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Convergence analysis of variational inequality and fixed point problems for pseudo-contractive mapping with Lipschitz assumption

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Abstract

In this paper, we consider and study variational inequality and fixed point problems for pseudo-contractive mapping. It is proven that the sequences generated by the proposed iterative algorithm converges strongly to the common solution of the variational inequality and fixed point problems. Numerical example illustrates the theoretical result.

Keywords: variational inequality problem, fixed point problem, extra-gradient method, pseudo-contractive mapping, strong convergence.
2010 MSC: 47J25, 47H09, 47H10, 65J15.

1. Introduction

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, let C be a nonempty closed convex subset of H. For a given nonlinear operator $A : C \to H$, the following classical variational inequality problem $VI(C, A)$ is formulated as finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

which was introduced and studied by Stampacchia [11] in 1964. It is well known that variational inequality theory has been researched extensively due to its applications in pure and applied science, such as in industry, economics, optimization, and partial differential equations; see, [10, 8, 16, 2, 3, 12, 9, 4, 13, 6] and the

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references therein. It is known that variational inequality (1.1) is equivalent to the fixed point problem, that is, in a Hilbert space, $x^* \in C$ is a solution of variational inequality if and only if $x^* \in C$ satisfies the relation

$$x^* = P_C\left(x^* - \lambda A x^*\right), \quad \forall x \in C, \tag{1.2}$$

where P_C is the projection of H onto a closed convex subset C and $\lambda > 0$ is a constant. This alternative equivalent formulation of the variational inequality problem has played an important and fundamental part in the existence of a solution and in developing several numerical iterative methods for solving the variational inequalities and related optimization problems. In 1976, Korpelevich [7] suggested an iterative algorithm for solving the variational inequality (1.1) in Euclidean space:

$$\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right), \\ x_{n+1} = P_C\left(x_n - \lambda A y_n\right), \quad n \geq 1, \end{cases}$$

with $\lambda > 0$ a given number, which is known as the extra-gradient method. C is a nonempty closed convex subset of Euclidean space $\mathbb{R}^n$ and A is a monotone and Lipschitz continuous mapping of C into $\mathbb{R}^n$. She showed that if $VI(C, A)$ is nonempty, then the generated $\{x_n\}$ converges to the solution of the problem (1.1). The extra-gradient iterative process was successfully generalized and extended not only to Euclidean but also to Hilbert and Banach spaces; see, *e.g.*, the recent references of [4, 6].

Nadezhkina and Takahashi [8] introduced the following iterative scheme in Hilbert space H, the iterative process is based on the so-called extra-gradient method:

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C\left(x_n - \lambda_n A y_n\right), \quad n \geq 1. \end{cases} \tag{1.3}$$

They proved that sequence $\{x_n\}$ generated by (1.3) converges weakly to the common element of the set of fixed points of a nonexpansive mapping $S : C \to C$ and the set of solutions to the variational inequality problem for a monotone, Lipschitz-continuous mapping $A : C \to H$. Subsequently, Yao [13] suggested and analyzed a new method which coupled extra-gradient method with Mann iteration for solving some variational inequality problem and fixed point problem in Hilbert space. It is shown that the proposed method below has strong convergence in a general Hilbert space.

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right), \\ z_n = P_C\left((1-\alpha_n)(y_n - \lambda_n A y_n)\right), \\ x_{n+1} = \beta x_n + (1-\beta) S z_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where A is an α-inverse strongly monotone mapping, S is a nonexpansive mapping.

In this paper, inspired and motivated by [8, 13], we will extend their results from nonexpansive mapping to pseudo-contractive mapping and suggest an iterative method for solving variational inequality and fixed point problems. We prove that the presented sequence converges strongly to a common solution of the variational inequality and fixed point problems.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The notation $Fix(T)$ denotes the set of fixed points of the mapping T, that is, $Fix(T) = \{x \in H : Tx = x\}$. Projections are an important tool for our work in this paper. Recall that the (nearest point or metric) projection from H onto C, denote by P_C, is defined in such a way that, for each $x \in H$, P_Cx is the unique point in C with the property

$$\|x - P_Cx\| = \min\{\|x - y\| : y \in C\}.$$

Some properties of projections are gathered in the following proposition.

Proposition 1. *Given $x \in H$ and $z \in C$.*
(1) $z = P_Cx \Leftrightarrow \langle x - z, y - z\rangle \leq 0$ for all $y \in C$.
(2) $z = P_Cx \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ for all $y \in C$.
(3) $\langle x - y, P_Cx - P_Cy\rangle \geq \|P_Cx - P_Cy\|^2$ for all $y \in H$, which hence implies that P_C is nonexpansive.

We also need other sorts of nonlinear operators which are introduced blow.

Definition 1. *A nonlinear operator $T : H \to H$ is said to be*
(1) L-Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H;$$

If L=1, we call T nonexpansive.
(2) monotone if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in H;$$

(3) β-strongly monotone, with $\beta > 0$, if

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in H;$$

(4) ν-inverse strongly monotone (ν-ism), with $\nu > 0$, if

$$\langle x - y, Tx - Ty\rangle \geq \nu \|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Inverse strongly monotone (also referred to as co-coercive) operators have been widely applied in solving practical problems in various fields, for instance, in traffic assignment problems; see, for example, [1, 5].

On the other hand, in a real Hilbert space H, a mapping $T : C \to C$ is called *pseudo-contractive* if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

The interest of pseudo-contractive mappings lies in their connection with monotone operators; namely, T is pseudo-contractive mapping if and only if the complement $I - T$ is a monotone operator. It is well known that T is a pseudo-contractive mapping if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{2.1}$$

For all x, $y \in H$, the following conclusions hold

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \quad t \in [0, 1], \tag{2.2}$$

Lemma 1. *[17] Let $\mathcal{H}$ be a real Hilbert space, C a closed convex subset of $\mathcal{H}$. Let $T : C \to C$ be a continuous pseudo-contractive mapping. Then*

(1) $Fix(T)$ is a closed convex subset of C,

(2) $(I - T)$ is demiclosed at zero.

Lemma 2. *[14] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1-\gamma_n)a_n + \sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\sigma_n\}$ are such that
(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(2) either $\limsup_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ or $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.
Then $\{a_n\}$ converges to zero.

Lemma 3. *[15] Let $\{u_n\}$ be a sequence of real numbers. Assume $\{u_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \leq u_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leq n : u_{n_i} < u_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq N_0$,

$$\max\{u_{\tau(n)}, u_n\} \leq u_{\tau(n)+1}.$$

3. Convergence Analysis

In this section, we will introduce our algorithm for solving variational inequality and fixed point problems and analyze the convergence. Next, we give a numerical example to illustrate the theoretical result. The set of solutions of variational inequality and fixed point problems are denoted by $F = Fix(T) \cap VI(C,A)$.

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a σ-inverse strongly monotone mapping and T be a L-Lipschitz-pseudo-contractive mapping from C to itself. For given $x_1 \in C$ arbitrarily, define a sequence $\{x_n\}$, $n \geq 1$, iteratively by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right), \\ z_n = P_C\left((1-\alpha_n)(y_n - \lambda_n A y_n)\right), \\ x_{n+1} = (1-\beta_n)z_n + \beta_n T((1-\gamma_n)z_n + \gamma_n T z_n), \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$, $\{\lambda_n\}$ is a sequence in $(0,2\sigma)$. Assume that the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < c < \beta_n < e < \gamma_n < d < \frac{1}{\sqrt{1+L^2}+1}$.

Suppose in addition that $F = Fix(T) \cap VI(C,A) \neq \emptyset$, then $\{x_n\}$ defined by (3.1) converges strongly to $P_F(0)$.

Proof. Taking $x^* \in F$, we have $x^* \in VI(C,A)$, then we get $x^* = P_C(x^* - \lambda A x^*)$ for all $\lambda > 0$. From (3.1), the definition of $\{\lambda_n\}$ and the nonexpansivity of P_C, we find

$$\begin{aligned} \|y_n - x^*\|^2 =& \|P_C\left(x_n - \lambda_n A x_n\right) - P_C\left(x^* - \lambda_n A x^*\right)\|^2 \\ \leq& \|(I-\lambda_n A)x_n - (I-\lambda_n A)x^*\|^2 \\ \leq& \|x_n - x^*\|^2 - 2\lambda_n\langle x_n - x^*, Ax_n - Ax^*\rangle + \lambda_n^2\|Ax_n - Ax^*\|^2 \\ \leq& \|x_n - x^*\|^2 - 2\lambda_n\sigma\|Ax_n - Ax^*\|^2 + \lambda_n^2\|Ax_n - Ax^*\|^2 \\ =& \|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\sigma)\|Ax_n - Ax^*\|^2 \\ \leq& \|x_n - x^*\|^2, \end{aligned} \tag{3.2}$$

thus we can see the nonexpansivity of $I-\lambda_n A$ from (3.2). In particular, choosing $\lambda=\lambda_n(1-\alpha_n)$, we gain

$$x^*=P_C(x^*-\lambda_n(1-\alpha_n)Ax^*)=P_C(\alpha_n x^*+(1-\alpha_n)(x^*-\lambda_n Ax^*)),\quad n\geq 1,$$

then,

$$\begin{aligned}\|z_n-x^*\|^2=&\|P_C((1-\alpha_n)(y_n-\lambda_n Ay_n))-P_C(\alpha_n x^*+(1-\alpha_n)(x^*-\lambda_n Ax^*))\|^2\\ \leq&\|((1-\alpha_n)(y_n-\lambda_n Ay_n))-(\alpha_n x^*+(1-\alpha_n)(x^*-\lambda_n Ax^*))\|^2\\ \leq&\alpha_n\|x^*\|^2+(1-\alpha_n)\|(I-\lambda_n A)y_n-(I-\lambda_n A)x^*\|^2\\ \leq&\alpha_n\|x^*\|^2+(1-\alpha_n)\|y_n-x^*\|^2\\ \leq&\alpha_n\|x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2.\end{aligned}\tag{3.3}$$

From (2.1), we have

$$\|Tz_n-x^*\|^2\leq\|z_n-x^*\|^2+\|z_n-Tz_n\|^2,\tag{3.4}$$

and

$$\begin{aligned}&\|T((1-\gamma_n)z_n+\gamma_n Tz_n)-x^*\|^2\\ \leq&\|(1-\gamma_n)(z_n-x^*)+\gamma_n(Tz_n-x^*)\|^2+\|(1-\gamma_n)z_n+\gamma_n Tz_n-T\left((1-\gamma_n)z_n+\gamma_n Tz_n\right)\|^2.\end{aligned}\tag{3.5}$$

By (2.2) and (3.4), we receive

$$\begin{aligned}\|(1-\gamma_n)(z_n-x^*)+\gamma_n(Tz_n-x^*)\|^2=&(1-\gamma_n)\|z_n-x^*\|^2+\gamma_n\|Tz_n-x^*\|^2-\gamma_n(1-\gamma_n)\|z_n-Tz_n\|^2\\ \leq&\|z_n-x^*\|^2+\gamma_n^2\|z_n-Tz_n\|^2.\end{aligned}\tag{3.6}$$

Also from (2.2), we deduce

$$\begin{aligned}&\|(1-\gamma_n)z_n+\gamma_n Tz_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2\\ =&(1-\gamma_n)\|z_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2+\gamma_n\|Tz_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2-\gamma_n(1-\gamma_n)\|z_n-Tz_n\|^2\\ \leq&(1-\gamma_n)\|z_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2+\gamma_n^3L^2\|z_n-Tz_n\|^2-\gamma_n(1-\gamma_n)\|z_n-Tz_n\|^2\\ =&(1-\gamma_n)\|z_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2+(\gamma_n^3L^2+\gamma_n^2-\gamma_n)\|z_n-Tz_n\|^2.\end{aligned}\tag{3.7}$$

Utilizing above two inequalities (3.6) and (3.7), we derive that

$$\begin{aligned}\|T((1-\gamma_n)z_n+\gamma_n Tz_n)-x^*\|^2\leq&\|z_n-x^*\|^2+(1-\gamma_n)\|z_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2\\ &-\gamma_n(1-2\gamma_n-\gamma_n^2L^2)\|z_n-Tz_n\|^2.\end{aligned}\tag{3.8}$$

From condition (ii), we can see

$$\|T((1-\gamma_n)z_n+\gamma_n Tz_n)-x^*\|^2\leq\|z_n-x^*\|^2+(1-\gamma_n)\|z_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2.\tag{3.9}$$

By (2.2), (3.1), (3.9) and condition (ii), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2&=\|(1-\beta_n)z_n+\beta_n T((1-\gamma_n)z_n+\gamma_n Tz_n)-x^*\|^2\\ &=(1-\beta_n)\|z_n-x^*\|^2+\beta_n\|T((1-\gamma_n)z_n+\gamma_n Tz_n)-x^*\|^2\\ &\quad-\beta_n(1-\beta_n)\|z_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2\\ &\leq\|z_n-x^*\|^2-\beta_n(\gamma_n-\beta_n)\|z_n-T((1-\gamma_n)z_n+\gamma_n Tz_n)\|^2\\ &\leq\|z_n-x^*\|^2.\end{aligned}\tag{3.10}$$

This together with (3.3) implies that

$$\|x_{n+1}-x^*\|^2\leq\|z_n-x^*\|^2\leq\alpha_n\|x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2\leq\max\{\|x^*\|^2,\|x_1-x^*\|^2\}.$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{z_n\}$.

Returning to (3.10), we deduce

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 - \beta_n(\gamma_n - \beta_n)\|z_n - T((1-\gamma_n)z_n + \gamma_n T z_n)\|^2 \\ &\leq \alpha_n\|x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \beta_n(\gamma_n - \beta_n)\|z_n - T((1-\gamma_n)z_n + \gamma_n T z_n)\|^2.\end{aligned}$$

It follows that

$$\beta_n(\gamma_n - \beta_n)\|z_n - T((1-\alpha_n)z_n + \alpha_n T z_n)\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|x^*\|^2.$$

Next, we consider two possible cases.

Case 1. Assume there exists some integer $m > 0$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \geq m$. In this case, we know that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. This together with condition (ii) that

$$\lim_{n\to\infty} \|z_n - T((1-\alpha_n)z_n + \alpha_n T z_n)\| = 0. \tag{3.11}$$

Observe that

$$\begin{aligned}\|z_n - Tz_n\| &\leq \|z_n - T((1-\gamma_n)z_n + \gamma_n T z_n)\| + \|T((1-\gamma_n)z_n + \gamma_n T z_n) - Tz_n\| \\ &\leq \|z_n - T((1-\gamma_n)z_n + \gamma_n T z_n)\| + \gamma_n L\|z_n - Tz_n\|.\end{aligned}$$

Thus,

$$\|z_n - Tz_n\| \leq \frac{1}{1-\gamma_n L} \|z_n - T((1-\alpha_n)z_n + \alpha_n T z_n)\|.$$

This together with (3.11) implies that

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.12}$$

From (3.2) and (3.3), we achieve

$$\begin{aligned}\|y_n - x^*\|^2 =&\|P_C(x_n - \lambda_n A x_n) - P_C(x^* - \lambda_n A x^*)\|^2 \\ \leq&\|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\|^2 \\ =&\|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\sigma)\|Ax_n - Ax^*\|^2\end{aligned}$$

and

$$\begin{aligned}\|z_n - x^*\|^2 =&\|P_C((1-\alpha_n)(y_n - \lambda_n A y_n)) - P_C(\alpha_n x^* + (1-\alpha_n)(x^* - \lambda_n A x^*))\|^2 \\ \leq&\alpha_n\|x^*\|^2 + (1-\alpha_n)\|(I - \lambda_n A)y_n - (I - \lambda_n A)x^*\|^2 \\ \leq&\alpha_n\|x^*\|^2 + (1-\alpha_n)\left(\|y_n - x^*\|^2 + \lambda_n(\lambda_n - 2\sigma)\|Ay_n - Ax^*\|^2\right) \\ \leq&\alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\sigma)\left(\|Ax_n - Ax^*\|^2 + \|Ay_n - Ax^*\|^2\right).\end{aligned} \tag{3.13}$$

Substitute (3.13) into (3.10) to obtain

$$(1-\alpha_n)2\lambda_n(2\sigma - \lambda_n)\left(\|Ax_n - Ax^*\|^2 + \|Ay_n - Ax^*\|^2\right) \leq \alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

It follows that

$$\lim_{n\to\infty} \|Ax_n - Ax^*\| = \lim_{n\to\infty} \|Ay_n - Ax^*\| = 0.$$

By the firmly nonexpansivity of the metric project P_C and the nonexpansivity of $I-\lambda_n A$, we conclude that

$$\begin{aligned}\|y_n - x^*\|^2 =& \|P_C(x_n-\lambda_n Ax_n) - P_C(x^*-\lambda_n Ax^*)\|^2\\ \le& \langle (x_n-\lambda_n Ax_n) - (x^*-\lambda_n Ax^*), y_n - x^*\rangle\\ =& \frac{1}{2}\Big(\|(x_n-\lambda_n Ax_n) - (x^*-\lambda_n Ax^*)\|^2 + \|y_n-x^*\|^2\\ &- \|(x_n-\lambda_n Ax_n) - (x^*-\lambda_n Ax^*) - (y_n - x^*)\|^2\Big)\\ \le& \frac{1}{2}\Big(\|x_n-x^*\|^2 + \|y_n-x^*\|^2 - \|(x_n-y_n) - \lambda_n(Ax_n - Ax^*)\|^2\Big)\\ \le& \frac{1}{2}\left(\|x_n-x^*\|^2 + \|y_n-x^*\|^2 - \|x_n-y_n\|^2 + 2\lambda_n\|x_n-y_n\|\|Ax_n-Ax^*\|\right),\end{aligned}$$

and

$$\begin{aligned}\|z_n - x^*\|^2 =& \|P_C((1-\alpha_n)(y_n-\lambda_n Ay_n)) - P_C(x^*-\lambda_n Ax^*)\|^2\\ \le& \langle (1-\alpha_n)(y_n-\lambda_n Ay_n) - (x^*-\lambda_n Ax^*), z_n - x^*\rangle\\ =& \frac{1}{2}\Big(\|(y_n-\lambda_n Ay_n) - (x^*-\lambda_n Ax^*) - \alpha_n(I-\lambda_n A)y_n\|^2 + \|z_n-x^*\|^2\\ &- \|(y_n-\lambda_n Ay_n) - (x^*-\lambda_n Ax^*) - (z_n - x^*) - \alpha_n(I-\lambda_n A)y_n\|^2\Big)\\ \le& \frac{1}{2}\Big(\|(y_n-\lambda_n Ay_n) - (x^*-\lambda_n Ax^*)\|^2 + \alpha_n M + \|z_n-x^*\|^2)\\ &- (\|(y_n-z_n) - (\lambda_n Ay_n - \lambda_n Ax^*) - \alpha_n(I-\lambda_n A)y_n\|^2\Big)\\ \le& \frac{1}{2}\left(\|y_n-x^*\|^2 + \alpha_n M + \|z_n-x^*\|^2 - \|y_n-z_n\|^2\right.\\ &\left. + 2\lambda_n\|y_n-z_n\|\|Ay_n-Ax^*\| + 2\alpha_n\|y_n-z_n\|\|(I-\lambda_n A)y_n\|\right),\end{aligned}$$

where $M>0$ is some constant.

It follows that

$$\|y_n-x^*\|^2 \le \|x_n-x^*\|^2 - \|x_n-y_n\|^2 + 2\lambda_n\|x_n-y_n\|\|Ax_n-Ax^*\|,$$

and

$$\begin{aligned}\|z_n-x^*\|^2 \le& \|y_n-x^*\|^2 + \alpha_n M - \|y_n-z_n\|^2\\ &+ 2\lambda_n\|y_n-z_n\|\|Ay_n-Ax^*\| + 2\alpha_n\|y_n-z_n\|\|(I-\lambda_n A)y_n\|\\ \le& \|x_n-x^*\|^2 - \|x_n-y_n\|^2 - \|y_n-z_n\|^2 + 2\lambda_n\|x_n-y_n\|\|Ax_n-Ax^*\|\\ &+ \alpha_n M + 2\lambda_n\|y_n-z_n\|\|Ay_n-Ax^*\| + 2\alpha_n\|y_n-z_n\|\|(I-\lambda_n A)y_n\|.\end{aligned}$$

Returning to (3.10), we deduce

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le& \|z_n-x^*\|^2\\ \le& \|x_n-x^*\|^2 - \|x_n-y_n\|^2 - \|y_n-z_n\|^2 + 2\lambda_n\|x_n-y_n\|\|Ax_n-Ax^*\|\\ &+ \alpha_n M + 2\lambda_n\|y_n-z_n\|\|Ay_n-Ax^*\| + 2\alpha_n\|y_n-z_n\|\|(I-\lambda_n A)y_n\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_n-y_n\|^2 + \|y_n-z_n\|^2 \le& \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + 2\lambda_n\|x_n-y_n\|\|Ax_n-Ax^*\|\\ &+ \alpha_n M + 2\lambda_n\|y_n-z_n\|\|Ay_n-Ax^*\| + 2\alpha_n\|y_n-z_n\|\|(I-\lambda_n A)y_n\|.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|y_n-z_n\|=0. \tag{3.14}$$

Next, we prove $\limsup_{n\to\infty}\langle P_F(0),P_F(0)-z_n\rangle\le 0$.

We choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle P_F(0),P_F(0)-z_n\rangle=\limsup_{i\to\infty}\langle P_F(0),P_F(0)-z_{n_i}\rangle.$$

As $\{z_{n_i}\}$ is bounded, we have that a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ converges weakly to z. We may assume without loss of generality that $z_{n_i}\rightharpoonup z$. By (3.12) and Lemma 1, we estimate that $z\in Fix(T)$.

Define $Tu=\begin{cases}Au+N_Cu, & u\in C;\\ \emptyset, & u\notin C.\end{cases}$ Then T is maximal monotone. Let $(u,v)\in G(T)$. Since $v-Au\in N_Cu$ and $z_n\in C$, we get $\langle u-z_n,v-Au\rangle\ge 0$. On the other hand, we have

$$\langle u-z_n,z_n-(1-\alpha_n)(y_n-\lambda_nAy_n)\rangle\ge 0,$$

that is,

$$\langle u-z_n,\frac{z_n-y_n}{\lambda_n}+Ay_n+\frac{\alpha_n}{\lambda_n}(y_n-\lambda_nAy_n)\rangle\ge 0.$$

Therefore,

$$\begin{aligned}\langle u-z_{n_i},v\rangle\ge&\langle u-z_{n_i},Au\rangle\\ \ge&\langle u-z_{n_i},Au\rangle-\langle u-z_{n_i},\frac{z_{n_i}-y_{n_i}}{\lambda_{n_i}}+Ay_{n_i}+\frac{\alpha_n}{\lambda_{n_i}}(y_{n_i}-\lambda_{n_i}Ay_{n_i})\rangle\\ =&\langle u-z_{n_i},Au-Ay_{n_i}-\frac{z_{n_i}-y_{n_i}}{\lambda_{n_i}}-\frac{\alpha_n}{\lambda_{n_i}}(y_{n_i}-\lambda_{n_i}Ay_{n_i})\rangle\\ =&\langle u-z_{n_i},Au-Az_{n_i}\rangle+\langle u-z_{n_i},Az_{n_i}-Ay_{n_i}\rangle-\langle u-z_{n_i},\frac{z_{n_i}-y_{n_i}}{\lambda_{n_i}}+\frac{\alpha_n}{\lambda_{n_i}}(y_{n_i}-\lambda_{n_i}Ay_{n_i})\rangle\\ \ge&\langle u-z_{n_i},Az_{n_i}-Ay_{n_i}\rangle-\langle u-z_{n_i},\frac{z_{n_i}-y_{n_i}}{\lambda_{n_i}}+\frac{\alpha_n}{\lambda_{n_i}}(y_{n_i}-\lambda_{n_i}Ay_{n_i})\rangle.\end{aligned}$$

From (3.14), we attain $\langle u-z,v\rangle\ge 0$. Since T is maximal monotone, we achieve $z\in T^{-1}(0)$ and hence $z\in VI(C,A)$. Thus, we get $z\in F$. Therefore,

$$\limsup_{n\to\infty}\langle P_F(0),P_F(0)-z_n\rangle=\limsup_{i\to\infty}\langle P_F(0),P_F(0)-z_{n_i}\rangle=\langle P_F(0),P_F(0)-z\rangle\le 0.$$

Finally,

$$\begin{aligned}\|z_n-P_F(0)\|^2=&\|P_C((1-\alpha_n)(y_n-\lambda_nAy_n))-P_C(\alpha_nP_F(0)+(1-\alpha_n)(P_F(0)-\lambda_nAP_F(0)))\|^2\\ \le&\langle\alpha_n(-P_F(0))+(1-\alpha_n)\left((y_n-\lambda_nAy_n)-(P_F(0)-\lambda_nAP_F(0))\right),z_n-P_F(0)\rangle\\ \le&\alpha_n\langle P_F(0),P_F(0)-z_n\rangle+(1-\alpha_n)\|(y_n-\lambda_nAy_n)-(P_F(0)-\lambda_nAP_F(0))\|\|z_n-P_F(0)\|\\ \le&\alpha_n\langle P_F(0),P_F(0)-z_n\rangle+(1-\alpha_n)\|y_n-P_F(0)\|\|z_n-P_F(0)\|\\ \le&\alpha_n\langle P_F(0),P_F(0)-z_n\rangle+\frac{1-\alpha_n}{2}\left(\|y_n-P_F(0)\|^2+\|z_n-P_F(0)\|^2\right),\end{aligned}$$

that is,

$$\begin{aligned}\|z_n-P_F(0)\|^2\le&(1-\alpha_n)\|y_n-P_F(0)\|^2+2\alpha_n\langle P_F(0),P_F(0)-z_n\rangle\\ \le&(1-\alpha_n)\|x_n-P_F(0)\|^2+2\alpha_n\langle P_F(0),P_F(0)-z_n\rangle.\end{aligned}$$

From (3.10), we obtain

$$\|x_{n+1} - P_F(0)\|^2 \leq (1-\alpha_n)\|x_n - P_F(0)\|^2 + 2\alpha_n\langle P_F(0), P_F(0) - z_n\rangle. \tag{3.15}$$

We apply Lemma 2 to the last inequality to deduce that $x_n \to P_F(0)$.

Case 2. Assume there exists an integers n_0, such that

$$\|x_{n_0} - P_F(0)\| \leq \|x_{n_0+1} - P_F(0)\|.$$

Set $u_n = \{\|x_n - P_F(0)\|\}$, then we have

$$u_{n_0} \leq u_{n_0+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \geq n_0$ as follows:

$$\tau(n) = \max\{l \geq 1 : n_0 \leq l \leq n, u_l \leq u_{l+1}\}.$$

It is clear that $\tau(n)$ is non-decresing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty$$

and

$$u_{\tau(n)} \leq u_{\tau(n)+1}$$

for all $n \geq n_0$.

By a similar argument to that of Case 1, we can obtain that

$$\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = \lim_{n\to\infty} \|y_{\tau(n)} - z_{\tau(n)}\| = \lim_{n\to\infty} \|z_{\tau(n)} - Tz_{\tau(n)}\| = 0.$$

This implies that

$$\omega_w(z_{\tau(n)}) \subset F.$$

Thus, we obtain

$$\limsup_{n\to\infty} \left\langle P_F(0), P_F(0) - z_{\tau(n)}\right\rangle \leq 0. \tag{3.16}$$

Since $u_{\tau(n)} \leq u_{\tau(n)+1}$, we have from (3.15) that

$$u^2_{\tau(n)} \leq u^2_{\tau(n)+1} \leq (1-\alpha_{\tau(n)})u^2_{\tau(n)} + 2\alpha_n\langle P_F(0), P_F(0) - z_{\tau(n)}\rangle. \tag{3.17}$$

It follows that

$$u^2_{\tau(n)} \leq 2\langle P_F(0), P_F(0) - z_{\tau(n)}\rangle. \tag{3.18}$$

Combining (3.16) and (3.18), we have

$$\limsup_{n\to\infty} u_{\tau(n)} \leq 0,$$

and hence

$$\lim_{n\to\infty} u_{\tau(n)} = 0. \tag{3.19}$$

By (3.17), we obtain

$$\limsup_{n\to\infty} u^2_{\tau(n)+1} \leq \limsup_{n\to\infty} u^2_{\tau(n)}.$$

This together with (3.19) implies that

$$\lim_{n\to\infty} u_{\tau(n)+1} = 0.$$

Applying Lemma 3 to get

$$0 \leq u_n \leq \max\{u_{\tau(n)}, u_{\tau(n)+1}\}.$$

Therefore, $u_n \to 0$. That is, $x_n \to P_F(0)$. This completes the proof. □

Next, we consider a numerical example below to illustrate the theoretical result.

4. Numerical example

Let $H = \mathbf{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbf{R}$ and the standard norm $|\cdot|$. Let $C = [0, +\infty)$ and $Tx = x - 1 + \frac{1}{x+1}$ for all $x \in C$. It is easy to see that $Fix(T) = \{0\}$ and

$$\begin{aligned}\langle Tx - Ty, x - y\rangle &= \langle x - 1 + \frac{1}{x+1} - y + 1 - \frac{1}{y+1}, x - y\rangle \\ &\le \left(1 - \frac{1}{(x+1)(y+1)}\right)|x-y|^2 \\ &\le |x-y|^2,\end{aligned}$$

hence, T is a Lipschitzian pseudo-contractive mapping.

Let $Ax = \frac{1}{3}x$ for all $x \in \mathbf{R}$. Let $\lambda_n = 1$, $\alpha_n = \frac{1}{7}$, $\beta_n = \frac{1}{8}$, $\gamma_n = \frac{1}{9}$. Let the sequence $\{x_n\}$ be generated iteratively by (3.1), then the sequence $\{x_n\}$ converges to $0 = P_F(0)$. We now rewrite (3.1) as follows:

$$\begin{cases} y_n = P_C\left(\frac{2}{3}x_n\right), \\ z_n = P_C\left(\frac{4}{7}y_n\right), \\ x_{n+1} = z_n + \frac{1}{72z_n+72} + \frac{1}{8z_n+\frac{64}{9}+\frac{8}{9z_n+9}} - \frac{5}{36}, \quad n \ge 1. \end{cases} \tag{4.1}$$

For every $n \ge 1$ and A, T, α_n, β_n, γ_n and λ_n satisfy all conditions in Theorem 1. Since the sequence $\{x_n\}$ is generated by (4.1), from Theorem 1, we find that the sequence $\{x_n\}$ converges strongly to 0. Choosing initial values $x_1 = 8$ and $x_1 = 3$, respectively, we see that Figure 1 demonstrate Theorem 1.

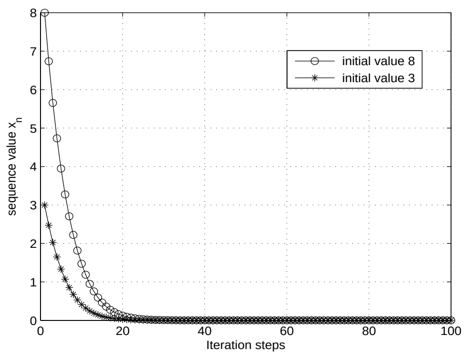

Figure 1: The convergence of x_n with initial values 8 and 3, respectively.

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