Gazi University Journal of Science
GU J Sci
25(3):655-659 (2012)

ORIGINAL ARTICLE

Coincidence Points of Hybrid Functions on Cone Metric Spaces

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Received: 26/03/2011 Accepted:04/07/2011

ABSTRACT

In this paper, we obtain a common coincidence point theorem for two pairs of hybrid functions on cone metric spaces.

Key words: Cone metric spaces,coincidence points, multi functions.
2000 Mathematics Subject Classification : 47H10,54H25.

1. INTRODUCTION

In 2007,Huang and Zhang defined cone metric spaces by substituting an ordered normed space for the real numbers([9]). In 2008,Rezapour and Hamlbarani characterized types of cones ([17]). Some interesting works about fixed point and common fixed point results on cone metric spaces are [1-8,10,11,13-24] etc.

In this paper, we prove a common coincidence point theorem for two pairs of hybrid functions on cone metric spaces. our result generalizes and improves the theorems of [18,19]. First we give some known definitions and lemmas.

Let E be a real Banach space and P a subset of E. P is called a cone whenever

(i) P is closed, non empty and $P \neq \{0\}$

(ii) $ax + by$ for all $x, y \in P$ and non negative real numbers a and b

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in \text{int } P$, where int P denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$\|x\| \leq M \|y\|$. The least positive number satisfying the above inequality is called the normal constant of P. Rezapour and Hamlbarani [17] observed that there are no normal cones with $M < 1$.Hene $M \geq 1$.

Definition 1.1. [9]: Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.2. [9]: Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X.Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 << c$, there is a natural

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number N such that $d(x_n, x) \ll c$ for all $n \geq N$. we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$,

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such

that $d(x_n, x_m) \ll c$ for all $n, m \geq N$,

(iii)(X, d) is a complete metric space if every Cauchy sequence in X is convergent in X.

Definition 1.3. [18]: Let (X, d) be a cone metric space and $B \subseteq X$.

(i) A point $b \in B$ is called an interior point of B whenever there is a $0 \ll p$ such

that $N(b, p) \subseteq B$, where $N(b, p) = \{y \in X / d(y, b) \ll p\}$.

(ii) A subset $A \subseteq X$ is called open if each element of A is an interior point of A.

The family $\beta = \{N(x, p) : x \in X, 0 \ll p\}$ is a sub basis for a topology on X.

we denote this cone topology by τ_c. Then τ_c is Hausdorff and first countable.

Recently Rezapour and Haghi [18] proved the following

Lemma 1.4. (Lemma 2.1, [18]) : Let (X, d) be a cone metric space, P a normal cone with normal constant M = 1 and A a compact set in (X, τ_c). Then for every $x \in X$, there exists $a_0 \in A$ such that

$$\|d(x, a_0)\| = \inf_{a \in A} \|d(x, a)\|.$$

Lemma 1.5. [Lemma 2.2, [19]] : Let (X, d) be a cone metric space, P a normal cone with normal constant M = 1 and A, B two compact sets in (X, τ_c). Then

$$\sup_{x \in B} d^1(x,A) < \infty, \text{ where } d^1(x,A) = \inf_{a \in A} \|d(x, a)\|.$$

Definition 1.6. [18]:Let (X, d) be a cone metric space, P a normal cone with normal constant M = 1, $\mathcal{F}_c(X)$ be the set of all compact subsets of (X, τ_c) and $A \in \mathcal{F}c(X)$. Define $h_A : \mathcal{F}c(X) \to [0, \infty)$ and

$d_H : \mathcal{F}c(X) \times \mathcal{F}c(X) \to [0, \infty)$ by $h_A(B) = \sup_{x \in A} d^1(x, B)$ and $d_H(A, B) = \max\{h_A(B), h_B(A)\}$ respectively. For each $A,B \in \mathcal{F}c(X)$ and $x, y \in A$, we have

(i) $d^1(x,A) \leq \|d(x, y)\| + d^1(y,A)$

(ii) $d^1(x,A) \leq d^1(x,B) + h_B(A)$

(iii) $d^1(x,A) \leq \|d(x, y)\| + d^1(y,B) + h_B(A)$.

Definition 1.7. : Let $f : X \to X$ and $F : X \to \mathcal{F}c(X)$. f is said to be F-weakly commuting at $x \in X$ if $f^2 x \in F f x$.

Kamran [12] defined the above in metric spaces.

Definition 1.8. : Let φ denote the class of all functions $\phi : R^+ \to R^+$ such that ϕ is non decreasing, continuous and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.

It is clear that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ and hence, we have $\phi(t) < t$, for all $t > 0$.

Now we give our main result.

2. THE MAIN RESULT

Theorem 2.1. Let (X, d) be a complete cone metric space with normal constant M = 1. Let $F,G : X \to \mathcal{F}c(X)$ be two multifunctions and $f, g : X \to X$ be self maps satisfying

$$(2.1.1)\; d_H(Fx,Gy) \leq \phi\left(\max\left\{ \begin{array}{l} \|d(fx, gy)\|, d'(fx, Fx), d'(gy, Gy), \\ \frac{1}{2}[d'(fx, Gy) + d'(gy, Fx)] \end{array} \right\} \right)$$

for all $x, y \in X$ and $\phi \in$ φ,

(2.1.2) $Fx \subseteq g(X), G(x) \subseteq f(X)$ for all $x \in X$,

(2.1.3) one of f(X) and g(X) is a complete subset of X and

(2.1.4) f is F-weakly commuting and g is G-weakly commuting at their coincidence points.

Then the pairs (f, F) and (g,G) have the same coincidence point in X.

Proof. Let $x_0 \in X$.Then by Lemma 1.4, there exists $gx_1 \in Fx_0$ such that

$d^1(fx_0, Fx_0) = \|d(fx_0, gx_1)\|$. Again by Lemma 1.4, there exists $fx_2 \in Gx_1$ such

that $d^1(gx_1, Gx_1) = \|d(gx_1, fx_2)\|$.

Continuing in this way, we get the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$d^1(y_{2n-1}, Fx_{2n}) = \|d(y_{2n-1}, y_{2n})\|$ and $d^1(y_{2n}, Gx_{2n+1}) = \|d(y_{2n}, y_{2n+1})\|$,where

$y_{2n} = gx_{2n+1} \in Fx_{2n}$ and

$y_{2n+1} = fx_{2n+2} \in Gx_{2n+1}$ n = 0, 1, 2.....

Case(i): Suppose $y_{2n} = y_{2n+1}$ for some n.

Assume that $y_{2n+1} \neq y_{2n+2}$.

$$\|d(y_{2n+1}, y_{2n+2})\| = d^1(y_{2n+1}, Fx_{2n+2})$$
$$\leq d_H(Fx_{2n+2}, Gx_{2n+1})$$

$$\le \phi\left(\max\left\{\begin{array}{l}\|d(y_{2n+1},y_{2n})\|, d'(y_{2n+1},Fx_{2n+2}), d'(y_{2n},Gx_{2n+1}),\\ \frac{1}{2}[d'(y_{2n+1},Gx_{2n+1})+d'(y_{2n},Fx_{2n+2})\end{array}\right\}\right)$$

$$\le \phi\left(\max\left\{\begin{array}{l}0, \|d(y_{2n+1},y_{2n+2})\|, 0\\ \frac{1}{2}[0+0+\|d(y_{2n+1},y_{2n+2})\|\ ]\end{array}\right\}\right)$$

$$= \phi\ (\|d(y_{2n+1},y_{2n+2})\|)$$

$$< \|d(y_{2n+1},y_{2n+2})\|.$$

It is a contradiction. Hence $y_{2n+1} = y_{2n+2}$.

Continuing in this way, we have $y_n = y_{n+k}$ for all k = 1, 2, 3, ... Hence $\{y_n\}$ is a Cauchy sequence in X.

Case (ii): Suppose that $y_n \ne y_{n+1}$ for all n. Now

$$\|d(y_{2n+1},y_{2n+2})\| = d^1(y_{2n},Gx_{2n+1})$$

$$\le d_H(Fx_{2n},Gx_{2n+1})$$

$$\le \phi\left(\max\left\{\begin{array}{l}\|d(y_{2n-1},y_{2n})\|, d'(y_{2n-1},Fx_{2n}), d'(y_{2n},Gx_{2n+1}),\\ \frac{1}{2}[d'(y_{2n-1},Gx_{2n+1})+d'(y_{2n},Fx_{2n})\end{array}\right\}\right)$$

$$\le \phi\left(\max\left\{\begin{array}{l}\|d(y_{2n-1},y_{2n})\|, \|d(y_{2n-1},y_{2n})\|, \|d(y_{2n},y_{2n+1})\|,\\ \frac{1}{2}[\|d(y_{2n-1},y_{2n+1})\|+0\ ]\end{array}\right\}\right)$$

$$\le \phi\left(\max\left\{\begin{array}{l}\|d(y_{2n-1},y_{2n})\|, \|d(y_{2n},y_{2n+1})\|,\\ \frac{1}{2}[\|d(y_{2n-1},y_{2n})\|+\|d(y_{2n},y_{2n+1})\|\ ]\end{array}\right\}\right)$$

$$= \phi\ (\|d(y_{2n-1},y_{2n})\|).$$

Similarly we can show that $\|d(y_{2n-1},y_{2n})\| \le \phi(\|d(y_{2n-2},y_{2n-1})\|)$.

Thus

$$\|d(y_n,y_{n+1})\| \le \phi\ (\|d(y_{n-1},y_n)\|) \le \phi^2\ (\|d(y_{n-2},y_{n-1})\|) \le \ldots\ldots\le \phi^n\ (\|d(y_0,y_1)\|)$$

Now for n > m we have

$$\|d(y_n,y_m)\| \le \sum_{i=m+1}^{n}\|d(y_i,y_{i-1})\|$$

$$\le \phi^m(\|d(y_0,y_1)\|)+\phi^{m+1}(\|d(y_0,y_1)\|)+\ldots\ldots\ldots\ldots+\phi^n(\|d(y_0,y_1)\|).$$

$$\le \sum_{i=m}^{\infty}\varphi^i(\|d(y_0,y_1)\|)$$

$\to 0$ as $m \to \infty$, since $\sum_{n=1}^{\infty}\varphi^n(t) < \infty$ for all $t > 0$.

This is implies that $\lim_{m,n\to\infty}\|d(y_n,y_m)\| = 0$.

By Lemma 4,([2]),$\{y_n\}$ is a Cauchy sequence in X.

Suppose g(X) is complete.

Then $y_{2n} = g x_{2n+1} \to p = g v \in g(X)$ for some p and $v \in X$.

Since $\{y_n\}$ is Cauchy, we have $y_{2n+1} \to p$.

$$d'(p,Gv) \le \|d(p,y_{2n})\| + d'(y_{2n},Gv)$$

$$\le \|d(p,y_{2n})\| + d_H(Fx_{2n},Gv)$$

$$\le \|d(p,y_{2n})\| + \phi\left(\max\left\{\begin{array}{l}\|d(y_{2n-1},gv)\|, d'(y_{2n-1},Fx_{2n}), d'(gv,Gv),\\ \frac{1}{2}[d'(y_{2n-1},Gv)+d'(gv,Fx_{2n})\ ]\end{array}\right\}\right)$$

$$\le \|d(p,y_{2n})\| + \phi\left(\max\left\{\begin{array}{l}\|d(y_{2n-1},p)\|, \|d(y_{2n-1},y_{2n})\|, d'(p,Gv),\\ \frac{1}{2}[\|d(y_{2n-1},p)\|+d'(p,Gv)+\|d(p,y_{2n})\|]\end{array}\right\}\right)$$

Letting $n \to \infty$,we get

$d^1(p,Gv) \le \phi(d^1(p,Gv))$ so that $d^1(p,Gv) = 0$.

Hence $p \in Gv$. Thus $gv = p \in Gv$.

Since g is G-weakly commuting at the coincidence point v, we have $gp = g^2v \in Ggv = Gp$.Thus p is a coincidence point of g and G . Since $Gv \subseteq f(X)$,there exists $w \in X$ such that $p = gv = fw \in Gv$.

$$d'(p,Fw) \le \|d(p,y_{2n+1})\| + d'(y_{2n+1},Fw)$$

$$\le \|d(p,y_{2n+1})\| + d_H(Fw,Gx_{2n+1})$$

$$\le \|d(p,y_{2n+1})\| + \phi$$

$$\left(\max\left\{ \begin{array}{l} \|d(fw, y_{2n})\|, d'(fw, Fw), d'(y_{2n}, Gx_{2n+1}), \\ \frac{1}{2}[d'(fw, Gx_{2n+1}) + d'(y_{2n}, Fw)] \end{array} \right\} \right)$$

$$\leq \|d(p, y_{2n+1})\| + \phi$$

$$\left(\max\left\{ \begin{array}{l} \|d(p, y_{2n})\|, d'(p, Fw), \|d(y_{2n}, y_{2n+1})\|, \\ \frac{1}{2}\left[\|d(p, y_{2n+1})\| + \|d(y_{2n}, p)\| + d'(p, Fw)\right] \end{array} \right\} \right)$$

Letting $n \to \infty$,we get

$d^1(p, Fw) \leq \phi(d^1(p, Fw))$ so that $d^1(p, Fw) = 0$.

Hence $p \in Fw$. Thus $fw = p \in Fw$.

Since f is F- weakly commuting at the coincidence point w, we have $fp = ffw \in Ffw = Fp$.

Thus p is a coincidence point of f and F . Hence, the pairs (f, F) and (g, G) have the same coincidence point.

By putting f = g = I (the identity map) in Theorem 2.1,we have

Corollary 2.2. Let (X, d) be a complete cone metric space with normal constant M = 1.Let $F,G : X \to \mathcal{F}c(X)$ be two multi functions satisfying

(2.2.1) $d_H(Fx,Gy) \leq \alpha$

$$\left(\max\left\{ \begin{array}{l} \|d(x, y)\|, d'(x, Fx), d'(y, Gy), \\ \frac{1}{2}[d'(x, Gy) + d'(y, Fx)] \end{array} \right\} \right)$$

for all $x, y \in X$, where $\alpha \in [0, 1)$.

Then F and G have a common fixed point in X.

Corrolary 2.2 is a generalization and improvement of Theorems 1 and 2 of [19] for a pair of multi functions and of Theorems 2.6 and 2.7 of [18]for a single multi function with G = F.

REFERENCES

[1] .Abbas M, and Jungck, G., "Common fixed point result for non commuting mappings without continuity in cone metric spaces", ***J.Math.Anal.Appl. 341:***416-420(2008).

[2] Azam, A., Arshad M., and Beg., I., "Common fixed points of two maps in cone metric spaces", ***Rend.Ciric.Mat.Palermo***,57:433-441 (2008),

[3] A.Azam,M.Arshad, "Common fixed points of generalized contractive maps in cone uniform spaces" , ***Bull.Iranian Math.Soc***., 35(2) 255-264. (2009).

[4] Arshad, M., Azam, A., Vetro, P., "Some common fixed point results in cone uniform spaces", ***Fixed Point Theory and App***l. 2009,Article ID 493965,11 Pages,doi: 10.1155/2009/493965.

[5] Bari C.Di, and Vetro. P., ϕ-pairs and common fixed points in cone metric spaces", ***Rend.Ciric.Mat. Palermo***,57:279-285 (2008).

[6] Bari C.Di, and Vetro P., "Weakly ϕ-pairs and common fixed points in cone metric spaces" ***Ciric.Mat.Palermo***,58 125-132 (2009).

[7] Haghi, R.H., Rezapour, Sh., "Fixed points of multifunctions on regular cone metric spaces", ***Expo.Math***.,28:71-77 (2010),

[8] Harjani, J., Sadarangani, K., "Generalized contractions in partially ordered metric spaces and Applications to ordinary differential equations", ***Nonlinear Analysis***,72(2010),1188-1197.

[9] L.G.Huang and X.Zhang., "Cone metric spaces and fixed point theorems of contractive mappings", ***J.Math.Anal.Appl***.332(2007), 1468-1476.

[10] D.Ilic and V.Rakocevic., "Common fixed point result for maps on cone metric spaces", ***J.Math. Anal.Appl***.341(2008), 867-882.

[11]S.Jankovic,Z.Kadelburg,S.Radonevic.B.E.Rhoades, "Assad-Kirk type fixed point theorems for a pair of nonself mappings on cone metric spaces",Fixed point Theory and Appl.Vol.2009,Article ID 761086,16 Pages,doi:10.1155/2009/761086.

[12] T.Kamaran., "Coincidence and fixed points for hybrid strict contractions" , ***J.Math.Anal.Appl***. 299464-468(2007).

[13] D.Kim,D.Wardowski, "Dynamics processes and fixed points of set-valued nonlinear contractions in cone metric spaces"***,Nonlinear Analysis*** 71,5170-5175(2009).

[14] Z.Kadelburg,S.Radonevic,B.Rosic, "Strict contractive conditions and common fixed point theorems in cone metric spaces",***Fixed point Theory and Appl.***Vol.2009,Article ID 173838,14 Pages,doi:10.1155/2009/173838.

[15] Z.Kadelburg,S.Radonevic,V.Rakocevic, "Remarks on quasi-contraction on a cone metric space", ***Appl.Math.Lett***.,22:1674-1679(2009).

[16] Radonevic, S., "Common fixed points under contractive conditions in cone metric spaces", C***omputer and Math.with Appl***.,581273-1278(2009).

[17] .Rezapour Sh., and.Hamalbarani., R., "Some notes on paer "cone metric spaces and fixed point theorems of contractive mappings", ***J.Math.Anal.Appl.*** 345(2008), 719-724.

[18] Rezapour Sh, and Haghi., R.H, "Fixed points of multi functions on cone metric spaces" ***, Numerical Functional Analysis and optimization***,30:7(2009),825-832. DOI:10.1080-01630560903123346.

[19] Rezapour S., Khandani H., and Vaezpour., S., M., "Efficacy of cones on topological vector spaces and application to common fixed points of multi functions", ***Rend.Ciric.Mat. Palermo***,59 185-197(2010).

[20] Rezapour, Sh., Haghi, R.H., Shahzad, N., "Some notes on fixed points of quasi-contraction maps", ***Appl.Math.Lett***.,23:498-502(2010).

[21] Wei-Shih Du, "A note on cone metric fixed point theory and its equivalence",***Nonlinear Analysis***, 72:2259-2261(2010).

[22] Vetro., P., "Common fixed points in cone metric spaces" , ***Rend. Ciric. Mat. Palermo,*** 56:464-468 (2007).

[23] Wlodarczyk, K., Plebaniak, R., Obczynski, C., "Convergence theorems,best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces", ***Nonlinear Analysis*** 72:794-805(2010).

[24] Zhao, Z., Chen, X., "Fixed points of decreasing operators in ordered Banach spaces and applications to nonlinear second order elliptic equations",***Computer and Math.with Appl.*** 58:1223-1229 (2009).