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Chebyshev Polynomial Solutions of Certain Second Order Non-Linear Differential Equations

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ABSTRACT

The purpose of this study is to give a Chebyshev polynomial approximation for the solution of second-order non-linear differential equations with variable coefficients. For this purpose, Chebyshev matrix method is introduced. This method is based on taking the truncated Chebyshev expansions of the functions in the non-linear differential equations. Hence, the result matrix equation can be solved and the unknown Chebyshev coefficients can be found approximately. Additionally, the mentioned method is illustrated by two examples.

Key Words: Non-linear differential equations, Chebyshev-matrix method, Approximate solution of non-linear ordinary differential equations.

1. INTRODUCTION

The Chebyshev matrix method has been presented by Keşan [2] to solve linear differential equations. This method has been also used by Köroğlu [3] to solve linear Fredholm integrodifferential equations. Additionally, Cantürk-Günhan [1] has extended this method to solve non-linear differential and integral equations. In this work, we adapt Chebyshev -Matrix method to second-order non-linear differential equations. It is presented as follow:

$$\sum_{k=1}^{s} P_k(x)(y'')^k + \sum_{k=1}^{m} Q_k(x)(y')^k + \sum_{k=1}^{n} R_k(x)(y)^k = F(x) \quad \mathrm{s, m, n} = 1,2,... \tag{1}$$

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where $P_k(x),\ Q_k(x),\ R_k(x)$ and $F(x)$ are functions having Chebyshevexpansions on an interval $a \le x \le b$, under the given conditions, which are

$$\sum_{i=0}^{2}\left[a_i y^{(i)}(a) + b_i y^{(i)}(b) + c_i y^{(i)}(c)\right] = \lambda \qquad (2)$$

$$\sum_{i=0}^{2}\left[\alpha_i y^{(i)}(a) + \beta_i y^{(i)}(b) + \gamma_i y^{(i)}(c)\right] = \mu$$

where $a \le c \le b$, provide that the real coefficients $a_i, b_i, \alpha_i, \beta_i, \lambda$ and μ are appropriate constants; and the solution is expressed in the form

$$y(x) = \sum_{r=0}^{n}{}' \mathrm{a}_\mathrm{r} T_r(x) \qquad (3)$$

under the certain conditions $r = 0,1,2,\ldots$. And, $\sum{}'$ denotes a sum whose first term is halved.

2. FUNDAMENTAL RELATIONS

Let us assume that the function $y(x)$ its nth derivative with respect to x, respectively, can be expanded in Chebyshev series

$$y(x) = \sum_{r=0}^{\infty}{}' a_r T_r(x) \quad \text{and}$$

$$y^{(n)}(x) = \sum_{r=0}^{\infty}{}' a_r^{(n)} T_r(x).$$

The recurrence relation between the Chebyshev coefficients $a_r^{(n)}$ and $a_r^{(n+1)}$ of $y^{(n)}(x)$ and $y^{(n+1)}(x)$, is given by

$$a_r^{(n+1)} = 2\sum_{i=0}^{\infty} (r+2i+1) a_{r+2i+1}^{(n)} \qquad (4)$$

Now let us take r=0,1,…,N and assume $a_r^{(n)} = 0$ for $r > N$. Then the system (4) can be transformed into the matrix form

$$A^{(n+1)} = 2MA^{(n)},\ n = 0,1,2,\ldots \qquad (5)$$

where

$$A^{(n)} = \left[\frac{1}{2}a_0^{(n)} \quad a_1^{(n)} \quad \ldots \quad a_N^{(n)}\right]^T$$

for add N

$$M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & . & . & . & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & . & . & . & N \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & N \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}_{(N+1)x(N+1)}$$

and for even N

$$M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & . & . & . & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & . & . & . & N \\ 0 & 0 & 0 & 3 & 0 & 5 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & N \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}_{(N+1)x(N+1)}$$

It follows from relation (5) that

$$A^{(n)} = 2MA^{(n-1)} = 2^n M^n A. \qquad (6)$$

where clearly

$$A^{(0)} = \left[\frac{1}{2}a_0 \quad a_1 \quad \ldots \quad a_N\right]^T.$$

The matrix equation (6) gives a relation between the Chebyshev coefficient matrix A of $y(x)$ and the Chebyshev coefficient matrix $A^{(n)}$ of the nth derivative of $y(x)$.

We can also assume that

$$[y(x)]^n = \sum_{r=0}^{nN}{}' a_r^n T_r(x),$$

$$[y'(x)]^m = \sum_{r=0}^{mN}{}' a_r^{(1)m} T_r(x), \text{ and}$$

$$[y''(x)]^s = \sum_{r=0}^{sN}{}' a_r^{(2)s} T_r(x)$$

The recurrence relation between the Chebyshev coefficients $a_r^n, a_r^{(1)m}, a_r^{(2)s}$ and a_r of $[y(x)]^n$, $[y'(x)]^m, [y''(x)]^s$ and $[y(x)]$, is given by

$$a_r^n = \sum_{i=0}^{nN}{}' a_i (a_{i+r}^{n-1} + a_{r-i}^{n-1}), \quad \text{where } i+r \le nN$$

$$a_r^{(1)m} = \sum_{i=0}^{mN}{}' a_i^{(1)} (a_{i+r}^{(1),m-1} + a_{r-i}^{(1),m-1}), \quad \text{where } i+r \le mN$$

$$a_r^{(2)s} = \sum_{i=0}^{sN}{}' a_i^{(2)} (a_{i+r}^{(2),m-2} + a_{r-i}^{(2),m-2}), \quad \text{where } i+r \le sN$$

and $a_{r-i}^{n-1} = 0, a_{r-i}^{(1),m-1} = 0, a_{r-i}^{(2),m-2} = 0$ for $r - i \le 0$, ($a_i = a_i^1, a_i^{(1)} = a_i^{(1),1}, a_i^{(2)} = a_i^{(2),1}$).

Now let us take $r = 0,1,...,N$ and assume $a_r^n = 0,\ a_r^{(1)m} = 0, a_r^{(2)s} = 0$ and $a_r = 0$ for $r > N$. Then the system can be transformed into matrix form

$$A^n = \begin{bmatrix} \frac{1}{2}a_0^n & a_1^n & \dots & a_N^n \end{bmatrix}^T,$$

$$A^{(1)m} = \begin{bmatrix} \frac{1}{2}a_0^{(1)m} & a_1^{(1)m} & \dots & a_N^{(1)m} \end{bmatrix}^T,$$

$$A^{(2)s} = \begin{bmatrix} \frac{1}{2}a_0^{(2)s} & a_1^{(2)s} & \dots & a_N^{(2)s} \end{bmatrix}^T.$$

3. METHOD OF SOLUTION

To obtain the solution of Eq. (1) in the form of expression (3) we first reduce Eq.(1) to a differential equation whose coefficients are polynomials. For this purpose, we assume that the functions $P_k(x), Q_k(x),$ and $R_k(x)$ can be expressed in the forms

$$P_k(x) = \sum_{i=0}^{I} p_i x^i \qquad Q_k(x) = \sum_{i=0}^{I} q_i x^i$$

$$R_k(x) = \sum_{i=0}^{I} r_i x^i \tag{7}$$

which are Taylor polynomials of degree *I.* By using the expressions (7) in Eq. (1), we get

$$\sum_{k=1}^{s}\sum_{i=0}^{I} p_{i,k} x^i [y''(x)]^k + \sum_{k=1}^{m}\sum_{i=0}^{I} q_{i,k} x^k [y'(x)]^k + \sum_{k=1}^{n}\sum_{i=0}^{I} r_{i,k} x^i [y(x)]^k = f(x) \tag{8}$$

The matrix representation of Chebyshev expansions of terms

$$x^p [y''(x)]^s = T(x) M_p A^{(2)s},$$

$$x^p [y'(x)]^m = T(x) M_p A^{(1)m},$$

$$x^p [y(x)]^n = T(x) M_p A^n,$$

$$p = 0,1,...,I,\ s,m,n = 1,2,... \tag{9}$$

where

$$T_x = [T_0(x) \quad T_1(x) \quad T_2(x) \quad \dots \quad T_N(x)]$$

and $M_\varepsilon = [m_{ij}]$, $(i = 0,1,...,N+1$ and $j = 0,1,...N+1)$ is a matrix of size $(N+1)\times(N+1)$. The elements of M_p are given in [3].

Also we assume that the function $f(x)$ can be expanded as

$$f(x) = \sum_{r=0}^{N} f_r T_r(x)$$

or in the matrix form

$$[f(x)] = T_x F \tag{10}$$

where

$$F=\left[\frac{1}{2}f_0 \quad f_1 \quad \ldots \quad f_N\right]^T.$$

Substitutting the expressions (9) and (10) into Eq. (8) and simplifying the result, we have the matrix equation

$$\sum_{k=1}^{s}\sum_{i=0}^{I}p_{i,k}M_iA^{(2)s}+\sum_{k=1}^{m}\sum_{i=0}^{I}q_{i,k}M_iA^{(1)m}+\sum_{k=1}^{n}\sum_{i=0}^{I}r_{i,k}M_iA^{n}=F \tag{11}$$

which corresponds to a system of $(N+1)$ algebraic equations for the unknown Chebyshev coefficients $a_r,\ r=0,1,\ldots,N$.

Briefly, we can write this equation in the form

$$W=F \tag{12}$$

So that

$$W=[w_t]=\sum_{k=1}^{s}\sum_{i=0}^{I}p_{i,k}M_iA^{(2)s}+\sum_{k=1}^{m}\sum_{i=0}^{I}q_{i,k}M_iA^{(1)m}+\sum_{k=1}^{n}\sum_{i=0}^{I}r_{i,k}M_iA^{n},$$

Then the augmented matrix of Eq.(12) becomes

$$[W;F]=\overline{W}=\begin{bmatrix} w_0 & ; & \frac{1}{2}f_0 \\ w_1 & ; & f_1 \\ . & ; & . \\ . & ; & . \\ w_N & ; & f_N \end{bmatrix} \tag{13}$$

Next we can obtain the corresponding matrix forms for the conditions (2) as follows: The expression (3) and its derivative are equivalent to the matrix equations

$$y^{(0)}(x)=[T_0(x) \quad T_1(x) \quad \ldots \quad T_N(x)]A,$$
$$y^{(1)}(x)=2[T_0(x) \quad T_1(x) \quad \ldots \quad T_N(x)]MA$$

and

$$y^{(2)}(x)=4[T_0(x) \quad T_1(x) \quad \ldots \quad T_N(x)]M^2A$$

where

$$A=\left[\frac{1}{2}a_0 \quad a_1 \quad \ldots \quad a_N\right].$$

By using these equations, the quantities $y^{(i)}(a), y^{(i)}(b)$ and $i=1,2$ can be written as

$$y^{(0)}(a)=[T_0(a) \quad T_1(a) \quad \ldots \quad T_N(a)]A$$
$$y^{(0)}(b)=[T_0(b) \quad T_1(b) \quad \ldots \quad T_N(b)]A \tag{14}$$
$$y^{(1)}(a)=2[T_0(a) \quad T_1(a) \quad \ldots \quad T_N(a)]MA$$
$$y^{(1)}(b)=2[T_0(b) \quad T_1(b) \quad \ldots \quad T_N(b)]MA$$
$$y^{(2)}(a)=4[T_0(a) \quad T_1(a) \quad \ldots \quad T_N(a)]M^2A$$
$$y^{(2)}(b)=4[T_0(b) \quad T_1(b) \quad \ldots \quad T_N(b)]M^2A$$

Substituting quantities (14) into Eq. (2) and then simplifying, we obtain the matrix forms of the first and second conditions defined in Eq. (2), respectively, as

$$U=[\lambda] \text{ and } V=[\mu]$$

or the augmented matrices, more clearly,

$$\overline{U}=[u_0 \quad ; \quad \lambda] \text{ and } \overline{V}=[v_0 \quad ; \quad \mu] \tag{15}$$

where u_j and v_j are related to the coefficients $a_i, b_i, \alpha_i, \beta_i$ in Eq. (2). Of course, we should be careful in the choice of coefficients of the conditions given by Eq. (2).

Consequently, by replacing the two rows matrices (10) by the last two rows of the augmented matrix (13), we have the required augmented matrix.

$$\overline{W}^* = \begin{bmatrix} w_0 & ; & \frac{1}{2}f_0 \\ w_1 & ; & f_1 \\ . & . & . \\ . & . & . \\ w_{N-2} & ; & f_{N-2} \\ u_0 & ; & \lambda \\ v_0 & ; & \mu \end{bmatrix} \tag{16}$$

and thus by solving the system of $(N+1)$ algebraic equations, the matrix A (thereby the coefficients a_r) is determined.

4. EXAMPLES

The method is demonstrated by following examples.

Example 4.1. Let us consider the initial value problem

$$y'' + y^2 = x^2 + 2x + 1, \qquad -1 \le x \le 1 \quad \text{and}$$
$$y(0) = 1,\ y'(0) = 1 \tag{17}$$

and approximate the solution $y(x)$ by the Chebyshev polynomial

$$y(x) = \sum_{r=0}^{4}{}' a_r T_r(x) \tag{18}$$

where $N = 2.$.

The matrix equation for (11) becomes

$$\left\{A^{(2)} - A^2\right\} = F\,. \tag{19}$$

$$\begin{bmatrix} 0 & 0 & 4 & 0 & 32 \\ 0 & 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 48 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{a_0}{2} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(\frac{1}{2}a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 \\ a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4 \\ \frac{1}{2}a_1^2 + a_0a_2 + a_1a_3 \\ a_0a_3 + a_1a_2 + a_1a_4 \\ \frac{1}{2}a_2^2 + a_1a_3 + a_0a_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 4a_2+32a_4+\frac{1}{2}(\frac{1}{2}a_0^2+a_1^2+a_2^2+a_3^2+a_4^2) & ;\frac{3}{2} \\ 24a_3+\left(a_0a_1+a_1a_2+a_2a_3+a_3a_4\right) & ;2 \\ 48a_4+\left(\frac{1}{2}a_1^2+a_0a_2+a_1a_3\right) & ;\frac{1}{2} \\ \left(a_0a_3+a_1a_2+a_1a_4\right) & ;0 \\ \left(\frac{1}{2}a_2^2+a_1a_3+a_0a_4\right) & ;0 \end{bmatrix}$$

and the augmented matrices corresponding to the conditions

$y(0)=1$, $y'(0)=1$

are obtain as $\overline{U}=\left[\frac{1}{2}a_0-a_2+a_4 \quad ; \quad 1\right]$ and $\overline{V}=\left[a_1-3a_3 \quad ; \quad 1\right]$.

$$\overline{W}^* = \begin{bmatrix} 4a_2+32a_4+\frac{1}{2}(\frac{1}{2}a_0^2+a_1^2+a_2^2+a_3^2+a_4^2) & ;\frac{3}{2} \\ 24a_3+\left(a_0a_1+a_1a_2+a_2a_3+a_3a_4\right) & ;2 \\ 48a_4+\left(\frac{1}{2}a_1^2+a_0a_2+a_1a_3\right) & ;\frac{1}{2} \\ \frac{1}{2}a_0-a_2+a_4 & ;1 \\ a_1-3a_3 & ;1 \end{bmatrix}$$

and has the solution

$a_0=2$, $a_1=1$, $a_2=0$, $a_3=0$.

and $a_4=0$ (20)

Substituting these, we have

$y(x)=T_0(x)+T_1(x)$.

Equation 17 has the same solution by the Taylor Method for series solutions to second order.

Example 4.2. Let us consider the boundary value problem

$4y''-2(y')^2+y=0$; $y(0)=-1, y'(0)=0$ (21)

It's exact solution is expressed in the same book as

$$y(x)=\frac{x^2}{8}-1.$$

Now, we shall assume the approximate solution $y(x)$ by the Chebyshev polynomial form

$$y(x)=\sum_{r=0}^{4}{}' a_r T_r(x) \quad (22)$$

where $N=2$.

The matrix equation for (11) becomes

$$\left\{4A^{(2)}-2A^{(1)2}+A\right\}=F. \quad (23)$$

$$4\left(\begin{bmatrix}0&0&4&0&32\\0&0&0&24&0\\0&0&0&0&48\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}\frac{a_0}{2}\\a_1\\a_2\\a_3\\a_4\end{bmatrix}\right)-2\begin{bmatrix}a_1^2+8a_2^2\\8a_1a_2\\8a_2^2\\0\\0\end{bmatrix}+\begin{bmatrix}\frac{a_0}{2}\\a_1\\a_2\\a_3\\a_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}.$$

$$\begin{bmatrix}\frac{1}{2}a_0-2(a_1^2+8(a_2^2-a_2)) & ; & 0\\ a_1(1-16a_2) & ; & 0\\ -16a_2^2+a_2 & ; & 0\\ a_3 & ; & 0\\ a_4 & ; & 0\end{bmatrix}$$

and the augmented matrices corresponding to the conditions

$$y(0)=-1, y'(0)=0,$$

are obtain as $\overline{U}=\begin{bmatrix}\frac{1}{2}a_0-a_2 & ; & -1\end{bmatrix}$ and $\overline{V}=\begin{bmatrix}a_1 & ; & 0\end{bmatrix}$.

$$\overline{W}^*=\begin{bmatrix}\frac{1}{2}a_0-2(a_1^2+8(a_2^2-a_2)) & ; & 0\\ a_1(1-16a_2) & ; & 0\\ -16a_2^2+a_2 & ; & 0\\ \frac{1}{2}a_0-a_2 & ; & -1\\ a_1=0 & ; & 0\end{bmatrix}$$

and has the solution

$$a_0=-\frac{15}{16},\ a_1=a_3=a_4=0 \text{ and } a_2=\frac{1}{16}.$$

Substituting these, we have

$$y(x)=-\frac{15}{16}T_0(x)+\frac{1}{16}T_2(x), \qquad (24)$$

where $T_0(x)=1$ and $T_2(x)=2x^2-1$.

It is safe to report that Chebyshev polynomial solution is congruent to exact solution of example 4.2.

5. CONCLUSIONS

In this paper, the usefulness of Chebyshev-matrix method for the polynomial solutions of second order nonlinear differential equations is discussed. These equations are usually difficult to solve analytically. In many cases, it is required to approximate solutions. A considerable advantage of the method is that Chebyshev coefficients of the solution are found very easily by using the computer programs. Following similar way, we can find the relations between Chebyshev coefficients, for the functions f(x) defined in $0 \le x \le 1$ and $0 \le x, y \le 1$, respectively. The method can be also extended to the polynomial solutions of second order nonlinear differential equations in general form.

REFERENCES

[1] Günhan, B.C., “Approximate solutions of non-linear differential and Integral equations by Chebyshev method”, Dissertation, Dokuz Eylül University, (2001).

[2] Keşan, C., “Taylor polynomial solutions of linear differential equations”, ***Appl. Math. Comput.***, 142: 155-165(2003).

[3] Köroğlu, H., “Chebyshev series solution of linear Fredholm integrodifferential equations”, ***Int. J. Math. Educ. Sci. Technol.***, 29 (4): 489-500(1998).