$$

So $\mathscr{K}$ is a $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$. By the definition of $\mathscr{K}$, (f_j,F_j) is cocontinuous with respect to (h_j,H_j) for each $j\in J$.

Let $\mathscr{K}'$ be a $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$ that makes (f_j,F_j) cocontinuous with respect to (h_j,H_j) for each $j\in J$. Then $G_j\in\mathscr{S}_j$ implies $h_j^{\leftarrow}\mathscr{K}_j(G_j)\subseteq\mathscr{K}'(f_j^{\leftarrow}G_j)$ for each $j\in J$. So $A=\bigcup_{j\in J_0} f_j^{\leftarrow}G_j \Rightarrow \bigcap_{j\in J_0} h_j^{\leftarrow}\mathscr{K}_j(G_j)\subseteq\bigcap_{j\in J_0}\mathscr{K}' f_j^{\leftarrow}G_j\subseteq\mathscr{K}'(\bigcup_{j\in J_0} f_j^{\leftarrow}G_j)=\mathscr{K}'(A)$ for all $A\in\mathscr{S}$. Hence,

$$\mathscr{K}(A)=\bigvee\{\bigcap_{j\in J_0} h_j^{\leftarrow}\mathscr{K}_j(G_j)\mid A=\bigcup_{j\in J_0} f_j^{\leftarrow}G_j,\ J_0\subseteq J,\ J_0\ is\ finite\}\subseteq\mathscr{K}'(A)$$

for all $A\in\mathscr{S}$. Therefore $\mathscr{K}$ is the coarsest $(V,\mathscr{V})$-graded cotopology on $(S,\mathscr{S})$ that makes (f_j,F_j) cocontinuous with respect to (h_j,H_j) for each $j\in J$.

Similarly it can be shown that $\mathscr{T}$ is the coarsest $(V,\mathscr{V})$-graded topology on $(S,\mathscr{S})$ that makes (f_j,F_j) cocontinuous with respect to (h_j,H_j) for each $j\in J$. □

Now, referring to [1], we investigate the outcomes of Theorem 2.1 in categorical aspects. If we consider the forgetful functor $\mathfrak{U}$: **dfGDitop** → **dfTex** × **dfTex** then (**dfGDitop**, $\mathfrak{U}$) is a concrete category over **dfTex** × **dfTex**.

Theorem 2.2. *The source*

$$(((f_j,F_j),(h_j,H_j)):(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})\to(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J})$$

in **dfGDitop** *is initial if and only if* $(\mathscr{T},\mathscr{K})$ *is the graded ditopology defined as in Theorem 2.1.*

Proof. Let the source

$$(((f_j,F_j),(h_j,H_j)):(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})\to(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J})$$

be initial. For each $j\in J$; (f_j,F_j) is bicontinuous with respect to (h_j,H_j) because $((f_j,F_j),(h_j,H_j))$ is a morphism in **dfGDitop**. So, $H_j^{\leftarrow}\mathscr{T}_j(G_j)\subseteq\mathscr{T}(F_j^{\leftarrow}G_j)$ and $h_j^{\leftarrow}\mathscr{K}_j(G_j)\subseteq\mathscr{K}(f_j^{\leftarrow}G_j)$ for all $G_j\in\mathscr{S}_j$, $j\in J$. If we denote the graded ditopology defined in Theorem 2.1 by $(\mathscr{T}^\star,\mathscr{K}^\star)$ then we have

$$A=\bigcap_{j\in J_0} F_j^{\leftarrow}G_j\Rightarrow\bigcap_{j\in J_0} H_j^{\leftarrow}\mathscr{T}_j(G_j)\subseteq\bigcap_{j\in J_0}\mathscr{T}(F_j^{\leftarrow}G_j)\subseteq\mathscr{T}(\bigcap_{j\in J_0} F_j^{\leftarrow}G_j)=\mathscr{T}(A)$$

and so, $\mathscr{T}^\star(A)\subseteq\mathscr{T}(A)$ for all $A\in\mathscr{S}$, i.e. $\mathscr{T}^\star\subseteq\mathscr{T}$.

Since $((i_S,I_S),(i_V,I_V))$ in **dfTex** × **dfTex** makes the right hand diagram commutative, it lifts to a morphism in **dfGDitop** such that the left hand diagram is commutative.

$$\begin{array}{ccc} (S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V}) & & \\ {\scriptstyle ((i_S,I_S),(i_V,I_V))}\uparrow & \searrow^{((f_j,F_j),(h_j,H_j))} & \\ (S,\mathscr{S},\mathscr{T}^\star,\mathscr{K}^\star,V,\mathscr{V}) & \xrightarrow[((f_j,F_j),(h_j,H_j))]{} & (S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j) \end{array} \qquad \begin{array}{ccc} ((S,\mathscr{S}),(V,\mathscr{V})) & & \\ {\scriptstyle ((i_S,I_S),(i_V,I_V))}\uparrow & \searrow^{((f_j,F_j),(h_j,H_j))} & \\ ((S,\mathscr{S}),(V,\mathscr{V})) & \xrightarrow[((f_j,F_j),(h_j,H_j))]{} & ((S_j,\mathscr{S}_j),(V_j,\mathscr{V}_j)) \end{array}$$

Since $((i_S,I_S),(i_V,I_V))$ is a morphism in **dfGDitop**, (i_S,I_S) is bicontinuous with respect to (i_V,I_V). Hence $I_V^{\leftarrow}\mathscr{T}(A) \subseteq \mathscr{T}^\star(I_S^{\leftarrow}A) \Rightarrow \mathscr{T}(A) \subseteq \mathscr{T}^\star(A)$ for all $A \in \mathscr{S}$, i.e. $\mathscr{T} \subseteq \mathscr{T}^\star$. Therefore we get $\mathscr{T} = \mathscr{T}^\star$. Similarly it can be shown that $\mathscr{K} = \mathscr{K}^\star$

Now we will show that

$$(((f_j,F_j),(h_j,H_j)) : (S,\mathscr{S},\mathscr{T}^\star,\mathscr{K}^\star,V,\mathscr{V}) \to (S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J})$$

is initial. Let $((k,K),(l,L))$ be a morphism in **dfTex** $\times$ **dfTex** that makes the right hand diagram commutative.

$$\begin{array}{ccc} (S,\mathscr{S},\mathscr{T}^\star,\mathscr{K}^\star,V,\mathscr{V}) & & \\ {\scriptstyle ((k,K),(l,L))}\uparrow & \searrow^{((f_j,F_j),(h_j,H_j))} & \\ (S',\mathscr{S}',\mathscr{T}',\mathscr{K}',V',\mathscr{V}') & \xrightarrow[((f'_j,F'_j),(h'_j,H'_j))]{} & (S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j) \end{array} \qquad \begin{array}{ccc} ((S,\mathscr{S}),(V,\mathscr{V})) & & \\ {\scriptstyle ((k,K),(l,L))}\uparrow & \searrow^{((f_j,F_j),(h_j,H_j))} & \\ ((S',\mathscr{S}'),(V',\mathscr{V}')) & \xrightarrow[((f'_j,F'_j),(h'_j,H'_j))]{} & ((S_j,\mathscr{S}_j),(V_j,\mathscr{V}_j)) \end{array}$$

Then, by using Proposition 1.3, Proposition 1.5 (1) and (GT_2) we have

$$\begin{aligned} L^{\leftarrow}(\mathscr{T}^\star(A)) &= L^{\leftarrow}(\bigvee\{\bigcap_{j\in J_0} H_j^{\leftarrow}\mathscr{T}_j(G_j) \mid A = \bigcap_{j\in J_0} F_j^{\leftarrow}G_j,\ J_0 \subseteq J\}) \\ &= \bigvee\{\bigcap_{j\in J_0} L^{\leftarrow}(H_j^{\leftarrow}\mathscr{T}_j(G_j)) \mid A = \bigcap_{j\in J_0} F_j^{\leftarrow}G_j,\ J_0 \subseteq J\} \\ &= \bigvee\{\bigcap_{j\in J_0} (H_j \circ L)^{\leftarrow}\mathscr{T}_j(G_j) \mid A = \bigcap_{j\in J_0} F_j^{\leftarrow}G_j,\ J_0 \subseteq J\} \\ &= \bigvee\{\bigcap_{j\in J_0} H_j'^{\leftarrow}\mathscr{T}_j(G_j) \mid A = \bigcap_{j\in J_0} F_j^{\leftarrow}G_j,\ J_0 \subseteq J\} \\ &\subseteq \bigcap_{j\in J_0} \mathscr{T}'(F_j'^{\leftarrow}G_j) = \bigcap_{j\in J_0} \mathscr{T}'((F_j \circ K)^{\leftarrow}G_j) \\ &= \bigcap_{j\in J_0} \mathscr{T}'^{\leftarrow} \circ F_j^{\leftarrow}G_j) \subseteq \mathscr{T}'^{\leftarrow}(\bigcap_{j\in J_0} F_j^{\leftarrow}G_j)) = \mathscr{T}'^{\leftarrow}A) \end{aligned}$$

for all $A \in \mathscr{S}$. Hence (k,K) is continuous with respect to (l,L). Similarly it can be shown that (k,K) is cocontinuous with respect to (l,L) and so (k,K) is bicontinuous with respect to (l,L). Therefore $((k,K),(l,L))$ is a morphism in **dfGDitop**, i.e. the left hand diagram is commutative. □

Definition 2.3. *The graded ditopology constructed in Theorem 2.1 is called the initial $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$ induced by*

$$((f_j,F_j),(h_j,H_j))_{j\in J} \ \ \textit{and} \ \ (S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J}.$$

Corollary 2.4. **dfGDitop** *is a topological structure over* **dfTex** $\times$ **dfTex** *with respect to the functor* $\mathfrak{U}$.

Proof. Let $(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)\in Ob$**dfGDitop** and $((f_j,F_j),(h_j,H_j)):((S,\mathscr{S}),(V,\mathscr{V}))\longrightarrow((S_j,\mathscr{S}_j),(V_j,\mathscr{V}_j))$ is a morphism in **dfTex** $\times$ **dfTex** for all $j\in J$. If $(\mathscr{T},\mathscr{K})$ is the graded ditopology defined in Theorem 2.1 then, considering Theorem 2.2, $(((f_j,F_j),(h_j,H_j)):(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})\to(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J})$ is the unique initial source, which satisfies

$$\begin{aligned}\mathfrak{U}((((f_j,F_j),(h_j,H_j)):(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})\to(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j))_{j\in J}))\\ =((f_j,F_j),(h_j,H_j)):((S,\mathscr{S}),(V,\mathscr{V}))\longrightarrow((S_j,\mathscr{S}_j),(V_j,\mathscr{V}_j))_{j\in J}).\end{aligned}$$

□

Definition 2.5. *Let* $(S,\mathscr{S})$ *and* $(V,\mathscr{V})$ *be the product textures of the textures* $(S_j,\mathscr{S}_j)_{j\in J}$ *and* $(V_j,\mathscr{V}_j)_{j\in J}$ *respectively then the initial* $(V,\mathscr{V})$*-graded ditopology on* $(S,\mathscr{S})$ *induced by the projection difunctions* $(\pi_j^S,\prod_j^S):(S,\mathscr{S})\to(S_j,\mathscr{S}_j)$ *and* $(\pi_j^V,\prod_j^V):(V,\mathscr{V})\to(V_j,\mathscr{V}_j)$ *is called the product graded ditopology of* $(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)_{j\in J}$.

Example 2.6. *Let* (τ,κ) *be the product ditopology of* $(S_j,\mathscr{S}_j,\tau_j,\kappa_j)_{j\in J}$. *For each* $j\in J$, *if we take* $(V_j,\mathscr{V}_j)=(1,\mathscr{P}(1))$ *then* $(S_j,\mathscr{S}_j,\tau_j^g,\kappa_j^g,V_j,\mathscr{V}_j)$ *is a graded ditopological texture space where* $\tau_j^g(A)=1\Leftrightarrow A\in\tau_j$ *and* $\kappa_j^g(A)=1\Leftrightarrow A\in\kappa_j, A\in\mathscr{S}_j$. *So, the product graded ditopology* $(\mathscr{T},\mathscr{K})$ *of* $(S_j,\mathscr{S}_j,\tau_j^g,\kappa_j^g,V_j,\mathscr{V}_j)_{j\in J}$ *equals the graded ditopology* (τ^g,κ^g) *corresponding to ditopology* (τ,κ). *Indeed, for all* $A\in\mathscr{S}$, *by the definition of* $\mathscr{T}$ *and* (GT_3) *we have*

$$\begin{aligned}&\tau^g(A)=1\Leftrightarrow A\in\tau\\ &\Leftrightarrow A=\bigvee_{i\in I}B_i,\ B_i=\bigcap_{j\in J_i}(\pi_j^{\leftarrow}G_j),\ J_i\subseteq J\ finite,\ G_j\in\tau_j\ \textit{for a index set } I\\ &\Leftrightarrow A=\bigvee_{i\in I}B_i,\ B_i=\bigcap_{j\in J_i}(\pi_j^{\leftarrow}G_j),\ J_i\subseteq J\ finite,\ \tau_j^g(G_j)=1\ \textit{for a index set } I\\ &\Leftrightarrow A=\bigvee_{i\in I}B_i,\ \mathscr{T}(B_i)=1\ \textit{for a index set } I\ \Leftrightarrow\mathscr{T}(A)=1\end{aligned}$$

3. Compactness in graded ditopological texture spaces

A. P. Šostak has developed the spectral approach for the study of various topological properties of fuzzy topological spaces in [12]. Accordingly, we use this effective approach to study compactness notion (in accordance with fuzzy idea) in graded ditopological texture spaces as a generalization of compactness in ditopological texture spaces.

Definition 3.1. *Let* $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ *be a graded ditopological texture space and* $A\in\mathscr{S}$. *The families defined by*

$$\mathscr{C}(A)=\{P_v\in\mathscr{V}\mid[\mathscr{U}\subseteq\mathscr{T}^v,\ A\subseteq\bigvee\mathscr{U}]\Rightarrow\exists\mathscr{U}_0\subseteq\mathscr{U}\ :A\subseteq\bigvee\mathscr{U}_0\}$$

$$\mathscr{C}^*(A) = \{P_v \in \mathscr{V} \mid [\mathscr{U} \subseteq \mathscr{K}^v,\ \bigwedge \mathscr{U} \subseteq A] \Rightarrow \exists \mathscr{U}_0 \subseteq \mathscr{U}\ : \bigwedge \mathscr{U}_0 \subseteq A\}$$

where $\mathscr{U}_0$ denotes a finite subfamily of $\mathscr{U}$, are called compactness and co-compactness spectrums of $A \in \mathscr{S}$ respectively. In particular, the compactness spectrum and the co-compactness spectrum of $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ are $\mathscr{C}(S)$ and $\mathscr{C}^*(\emptyset)$ respectively.

Proposition 3.2. *If $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, \sigma, V, \mathscr{V})$ is a complemented graded ditopological texture space then $\mathscr{C}(A) = \mathscr{C}^*(\sigma(A))$ for all $A \in \mathscr{S}$. In particular, $\mathscr{C}(S) = \mathscr{C}^*(\emptyset)$ i.e. the compactness and co-compactness spectrums of a complemented graded ditopological texture space are equal.*

Proof. Since $A \subseteq \bigvee \mathscr{U} \Leftrightarrow \sigma(A) \supseteq \sigma(\bigvee \mathscr{U}) = \bigwedge \sigma(\mathscr{U}) \Leftrightarrow \bigwedge \sigma(\mathscr{U}) \subseteq \sigma(A)$ and $U \in \mathscr{T}^v \Leftrightarrow \sigma(U) \in \mathscr{K}^v$ for all $U \in \mathscr{S}$ we get

$$\mathscr{C}(A) = \{P_v \in \mathscr{V} \mid [\mathscr{U} \subseteq \mathscr{T}^v,\ A \subseteq \bigvee \mathscr{U}] \Rightarrow \exists \mathscr{U}_0 \subseteq \mathscr{U}\ : A \subseteq \bigvee \mathscr{U}_0\}$$

$$= \{P_v \in \mathscr{V} \mid [\sigma(\mathscr{U}) \subseteq \mathscr{K}^v,\ \bigwedge \sigma(\mathscr{U}) \subseteq \sigma(A)] \Rightarrow \exists \sigma(\mathscr{U}_0) \subseteq \sigma(\mathscr{U})\ : \bigwedge \sigma(\mathscr{U}_0) \subseteq \sigma(A)\}$$

$$= \{P_v \in \mathscr{V} \mid [\mathscr{U}' \subseteq \mathscr{K}^v,\ \bigwedge \mathscr{U}' \subseteq \sigma(A)] \Rightarrow \exists \mathscr{U}_0' \subseteq \mathscr{U}\ : \bigwedge \mathscr{U}_0' \subseteq \sigma(A)\} = \mathscr{C}^*(\sigma(A))$$

where $\mathscr{U}' = \sigma(\mathscr{U})$ and $\mathscr{U}_0' = \sigma(\mathscr{U}_0)$. In particular, since $S = \sigma(\emptyset)$ we have $\mathscr{C}(S) = \mathscr{C}^*(\emptyset)$. □

Theorem 3.3. *Let $(S_k, \mathscr{S}_k, \mathscr{T}_k, \mathscr{K}_k, V_k, \mathscr{V}_k)_{k=1,2}$ be graded ditopological texture spaces and let $(f,F) : (S_1, \mathscr{S}_1) \to (S_2, \mathscr{S}_2)$, $(h,H) : (V_1, \mathscr{V}_1) \to (V_2, \mathscr{V}_2)$ be difunctions. For all $A \in \mathscr{S}_1$*

(1) *If (f,F) is continuous with respect to (h,H) then,*

$$P_{v_1} \in \mathscr{C}_1(A) \Rightarrow P_{v_2} \in \mathscr{C}_2(f^{\to}A)$$

(2) *If (f,F) is cocontinuous with respect to (h,H) then,*

$$P_{v_1} \in \mathscr{C}_1^*(A) \Rightarrow P_{v_2} \in \mathscr{C}_2^*(F^{\to}A)$$

where $P_{v_1} \in \mathscr{V}_1$, $P_{v_2} \in \mathscr{V}_2$ with $P_{v_1} \subseteq h^{\leftarrow}P_{v_2}$.

Proof. Let $P_{v_1} \in \mathscr{C}_1(A)$ and $P_{v_1} \subseteq h^{\leftarrow}P_{v_2}$. If $\mathscr{U} \subseteq \mathscr{T}_2^{v_2}$ and $f^{\to}A \subseteq \bigvee \mathscr{U}$ then $A \subseteq F^{\leftarrow}(f^{\to}A) \subseteq F^{\leftarrow}(\bigvee \mathscr{U}) = \bigvee F^{\leftarrow}\mathscr{U} = \bigvee_{U \in \mathscr{U}} F^{\leftarrow}U$. Moreover, $P_{v_1} \subseteq h^{\leftarrow}P_{v_2} \subseteq h^{\leftarrow}(\mathscr{T}_2(\mathscr{U})) \subseteq \mathscr{T}_1(F^{\leftarrow}\mathscr{U})$ since (f,F) is continuous with respect to (h,H). Now, because of $P_{v_1} \in \mathscr{C}_1(A)$ there exists a finite subfamily $F^{\leftarrow}(\mathscr{U}_0) \subseteq F^{\leftarrow}(\mathscr{U})$ such that $A \subseteq \bigvee F^{\leftarrow}(\mathscr{U}_0)$. This follows $f^{\to}A \subseteq f^{\to} \bigvee F^{\leftarrow}(\mathscr{U}_0) = \bigvee f^{\to}(F^{\leftarrow}(\mathscr{U}_0)) \subseteq \bigvee \mathscr{U}_0$. Hence $P_{v_2} \in \mathscr{C}_2(f^{\to}A)$.

The proof of (2) is similar. □

Corollary 3.4. *Let the difunction (f,F) in Theorem 3.3 be surjective.*

(1) *If (f,F) is continuous with respect to (h,H) then,*

$$P_{v_1} \in \mathscr{C}_1(S_1) \Rightarrow P_{v_2} \in \mathscr{C}_2(S_2)$$

(2) *If* (f,F) *is cocontinuous with respect to* (h,H) *then,*

$$P_{v_1} \in \mathscr{C}_1^*(\emptyset) \Rightarrow P_{v_2} \in \mathscr{C}_2^*(\emptyset)$$

where $P_{v_1} \in \mathscr{V}_1$, $P_{v_2} \in \mathscr{V}_2$ *with* $P_{v_1} \subseteq h^{\leftarrow} P_{v_2}$.

Proof. Immediate from $f^{\rightarrow} S_1 = S_2$ and $F^{\rightarrow}\emptyset = \emptyset$. □

Corollary 3.5. *Let* $(S_j, \mathscr{S}_j, \mathscr{T}_j, \mathscr{K}_j, V_j, \mathscr{V}_j)_{j\in J}$ *be non-empty graded ditopological texture spaces and* $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ *their product graded ditopological texture space. Then for all* $j \in J$*;*

(1) $P_v \in \mathscr{C}(S) \Rightarrow P_{v_j} \in \mathscr{C}_j(S_j)$
(2) $P_v \in \mathscr{C}^*(\emptyset) \Rightarrow P_{v_j} \in \mathscr{C}_j^*(\emptyset)$

where $P_v = \prod_{j\in J} P_{v_j} \in \mathscr{V}$ *and* $P_{v_j} \in \mathscr{V}_j$.

Proof. We have $P_v \subseteq \pi_j^{V\leftarrow}(\pi_j^{V\rightarrow} P_v) = \pi_j^{V\leftarrow}(P_{v_j})$ for all $j \in J$ and $v \in V$ by Proposition 1.5 (3). So, the proof follows from Corollary 3.4. □

Theorem 3.6. ***(Tychonoff Theorem)*** *Let* $(S_j, \mathscr{S}_j, \mathscr{T}_j, \mathscr{K}_j, V_j, \mathscr{V}_j)_{j\in J}$ *be non-empty graded ditopological texture spaces and* $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ *their product graded ditopological texture space. If* $(V_j, \mathscr{V}_j)_{j\in J}$ *are plain textures then;*

(1) $P_v \in \mathscr{C}(S) \Leftrightarrow \forall j \in J\ P_{v_j} \in \mathscr{C}_j(S_j)$
(2) $P_v \in \mathscr{C}^*(\emptyset) \Leftrightarrow \forall j \in J\ P_{v_j} \in \mathscr{C}_j^*(\emptyset)$

where $P_v = \prod_{j\in J} P_{v_j} \in \mathscr{V}$ *and* $P_{v_j} \in \mathscr{V}_j$.

Proof. The necessity comes from Corollary 3.5. For sufficiency let $P_v \in \mathscr{V}$ and $P_{v_j} = \pi_j^{v\rightarrow} P_v \in \mathscr{C}_j(S_j)$ for all $j \in J$. If $\mathscr{U} \subseteq \mathscr{T}^v$ and $S = \bigvee \mathscr{U}$ then we get for all $j \in J$

$$S_j = \pi_j^{s\rightarrow}(S) = \pi_j^{s\rightarrow}(\bigvee \mathscr{U}) = \bigvee_{U\in\mathscr{U}} \pi_j^{s\rightarrow} U$$

On the other hand, since $\mathscr{U} \subseteq \mathscr{T}^v$, $U \in \mathscr{U}$ implies

$$P_v \subseteq \mathscr{T}(U) = \bigvee \{ \bigcap_{j\in J_0} \Pi_j^{v\leftarrow} \mathscr{T}_j(G_j) \mid U = \bigcap_{j\in J_0} \Pi_j^{s\leftarrow} G_j,\ J_0 \subseteq J,\ J_0 \text{ is finite}\}$$

Since $(V_j, \mathscr{V}_j)_{j\in J}$ are plain, $(V, \mathscr{V})$ is also plain by Proposition 1.10. Hence, $P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{v\leftarrow} \mathscr{T}_j(G_j^U)$ for some $U = \bigcap_{j\in J_0} \Pi_j^{s\leftarrow} G_j^U$ with $G_j^U \in \mathscr{S}_j$, $j \in J_0$. From Proposition 1.5 (3) we have

$$P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{v\leftarrow} \mathscr{T}_j(G_j^U) \Rightarrow \forall j \in J_0\ \ P_v \subseteq \Pi_j^{v\leftarrow} \mathscr{T}_j(G_j^U)$$

$$\Rightarrow \forall j \in J_0\ \ P_{v_j} = \pi_j^{v\rightarrow} P_v \subseteq \pi_j^{v\rightarrow}(\Pi_j^{v\leftarrow} \mathscr{T}_j(G_j^U)) \subseteq \mathscr{T}_j(G_j^U) \Rightarrow P_{v_j} \subseteq \mathscr{T}_j(G_j^U).$$

Since $U = \bigcap_{j\in J_0} \Pi_j^{s\leftarrow} G_j^U = \bigcap_{j\in J_0} E(j, G_j^U)$ we get $\pi_j^{s\rightarrow} U = \pi_j^{s\rightarrow}(\bigcap_{j\in J_0} E(j, G_j^U)) = G_j^U$ by Proposition 1.9 (1). So, considering (1) we have $S_j = \bigvee_{U\in\mathscr{U}} \pi_j^{s\rightarrow} U = \bigvee_{U\in\mathscr{U}} G_j^U$. Since

$G_j^U \in \mathscr{T}_j^{v_j}$ and $P_{v_j} \in \mathscr{C}_j(S_j)$ we get

$$\exists \mathscr{U}_0 \subseteq \mathscr{U} \;:\; S_j \subseteq \bigvee_{U \in \mathscr{U}_0} G_j^U \; (j \in J_0).$$

Thus, $S = \Pi_j^{s\leftarrow} S_j \subseteq \Pi_j^{s\leftarrow}(\bigvee_{U\in\mathscr{U}_0} G_j^U) \subseteq \bigvee_{U\in\mathscr{U}_0} \Pi_j^{s\leftarrow} G_j^U$ for all $j \in J_0$ and so,

$$\bigotimes_{j\in J} S_j = S \subseteq \bigcap_{j\in J_0} \big(\bigvee_{U\in\mathscr{U}_0} \Pi_j^{s\leftarrow} G_j^U\big) = \bigvee_{U\in\mathscr{U}_0} \big(\bigcap_{j\in J_0} \Pi_j^{s\leftarrow} G_j^U\big) = \bigvee_{U\in\mathscr{U}_0} \big(\bigcap_{j\in J_0} E(j, G_j^U)\big).$$

By the definition of $E(j, G_j^U)$;

$$j \notin J_0 \Rightarrow S_j = \pi_j^{s\rightarrow} S \subseteq \pi_j^{s\rightarrow}\big(\bigvee_{U\in\mathscr{U}_0} \bigcap_{j\in J_0} E(j, G_j^U)\big) = \pi_j^{s\rightarrow}\big(\bigvee_{U\in\mathscr{U}_0} U\big)$$

$$j \in J_0 \Rightarrow S_j = \pi_j^{s\rightarrow} S \subseteq \pi_j^{s\rightarrow}\big(\bigvee_{U\in\mathscr{U}_0} \bigcap_{j\in J_0} E(j, G_j^U)\big) = \pi_j^{s\rightarrow}\big(\bigvee_{U\in\mathscr{U}_0} U\big)$$

and hence, $S_j = \pi_j^{s\rightarrow} S \subseteq \pi_j^{s\rightarrow}(\bigvee_{U\in\mathscr{U}_0} U)$ for all $j \in J$. That means $S \subseteq \bigvee_{U\in\mathscr{U}_0} U$ and as a result $P_v \in \mathscr{C}(S)$. □

Definition 3.7. *For a graded ditopological texture space $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$, the families defined by*

$$\Omega = \{P_v \in \mathscr{V} \mid [A \in \mathscr{S},\; A \neq S] \Rightarrow [P_v \subseteq \mathscr{K}(A) \Rightarrow P_v \in \mathscr{C}(A)]\}$$

$$\Omega^* = \{P_v \in \mathscr{V} \mid [A \in \mathscr{S},\; A \neq \emptyset] \Rightarrow [P_v \subseteq \mathscr{T}(A) \Rightarrow P_v \in \mathscr{C}^*(A)]\}$$

are called stableness and costableness spectrums of $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ respectively.

Proposition 3.8. *For a complemented graded ditopological texture space*

Proposition 3.9. *For a complemented graded ditopological texture space $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, \sigma, V, \mathscr{V})$, $\Omega = \Omega^*$.*

Proof. Let $P_v \in \Omega^*$ and $A \in \mathscr{S}$, $A \neq S$. Then, $P_v \subseteq \mathscr{K}(A) \Rightarrow P_v \subseteq (\mathscr{T} \circ \sigma)(A) \Rightarrow P_v \subseteq \mathscr{T}(\sigma(A))$. Since $\sigma(A) \neq \emptyset$ and $P_v \in \Omega^*$ we have $P_v \in \mathscr{C}^*(\sigma(A)) = \mathscr{C}(A)$ by Proposition 3.2. So, $P_v \in \Omega$. The other direction of the proof can be shown similarly. □

Proposition 3.10. *Let $(S_k, \mathscr{S}_k, \mathscr{T}_k, \mathscr{K}_k, V_k, \mathscr{V}_k)_{k=1,2}$ be graded ditopological texture spaces with stableness (costableness) spectrums Ω_1, Ω_2 (Ω_1^*, Ω_2^*) respectively. If $(f,F) : (S_1, \mathscr{S}_1) \to (S_2, \mathscr{S}_2)$, $(h,H) : (V_1, \mathscr{V}_1) \to (V_2, \mathscr{V}_2)$ are surjective difunctions and (f,F) is bicontinuous with respect to (h,H) then $P_{v_1} \in \Omega_1 \Rightarrow P_{v_2} \in \Omega_2$ and $P_{v_1} \in \Omega_1^* \Rightarrow P_{v_2} \in \Omega_2^*$ where $v_1 \in V_1$, $v_2 \in V_2$ with $P_{v_1} \subseteq h^{\leftarrow} P_{v_2}$.*

Proof. Let (f,F) be bicontinuous with respect to (h,H) and $P_{v_1} \in \Omega_1$ with $P_{v_1} \subseteq h^{\leftarrow} P_{v_2}$. For a set $A \in \mathscr{S}_2$ with $A \neq S_2$ we have; $P_{v_2} \subseteq \mathscr{K}_2(A) \Rightarrow P_{v_1} \subseteq h^{\leftarrow} P_{v_2} \subseteq h^{\leftarrow} \mathscr{K}_2(A) \subseteq \mathscr{K}_1(f^{\leftarrow} A)$ by the bicontinuity of (f,F) with respect to (h,H). On the other hand, $f^{\leftarrow} A \neq S_1$ since (f,F) is surjective and $A \neq S_2$. So, $P_{v_1} \subseteq \mathscr{K}_1(f^{\leftarrow} A)$ and $P_{v_1} \in \Omega_1$ imply $P_{v_1} \in \mathscr{C}_1(f^{\leftarrow} A)$.

Now, by using Theorem 3.3 we have $P_{v_2} \in \mathscr{C}_2(f^{\rightarrow}(f^{\leftarrow}A))$. Since (f,F) is surjective we get $f^{\rightarrow}(f^{\leftarrow}A) = f^{\rightarrow}(F^{\leftarrow}A) = A$ by Proposition 1.5 (4). Therefore we have $P_{v_2} \in \mathscr{C}_2(A)$ and that means $P_{v_2} \in \Omega_2$. □

Corollary 3.11. *Let $(S_j, \mathscr{S}_j, \mathscr{T}_j, \mathscr{K}_j, V_j, \mathscr{V}_j)_{j\in J}$ be non-empty graded ditopological texture spaces with stableness (costableness) spectrums Ω_j, (Ω_j^*) respectively and $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ their product graded ditopological texture space with stableness (costableness) spectrum Ω, (Ω^*) respectively. Then for all $j \in J$;*

(1) $P_v \in \Omega \Rightarrow P_{v_j} \in \Omega_j$
(2) $P_v \in \Omega^* \Rightarrow P_{v_j} \in \Omega_j^*$

where $P_v = \prod_{j\in J} P_{v_j} \in \mathscr{V}$ and $P_{v_j} \in \mathscr{V}_j$.

Proof. We have $P_v \subseteq \pi_j^{v\leftarrow}(\pi_j^{v\rightarrow} P_v) = \pi_j^{v\leftarrow}(P_{v_j})$ for all $j \in J$ and $v \in V$ by Proposition 1.5 (3). So, the proof follows from Proposition 3.10. □

The other direction of Corollary 3.11 (i.e. $\forall j \in J\ P_{v_j} \in \Omega_j \Rightarrow P_v \in \Omega$ and $\forall j \in J\ P_{v_j} \in \Omega_j^* \Rightarrow P_v \in \Omega^*$) is an open problem for now as in the ditopological case in [6]. So we use the method which based on the relationship between ditopological and graded ditopological case to prove Theorem 3.16.

Definition 3.12. *For a graded ditopological texture space $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$, the family defined by*

$$\mathscr{DC} = \mathscr{C}(S) \cap \mathscr{C}^*(\emptyset) \cap \Omega \cap \Omega^*$$

is called dicompactness spectrum of $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$.

Example 3.13. *Let $(S, \mathscr{S}, \tau, \kappa)$ be a ditopological texture space and $(V, \mathscr{V}) = (1, \mathscr{P}(1))$ the discrete texture on a singleton. If $(S, \mathscr{S}, \tau, \kappa)$ is compact (cocompact, dicompact) then for the graded ditopological texture space $(S, \mathscr{S}, \tau^g, \kappa^g, V, \mathscr{V})$, $P_v \in \mathscr{C}(S)$ ($P_v \in \mathscr{C}^*(S)$, $P_v \in \mathscr{DC}$) respectively for all $v \in V$, i.e. $v = 0$.*

Proposition 3.14. *Let $(S, \mathscr{S}, \mathscr{T}, \mathscr{K}, V, \mathscr{V})$ be a graded ditopological texture space. Then the following hold:*

(1) $P_v \in \mathscr{C}(S) \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is compact*
(2) $P_v \in \mathscr{C}^*(\emptyset) \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is cocompact*
(3) $P_v \in \Omega \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is stable*
(4) $P_v \in \Omega^* \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is costable*
(5) $P_v \in \mathscr{DC} \Leftrightarrow (S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ *is dicompact*

Example 3.15. *Let $(S, \mathscr{S} = \mathscr{P}(S))$ and $(V, \mathscr{V} = \mathscr{P}(V))$ be discrete textures where $S \neq \emptyset$ and $V = \{a, b, c\}$. Then the mappings $\mathscr{T}, \mathscr{K} : \mathscr{S} \to \mathscr{V}$ defined by*

$$\mathscr{T}(A) = \begin{cases} V, & A = \emptyset \text{ or } A = S \\ \{a\}, & \text{otherwise} \end{cases}$$

$$\mathscr{K}(A) = \begin{cases} V, & A = \emptyset \text{ or } A = S \\ \{b\}, & \text{otherwise} \end{cases}$$

for all $A \in \mathscr{S}$ form a $(V,\mathscr{V})$-graded ditopology on $(S,\mathscr{S})$. We have $\mathscr{T}^a = \mathscr{S} = \mathscr{P}(S)$, $\mathscr{T}^b = \mathscr{T}^c = \{S,\emptyset\}$, $\mathscr{K}^b = \mathscr{S} = \mathscr{P}(S)$, $\mathscr{K}^a = \mathscr{K}^c = \{S,\emptyset\}$. If S is finite then we have $\mathscr{C}(S) = \mathscr{C}^*(\emptyset) = \Omega = \Omega^* = \mathscr{DC} = \{P_a,P_b,P_c\} = \{\{a\},\{b\},\{c\}\}$.

If S is infinite then for an infinite subset $A \subseteq S$, $\mathscr{U} = \{P_x \mid x \in A\}$ implies $A \subseteq \bigvee \mathscr{U} = \bigvee_{x\in A}\{x\}$ however there is no finite subfamily $\mathscr{U}_0$ of $\mathscr{U}$ such that $A \subseteq \bigvee \mathscr{U}_0$. So we get $\mathscr{C}(S) = \Omega = \{P_b,P_c\}$. Similarly, for a subset $A \subseteq S$, if $S\setminus A$ is infinite then $\mathscr{U} = \{(S\setminus A)\setminus P_x \mid x \in (S\setminus A)\}$ implies $\bigwedge \mathscr{U} = \bigwedge_{x\in(S\setminus A)}((S\setminus A)\setminus P_x) = \emptyset \subseteq A$ however there is no finite subfamily $\mathscr{U}_0$ of $\mathscr{U}$ such that $\bigwedge \mathscr{U}_0 \subseteq A$. Hence we get $\mathscr{C}^*(\emptyset) = \Omega^* = \{P_a,P_c\}$. Therefore, if S is infinite then $\mathscr{DC} = \mathscr{C}(S)\cap\mathscr{C}^*(\emptyset)\cap\Omega\cap\Omega^* = \{P_c\} = \{\{c\}\}$ is obtained.

Theorem 3.16. *Let $(S_j,\mathscr{S}_j,\mathscr{T}_j,\mathscr{K}_j,V_j,\mathscr{V}_j)_{j\in J}$ be non-empty graded ditopological texture spaces and $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ their product graded ditopological texture space. If $(V_j,\mathscr{V}_j)_{j\in J}$ are plain textures then;*

$$P_v \in \mathscr{DC} \Leftrightarrow \forall j \in J\ P_{v_j} \in \mathscr{DC}_j$$

where $P_v = \prod_{j\in J} P_{v_j} \in \mathscr{V}$ and $P_{v_j} \in \mathscr{V}_j$.

Proof. ($\Rightarrow$): It is obvious from Theorem 3.6 and Corollary 3.11.
($\Leftarrow$): Let $P_{v_j} \in \mathscr{DC}_j$ for all $j \in J$ where $P_v = \prod_{j\in J} P_{v_j} \in \mathscr{V}$ and $P_{v_j} \in \mathscr{V}_j$. Then ditopological texture spaces $(S_j,\mathscr{S}_j,\mathscr{T}_j^{v_j},\mathscr{K}_j^{v_j})_{j\in J}$ are dicompact by Proposition 3.14. So, their product ditopological texture space $(S,\mathscr{S},\mathscr{T}_v,\mathscr{K}_v)$ is dicompact by Theorem 1.12.

Now, we show that $\mathscr{T}_v = \mathscr{T}^v$. Take $A \in \mathscr{T}_v$ then $A = \bigvee_{B\in\beta'} B$ where $\beta' \subseteq \beta$ and β is the base for the ditopology $(\mathscr{T}_v,\mathscr{K}_v)$. On the other hand, if $B \in \beta'$ there exists a finite subset $J_0 \subseteq J$ with $B = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j$ such that "$G_j \in \mathscr{T}_j^{v_j}$ for all $j \in J_0$". This follows,

$$\forall j \in J_0\ P_{v_j} \subseteq \mathscr{T}_j(G_j) \Rightarrow \forall j \in J_0\ \Pi_j^{V\leftarrow} P_{v_j} \subseteq \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)$$

$$\Rightarrow P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} P_{v_j} \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)$$

because $P_v \subseteq \Pi_j^{V\leftarrow}(\pi_j^{V\rightarrow} P_v) = \Pi_j^{V\leftarrow}(P_{v_j})$ for all $j \in J$ and $v \in V$ by Proposition 1.5 (3). Thus we have $P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)$ and $B = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j$ where $J_0 \subseteq J$ is a finite subset. So we get $P_v \subseteq \mathscr{T}(B)$ by the definition of $\mathscr{T}$. Using GT_3 we obtain that $\mathscr{T}(A) = \mathscr{T}(\bigvee_{B\in\beta'} B) \supseteq \bigcap_{B\in\beta'} \mathscr{T}(B) \supseteq P_v$ and so $A \in \mathscr{T}^v$.

If we take $A \in \mathscr{T}^v$ then $P_v \subseteq \mathscr{T}(A)$. Since $(V_j,\mathscr{V}_j)_{j\in J}$ are plain, $(V,\mathscr{V})$ is also plain by Proposition 1.10. Hence, considering the definition of $\mathscr{T}$ we have:

$$\exists J_0 \subseteq J \text{ finite with } A = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j\ :\ P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j).$$

Besides, considering Proposition 1.5 (3) we get

$$P_v \subseteq \bigcap_{j\in J_0} \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j) \Rightarrow \forall j \in J_0\ P_v \subseteq \Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)$$

$$\Rightarrow \forall j \in J_0\ P_{v_j} = \pi_j^{V\rightarrow} P_v \subseteq \pi_j^{V\rightarrow}(\Pi_j^{V\leftarrow} \mathscr{T}_j(G_j)) \subseteq \mathscr{T}_j(G_j).$$

Thus we obtain that

$$\exists J_0 \subseteq J \text{ finite with } A = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j \ :\ \text{“}\forall j \in J_0\ P_{v_j} \subseteq \mathscr{T}_j(G_j)\text{”}$$

$$\Rightarrow \exists J_0 \subseteq J \text{ finite with } A = \bigcap_{j\in J_0} \Pi_j^{S\leftarrow} G_j \ :\ \text{“}\forall j \in J_0\ G_j \in \mathscr{T}_j^{v_j}\text{”} \Rightarrow A \in \mathscr{T}_v$$

Similarly it can be shown that $\mathscr{K}_v = \mathscr{K}^v$. That means $(S, \mathscr{S}, \mathscr{T}^v, \mathscr{K}^v)$ is dicompact and so, $P_v \in \mathscr{DC}$. □

Note that in Theorem 3.16, the textures $(S_j, \mathscr{S}_j)_{j\in J}$ don't have to be plain unlike the textures $(V_j, \mathscr{V}_j)_{j\in J}$. It is an open problem whether Theorem 3.16 is still true in case $(V_j, \mathscr{V}_j)_{j\in J}$ are not plain.

4. Graded Difilters and dicompactness spectrum

Difilters on Textures: [14] Let $(S, \mathscr{S})$ be a texture. $\mathscr{F} \subseteq \mathscr{S}$ is called a filter on $(S, \mathscr{S})$ if (i) $\mathscr{F} \neq \emptyset$, (ii) $\emptyset \notin \mathscr{F}$, (iii) $F \in \mathscr{F},\ F \subseteq F' \in \mathscr{S} \Rightarrow F' \in \mathscr{F}$ and (iv) $F_1, F_2 \in \mathscr{F} \Rightarrow F_1 \cap F_2 \in \mathscr{F}$. $\mathscr{G} \subseteq \mathscr{S}$ is called a cofilter on $(S, \mathscr{S})$ if (i) $\mathscr{G} \neq \emptyset$, (ii) $S \notin \mathscr{G}$, (iii) $G \in \mathscr{G},\ G \supseteq G' \in \mathscr{S} \Rightarrow G' \in \mathscr{G}$, and (iv) $G_1, G_2 \in \mathscr{G} \Rightarrow G_1 \cup G_2 \in \mathscr{G}$. If $\mathscr{F}$ is a filter and $\mathscr{G}$ is a cofilter on $(S, \mathscr{S})$ then $\mathscr{F} \times \mathscr{G}$ is called a difilter on $(S, \mathscr{S})$. A difilter $\mathscr{F} \times \mathscr{G}$ on $(S, \mathscr{S})$ is called regular if $\mathscr{F} \cap \mathscr{G} = \emptyset$.

Proposition 4.1. [14] *If $\mathscr{F} \times \mathscr{G}$ is a difilter on $(S, \mathscr{S}, \tau, \kappa)$ then*

(a) *$\mathscr{F}$ converges to $s \in S^\flat$ ($\mathscr{F} \to s$) $\Leftrightarrow$ “$G \in \tau,\ G \nsubseteq Q_s \Rightarrow G \in \mathscr{F}$”*
(b) *$\mathscr{G}$ converges to s ($\mathscr{G} \to s$) $\Leftrightarrow$ “$K \in \kappa,\ P_s \nsubseteq K \Rightarrow K \in \mathscr{G}$”*
(c) *$\mathscr{F} \times \mathscr{G}$ is diconvergent if $\mathscr{F} \to s$ and $\mathscr{G} \to s'$ for some $s, s' \in S$ with $P_{s'} \nsubseteq Q_s$.*

A difilter $\mathscr{F} \times \mathscr{G}$ on $(S, \mathscr{S}, \tau, \kappa)$ is said to be diclustering if $A \in \mathscr{F} \Rightarrow P_{s'} \subseteq [A]$ and $B \in \mathscr{G} \Rightarrow]B[\subseteq Q_s$ for some $s, s' \in S$ with $P_{s'} \nsubseteq Q_s$.

Theorem 4.2. [14] *A regular difilter $\mathscr{F} \times \mathscr{G}$ on $(S, \mathscr{S})$ is maximal if and only if $\mathscr{F} \cup \mathscr{G} = S$.*

Proposition 4.3. [14] *A maximal regular difilter is diconvergent if and only if it is diclustering.*

Theorem 4.4. [14] *A ditopological texture space $(S, \mathscr{S}, \tau, \kappa)$ is dicompact if and only if every regular difilter on $(S, \mathscr{S}, \tau, \kappa)$ is diclustering if and only if every maximal regular difilter on $(S, \mathscr{S}, \tau, \kappa)$ is diconvergent.*

Graded difilters : [9] Let $(S, \mathscr{S})$ and $(V, \mathscr{V})$ be textures. A mapping $\mathfrak{F} : \mathscr{S} \to \mathscr{V}$ is called a $(V, \mathscr{V})$-graded filter on $(S, \mathscr{S})$ if (i) $\mathfrak{F}(\emptyset) = \emptyset$, (ii) $A_1 \subseteq A_2 \Rightarrow \mathfrak{F}(A_1) \subseteq \mathfrak{F}(A_2)$ and (iii) $\mathfrak{F}(A_1) \wedge \mathfrak{F}(A_2) \subseteq \mathfrak{F}(A_1 \cap A_2)$. A mapping $\mathfrak{G} : \mathscr{S} \to \mathscr{V}$ is called a $(V, \mathscr{V})$-graded cofilter on $(S, \mathscr{S})$ if (i) $\mathfrak{G}(S) = \emptyset$, (ii) $A_1 \subseteq A_2 \Rightarrow \mathfrak{G}(A_2) \subseteq \mathfrak{G}(A_1)$ and (iii) $\mathfrak{G}(A_1) \wedge \mathfrak{G}(A_2) \subseteq \mathfrak{G}(A_1 \cup A_2)$. If $\mathfrak{F}$ is a $(V, \mathscr{V})$-graded filter and $\mathfrak{G}$ $(V, \mathscr{V})$-graded cofilter on $(S, \mathscr{S})$ then the pair $(\mathfrak{F}, \mathfrak{G})$ is called a $(V, \mathscr{V})$-graded difilter on $(S, \mathscr{S})$.

$(\mathfrak{F},\mathfrak{G})$ is called regular if $\mathfrak{F}\wedge\mathfrak{G}=\emptyset$ i.e. $\mathfrak{F}(A)\wedge\mathfrak{G}(A)=\emptyset$ for all $A\in\mathscr{S}$.

The diconvergent graded difilters were defined in [9]. To avoid a long preliminaries we will give the following equivalent proposition instead of the definition.

Proposition 4.5. [9] *If $(\mathfrak{F},\mathfrak{G})$ is a $(V,\mathscr{V})$-graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ then*

(a) *$\mathfrak{F}$ converges to s ($\mathfrak{F}\to s$) $\Leftrightarrow$ "$A\nsubseteq Q_s \Rightarrow \mathscr{T}(A)\subseteq\mathfrak{F}(A)$"*
(b) *$\mathfrak{G}$ converges to s ($\mathfrak{G}\to s$) $\Leftrightarrow$ "$P_s\nsubseteq A \Rightarrow \mathscr{K}(A)\subseteq\mathfrak{G}(A)$"*
(c) *For $s,s'\in S$, $(\mathfrak{F},\mathfrak{G})$ is diconvergent if $P_{s'}\nsubseteq Q_s$, $\mathfrak{F}\to s$ and $\mathfrak{G}\to s'$.*

Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space, $A\in\mathscr{S}$ and $v\in V$. The set $\bigcap\{B\in\mathscr{S}|A\subseteq B,\ P_v\subseteq\mathscr{K}(B)\}\in\mathscr{S}$ is called v-closure of A and denoted by $[A]^v$. The set $\bigvee\{B\in\mathscr{S}|B\subseteq A,\ P_v\subseteq\mathscr{T}(B)\}\in\mathscr{S}$ is called v-interior of A and denoted by $]A[^v$. Note that for each $v\in V$, $[A]^v$ ($]A[^v$) is the closure (the interior) of A in the ditopological texture space $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$.

A regular graded difilter $(\mathfrak{F},\mathfrak{G})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is called diclustering if for all $A\in\mathscr{S}$, $v\in\mathfrak{F}(A)\Rightarrow P_s\subseteq[A]^v$ and $v\in\mathfrak{G}(A)\Rightarrow]A[^v\subseteq Q_{s'}$ for some $s,s'\in S$ with $P_s\nsubseteq Q_{s'}$.

Proposition 4.6. [9] *Let $(\mathfrak{F},\mathfrak{G})$ be a regular $(V,\mathscr{V})$-graded difilter on $(S,\mathscr{S})$. For the statements*

(1) *$(\mathfrak{F},\mathfrak{G})$ is a maximal regular $(V,\mathscr{V})$-graded difilter*
(2) *$\mathfrak{F}\vee\mathfrak{G}=V$ (i.e. $\forall A\in\mathscr{S},\mathfrak{F}(A)\vee\mathfrak{G}(A)=\mathfrak{F}(A)\cup\mathfrak{G}(A)=V$)*

$(1)\Leftarrow(2)$ and in case of $(V,\mathscr{V})$ is discrete, $(1)\Rightarrow(2)$ are hold.

if $(\mathfrak{F},\mathfrak{G})$ be a (regular) $(V,\mathscr{V})$-graded difilter on a texture $(S,\mathscr{S})$ then the families

$$\mathfrak{F}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathfrak{F}(A)\},\ \ \mathfrak{G}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathfrak{G}(A)\}$$

form a (regular) difilter $\mathfrak{F}^v\times\mathfrak{G}^v$ on $(S,\mathscr{S})$ for each $v\in V$ [9].

Proposition 4.7. [9] *Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space. Then, for the statements*

(a) *Every regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is diclustering.*
(b) *Every maximal regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is diconvergent.*

the implication $(b)\Rightarrow(a)$ and in case of $(V,\mathscr{V})$ is discrete, $(a)\Rightarrow(b)$ are hold.

Definition 4.8. *Let $(\mathfrak{F},\mathfrak{G})$ be a regular graded difilter on a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Then the family defined by*

$$\begin{aligned} Dcl(\mathfrak{F},\mathfrak{G})=\{P_v\mid \exists s,s'\in S \text{ with } P_s\nsubseteq Q_{s'}\ :\ \forall A\in\mathscr{S}\\ [v\in\mathfrak{G}(A)\Rightarrow]A[^v\subseteq Q_{s'} \text{ and } v\in\mathfrak{F}(A)\Rightarrow P_s\subseteq[A]^v]\}\end{aligned}$$

is called diclustering spectrum of $(\mathfrak{F},\mathfrak{G})$.

Example 4.9. *Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space and $v\in V$. If $\mathscr{F}\times\mathscr{G}$ is a a regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ then the mappings defined by*

$$\mathfrak{F}_{\mathscr{F}}(A)=\begin{cases} V, & A\in\mathscr{F}\\ \emptyset, & A\notin\mathscr{F}\end{cases}$$

$$\mathfrak{G}_{\mathscr{G}}(A)=\begin{cases} V, & A\in\mathscr{G}\\ \emptyset, & A\notin\mathscr{G}\end{cases}$$

for all $A\in\mathscr{S}$ form a regular graded difilter $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Moreover, $\mathfrak{F}^v_{\mathscr{F}}=\mathscr{F}$ and $\mathfrak{G}^v_{\mathscr{G}}=\mathscr{G}$.

Proposition 4.10. *For a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$, the following equation holds:*

$$\mathscr{DC}=\bigcap\{Dcl(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})\text{ is a regular graded difilter}\}$$

Proof. Let $P_v\in\bigcap\{Dcl(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a regular graded difilter$\}$ and take a regular difilter $\mathscr{F}\times\mathscr{G}$ on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$. Then $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ is a regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Since $P_v\in\bigcap\{Dcl(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a regular graded difilter$\}$ we have $P_v\in Dcl(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$. So we get

$$\exists s,s'\in S\text{ with }P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}\ [v\in\mathfrak{G}_{\mathscr{G}}(A)\Rightarrow]A[^v\subseteq Q_{s'}\text{ and }v\in\mathfrak{F}_{\mathscr{F}}(A)\Rightarrow P_s\subseteq[A]^v].$$

This follows that "$A\in\mathfrak{G}^v_{\mathscr{G}}\Rightarrow]A[^v\subseteq Q_{s'}$" and "$A\in\mathfrak{F}^v_{\mathscr{F}}\Rightarrow P_s\subseteq[A]^v$". Thus we have "$A\in\mathscr{G}\Rightarrow]A[\subseteq Q_{s'}$", "$A\in\mathscr{F}\Rightarrow P_s\subseteq[A]$" and this implies that $\mathscr{F}\times\mathscr{G}$ is diclustering. Since every regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is diclustering, $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is dicompact by Theorem 4.4 and that means $P_v\in\mathscr{DC}$.

On the other hand, let $P_v\in\mathscr{DC}$ and take a regular graded difilter $(\mathfrak{F},\mathfrak{G})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Then $(\mathfrak{F}^v\times\mathfrak{G}^v)$ is a regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$. Since $P_v\in\mathscr{DC}$, $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is dicompact and so, $(\mathfrak{F}^v\times\mathfrak{G}^v)$ is diclustering by Theorem 4.4. That means "$A\in\mathfrak{G}^v\Rightarrow]A[\subseteq Q_{s'}$", "$A\in\mathfrak{F}^v\Rightarrow P_s\subseteq[A]$" for some $s,s'\in S$ with $P_s\not\subseteq Q_{s'}$. Thus we get,

$$\exists s,s'\in S\text{ with }P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}\ [v\in\mathfrak{G}(A)\Rightarrow]A[^v\subseteq Q_{s'}\text{ and }v\in\mathfrak{F}(A)\Rightarrow P_s\subseteq[A]^v].$$

Hence we get $P_v\in\bigcap\{Dcl(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a regular graded difilter$\}$. □

Lemma 4.11. *Let $(V,\mathscr{V})$ be a discrete texture. If $(\mathfrak{F},\mathfrak{G})$ is a maximal regular graded difilter on a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ then the regular difilter $\mathfrak{F}^v\times\mathfrak{G}^v$ on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is maximal for all $v\in V$.*

Proof. Let $(\mathfrak{F},\mathfrak{G})$ be a maximal regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ and $v\in V$. Since $(V,\mathscr{V})$ is a discrete texture we have $\mathfrak{F}\vee\mathfrak{G}=V$ by Proposition 4.6. We also know that $\mathfrak{F}^v\times\mathfrak{G}^v$ is a regular difilter. Consider $\mathfrak{F}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathfrak{F}(A)\}$ and $\mathfrak{G}^v=\{A\in\mathscr{S}\mid P_v\subseteq\mathfrak{G}(A)\}$. Since $\mathfrak{F}\vee\mathfrak{G}=V$ and $(V,\mathscr{V})$ is discrete we get

$$A\in\mathscr{S}\Rightarrow P_v\subseteq\mathfrak{F}(A)\vee\mathfrak{G}(A)=\mathfrak{F}(A)\cup\mathfrak{G}(A)\Rightarrow P_v\subseteq\mathfrak{F}(A)\text{ or }P_v\subseteq\mathfrak{G}(A)$$
$$\Rightarrow A\in\mathfrak{F}^v\text{ or }A\in\mathfrak{G}^v\Rightarrow A\in\mathfrak{F}^v\cup\mathfrak{G}^v$$

for all $A\in\mathscr{S}$. That means $\mathfrak{F}^v\cup\mathfrak{G}^v=\mathscr{S}$ and so $\mathfrak{F}^v\times\mathfrak{G}^v$ is maximal by Theorem 4.2. □

Lemma 4.12. *Let $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ be a graded ditopological texture space and $v\in V$. If $\mathscr{F}\times\mathscr{G}$ is a maximal regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ then the regular graded difilter $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ is maximal.*

Proof. Let $\mathscr{F}\times\mathscr{G}$ be a maximal regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$. Then we have $\mathscr{F}\cup\mathscr{G}=\mathscr{S}$. So, for all $A\in\mathscr{S}$ we get $A\in\mathscr{F}$ or $A\in\mathscr{G}$. This follows $\mathfrak{F}_{\mathscr{F}}(A)=V$ or $\mathfrak{G}_{\mathscr{G}}(A)=V$. That means $\mathfrak{F}_{\mathscr{F}}\vee\mathfrak{G}_{\mathscr{G}}=V$. Hence $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ is maximal by Proposition 4.6. □

Definition 4.13. *Let $(\mathfrak{F},\mathfrak{G})$ be a graded difilter on a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. The family defined by*

$$Dcn(\mathfrak{F},\mathfrak{G})=\{P_v\mid \exists s,s'\in S \text{ with } P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}$$
$$[(A\in\mathscr{T}^v,\ A\not\subseteq Q_s)\Rightarrow A\in\mathfrak{F}^v \text{ and } (A\in\mathscr{K}^v,\ P_{s'}\not\subseteq A)\Rightarrow A\in\mathfrak{G}^v]\}$$

is called diconvergence spectrum of $(\mathfrak{F},\mathfrak{G})$.

Proposition 4.14. *Let $(V,\mathscr{V})$ be a discrete texture. For a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$, the following equation holds:*

$$\mathscr{DC}=\bigcap\{Dcn(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G}) \text{ is a maximal regular graded difilter}\}$$

Proof. Let $P_v\in\bigcap\{Dcn(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a maximal regular graded difilter$\}$ and $\mathscr{F}\times\mathscr{G}$ be a maximal regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$. Then $(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$ is a maximal regular graded difilter on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$ by Lemma 4.12.

Since $P_v\in\bigcap\{Dcn(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G})$ is a maximal regular graded difilter$\}$ we have $P_v\in Dcn(\mathfrak{F}_{\mathscr{F}},\mathfrak{G}_{\mathscr{G}})$. This follows

$$\exists s,s'\in S \text{ with } P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}$$
$$[(A\in\mathscr{T}^v,\ A\not\subseteq Q_s)\Rightarrow A\in\mathfrak{F}^v_{\mathscr{F}} \text{ and } (A\in\mathscr{K}^v,\ P_{s'}\not\subseteq A)\Rightarrow A\in\mathfrak{G}^v_{\mathscr{G}}].$$

Therefore we get "$(A\in\mathscr{T}^v,\ A\not\subseteq Q_s)\Rightarrow A\in\mathscr{F}$" and "$(A\in\mathscr{K}^v,\ P_{s'}\not\subseteq A)\Rightarrow A\in\mathscr{G}$". Considering Proposition 4.1, $\mathscr{F}\times\mathscr{G}$ is diconvergent and so, $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is dicompact by Theorem 4.4. Hence we get $P_v\in\mathscr{DC}$.

On the other hand, let $P_v\in\mathscr{DC}$ and take a maximal regular graded difilter $(\mathfrak{F},\mathfrak{G})$ on $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Then $\mathfrak{F}^v\times\mathfrak{G}^v$ is a maximal regular difilter on $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ by Lemma 4.11. Besides, $(S,\mathscr{S},\mathscr{T}^v,\mathscr{K}^v)$ is dicompact since $P_v\in\mathscr{DC}$. So, $\mathfrak{F}^v\times\mathfrak{G}^v$ is diconvergent by Theorem 4.4. Thus we have

$$\exists s,s'\in S \text{ with } P_s\not\subseteq Q_{s'}\ :\ \forall A\in\mathscr{S}$$
$$[(A\in\mathscr{T}^v,\ A\not\subseteq Q_s)\Rightarrow A\in\mathfrak{F}^v \text{ and } (A\in\mathscr{K}^v,\ P_{s'}\not\subseteq A)\Rightarrow A\in\mathfrak{G}^v].$$

Hence we get $P_v\in Dcn(\mathfrak{F},\mathfrak{G})$ i.e.,

$$P_v\in\bigcap\{Dcn(\mathfrak{F},\mathfrak{G})\mid(\mathfrak{F},\mathfrak{G}) \text{ is a maximal regular graded difilter}\}.$$

□

Corollary 4.15. *Let $(V,\mathscr{V})$ be a discrete texture. If $(\mathfrak{F},\mathfrak{G})$ is a maximal regular graded difilter on a graded ditopological texture space $(S,\mathscr{S},\mathscr{T},\mathscr{K},V,\mathscr{V})$. Then*

$$Dcn(\mathfrak{F},\mathfrak{G})=Dcl(\mathfrak{F},\mathfrak{G})$$

Proof. Considering Lemma 4.11 and Proposition 4.3 we have

$$P_v \in Dcn(\mathfrak{F},\mathfrak{G}) \Leftrightarrow \text{“}\exists s, s' \in S \text{ with } P_s \not\subseteq Q_{s'} \ : \ \forall A \in \mathscr{S}$$
$$[(A \in \mathscr{T}^v,\ A \not\subseteq Q_s) \Rightarrow A \in \mathfrak{F}^v \text{ and } (A \in \mathscr{K}^v,\ P_{s'} \not\subseteq A) \Rightarrow A \in \mathfrak{G}^v]\text{”}$$
$$\Leftrightarrow \text{“}\mathfrak{F}^v \to s,\ \mathfrak{G}^v \to s' \text{ and } P_s \not\subseteq Q_{s'}\text{”}$$
$$\Leftrightarrow \text{“}\exists s, s' \in S \text{ with } P_s \not\subseteq Q_{s'} \ : \ \forall A \in \mathscr{S}\ [A \in \mathfrak{G}^v \Rightarrow]A[^v \subseteq Q_{s'} \text{ and } A \in \mathfrak{F}^v \Rightarrow P_s \subseteq [A]^v]\text{”}$$
$$\Leftrightarrow P_v \in Dcl(\mathfrak{F},\mathfrak{G}).$$

□

ACKNOWLEDGEMENTS

The author would like to thank the referees for their helpful suggestions and comments.

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