Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2020, 13(2), 923-934
ISSN: 1307-9085, e-ISSN: 2149-4584
Araştırma Makalesi

Erzincan University
Journal of Science and Technology
2020, 13(2), 923-934
DOI: 10.18185/erzifbed.635545
Research Article

On Some Properties of Space S_w^α

Erdem TOKSOY[1*], Ayşe SANDIKÇI[2]

[1,2] Ondokuz Mayıs University, Faculty of Arts and Sciences, Department of Mathematics, Samsun, Turkey

Geliş / Received: 21/10/2019, Kabul / Accepted: 01/06/2020

Abstract

In this study, first of all we define spaces $S^\Theta(\mathbb{R}^d)$ and $S_w^\Theta(\mathbb{R}^d)$ and give examples of these spaces. After we define $S_w^\alpha(\mathbb{R}^d)$ to be the vector space of $f \in L_w^1(\mathbb{R}^d)$ such that the fractional Fourier transform $F_\alpha f$ belongs to $S_w^\Theta(\mathbb{R}^d)$. We endow this space with the sum norm $\|f\|_{S_w^\alpha} = \|f\|_{1,w} + \|F_\alpha f\|_{S_w^\Theta}$ and then show that it is a Banach space. We show that $S_w^\alpha(\mathbb{R}^d)$ is a Banach algebra and a Banach ideal on $L_w^1\left(\mathbb{R}^d\right)$ if the space $S_w^\Theta(\mathbb{R}^d)$ is solid. Furthermore, we prove that the space $S_w^\alpha(\mathbb{R}^d)$ is translation and character invaryant and also these operators are continuous. Finally, we discuss inclusion properties of these spaces.

Keywords: Fractional Fourier transform, convolution, Segal algebras.

S_w^α Uzayının Bazı Özellikleri

Öz

Bu çalışmada ilk olarak $S^\Theta(\mathbb{R}^d)$ ve $S_w^\Theta(\mathbb{R}^d)$ uzayları tanımlandı ve bu uzaylara örnekler verildi. Sonra $F_\alpha f$ kesirli Fourier dönüşümü $S_w^\Theta(\mathbb{R}^d)$ uzayında olan $f \in L_w^1(\mathbb{R}^d)$ fonksiyonlarının $S_w^\alpha(\mathbb{R}^d)$ vektör uzayı tanımlandı. Yine $S_w^\alpha(\mathbb{R}^d)$ uzayı üzerinde $\|f\|_{S_w^\alpha} = \|f\|_{1,w} + \|F_\alpha f\|_{S_w^\Theta}$ fonksiyonunun bir norm olduğu ifade edildikten sonra $S_w^\alpha(\mathbb{R}^d)$ uzayının Banach uzayı olduğu ve $S_w^\Theta(\mathbb{R}^d)$ uzayının bir katı (solid) uzay olması koşuluyla bu uzayın bir Banach cebiri ve $L_w^1(\mathbb{R}^d)$ uzayının bir Banach ideali olduğu gösterildi. Ayrıca $S_w^\alpha(\mathbb{R}^d)$ uzayının öteleme ve karakter işlemcileri altında değişmez olduğu ve bu operatörlerin sürekliliği ispatlandı. Son olarak bu uzayların kapsama özellikleri tartışıldı.

Anahtar Kelimeler: Kesirli Fourier dönüşümü, girişim işlemi, Segal cebirleri.

1. Introduction and Preliminaries

Throughout this article, we study on $\mathbb{R}^d$. We write the Lebesgue space $\left(L^p\left(\mathbb{R}^d\right), \|f\|_p\right)$, for $1 \le p < \infty$. A weight function w on $\mathbb{R}^d$ is a measurable and locally bounded *function* that satisfying $w(x) \ge 1$ and $w(x+y) \le w(x)w(y)$ for all $x, y \in \mathbb{R}^d$. We define, for $1 \le p < \infty$,

$$L_w^p\left(\mathbb{R}^d\right) = \left\{ f \mid fw \in L^p\left(\mathbb{R}^d\right) \right\}.$$

*Corresponding Author: erdem.toksoy@omu.edu.tr

It is well known that $L_w^p\left(\mathbb{R}^d\right)$ is a Banach space under the norm $\|f\|_{p,w}=\|fw\|_p$ (Reiter and Stegeman, 2000). Let w_1 and w_2 are two weight functions. We state that $w_1 \prec w_2$ if there exists $c>0$ such that $w_1(x)\leq cw_2(x)$ for all $x\in\mathbb{R}^d$ (Feichtinger and Gürkanlı, 1990).

Let A and B Banach algebras and $B\subseteq A$. B is called a Banach ideal of A if $\|f\|_B\geq\|f\|_A$ and $fg\in B$, with $\|fg\|_B\leq\|f\|_B\|g\|_A$ for all $f\in B$, $g\in A$ (Feichtinger vd., 1979). A Banach function space $\left(B,\|.\|_B\right)$ of measurable functions is called solid, if for every $f\in B$ and any measurable function g satisfying $|g(x)|\leq|f(x)|$ almost everywhere, $g\in B$ and $\|g\|_B\leq\|f\|_B$ (Feichtinger, 1977).

Let $\left(B(\mathbb{R}^d),\|.\|_B\right)$ be a complex valued measurable functions on $\mathbb{R}^d$. $B(\mathbb{R}^d)$ is called homogeneous Banach space if it is strongly translation invaryant (i.e $T_y f\in B(\mathbb{R}^d)$ and $\|T_y f\|_B=\|f\|_B$) where $T_y f(t)=f(t-y)$ for each y and the mapping $y\to T_y f$ from $\mathbb{R}^d$ into $B(\mathbb{R}^d)$ is continuous for each f in $B(\mathbb{R}^d)$.

A homogeneous Banach algebra $B(\mathbb{R}^d)$ is a subalgebra of $L^1\left(\mathbb{R}^d\right)$ such that $B(\mathbb{R}^d)$ is itself a Banach algebra with respect to a norm $\|.\|_1\leq\|.\|_B$. A homogeneous Banach algebra is called Segal algebra if it is dense in $L^1\left(\mathbb{R}^d\right)$ (Wang, 1977).

Let w be a weight function on $\mathbb{R}^d$. The space $S_w(\mathbb{R}^d)$ is subalgebra of $L^1_w(\mathbb{R}^d)$ satisfying the following conditions (Cigler, 1969):

i) $S_w(\mathbb{R}^d)$ is dense in $L_w^1(\mathbb{R}^d)$

ii) $S_w(\mathbb{R}^d)$ is a Banach algebra under some norm $\|.\|_{S_w}$ and the inequality $\|f\|_{1,w}\leq\|f\|_{S_w}$ holds for all $f\in S_w(\mathbb{R}^d)$

iii) $S_w(\mathbb{R}^d)$ is translation invaryant and the mapping $y\to T_y f$ from $\mathbb{R}^d$ into $S_w(\mathbb{R}^d)$ is continuous.

iv) For each $f\in S_w(\mathbb{R}^d)$ and all $y\in\mathbb{R}^d$, the inequality $\|T_y f\|_{S_w}\leq w(y)\|f\|_{S_w}$ holds.

We define the Fourier transform f (or Ff) of a function $f\in L^1(\mathbb{R})$ as

$$f(\omega)=Ff(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt.$$

The fractional Fourier transform is a generalization of the Fourier transform through an angle parameter α and can be considered as a rotation by an angle α in the time-frequency plane. The fractional Fourier transform with an angle α of a function $f\in L^1(\mathbb{R})$ is defined by

$$F_\alpha f(u)=\int_{-\infty}^{+\infty}K_\alpha(u,t)f(t)dt$$

where,

$$K_\alpha(u,t) = \begin{cases} Me^{\frac{i}{2}(u^2+t^2)\cot\alpha - iut\operatorname{cosec}\alpha}, & \text{if } \alpha \neq k\pi,\ k\in\mathbb{Z} \\ \delta(t-u), & \text{if } \alpha = 2k\pi,\ k\in\mathbb{Z} \\ \delta(t+u), & \text{if } \alpha = (2k+1)\pi,\ k\in\mathbb{Z} \end{cases}$$

$M=\sqrt{\dfrac{1-i\cot\alpha}{2\pi}}$ and δ be a Dirac delta function. The fractional Fourier transform with $\alpha=\dfrac{\pi}{2}$ corresponds to the Fourier transform, (Almeida, 1994; Almeida, 1997; Bultheel and Martinez, 2002; Namias, 1980; Ozaktas vd., 2001). The fractional Fourier transform can be extended for higher dimensions as Bultheel and Martinez (2002):

$$\left(F_{\alpha_1,\dots,\alpha_d} f\right)(u_1,\dots,u_d) = \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} K_{\alpha_1,\dots,\alpha_d}\left(u_1,\dots,u_d;t_1,\dots,t_d\right) \times f\left(t_1,\dots,t_d\right)dt_1\dots dt_d$$

or shortly

$$F_\alpha f(u) = \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} K_\alpha\left(u,t\right)f\left(t\right)dt,$$

where

$$K_\alpha\left(u,t\right) = K_{\alpha_1,\dots,\alpha_d}\left(u_1,\dots,u_d;t_1,\dots,t_d\right) = K_{\alpha_1}\left(u_1,t_1\right)K_{\alpha_2}\left(u_2,t_2\right)\dots K_{\alpha_d}\left(u_d,t_d\right).$$

Let $\alpha=\left(\alpha_1,\alpha_2,\dots,\alpha_d\right)$ where $\alpha_i \neq k\pi$ for each index i with $1\le i\le d$ and $k\in\mathbb{Z}$. The Θ convolution operation define as:

$$\left(f\Theta g\right)\left(x\right) = \int_{\mathbb{R}^d} f\left(y\right)g\left(x-y\right)e^{\sum_{j=1}^{d} iy_j\left(y_j-x_j\right)\cot\alpha_j}dy$$

for all $f,g\in L^1\left(\mathbb{R}^d\right)$ (Toksoy and Sandıkçı, 2015).

Troughout this paper, unless otherwise indicated, we get $\alpha=\left(\alpha_1,\alpha_2,\dots,\alpha_d\right)$ where $\alpha_i\neq k\pi$ for each index i with $1\le i\le d$ and $k\in\mathbb{Z}$. In this work, we will define a space $S_w^\alpha(\mathbb{R}^d)$ is analogous with the space $S_{w,\omega}(\mathbb{R}^d)$ which is define in (Doğan and Gürkanlı, 2000). Since the space $S_w^\alpha(\mathbb{R}^d)$ is defined by using Θ convolution operation and fractional Fourier transform, then this space is a generalization of the space $S_{w,w}(\mathbb{R}^d)$. If we take $\alpha_i=\dfrac{\pi}{2}$ for each index i with $1\le i\le d$, the Θ convolution corresponds to ordinary convolution and also the fractional Fourier transform corresponds to the Fourier transform. An angle parameter α provide us more overview of results that established in (Doğan and Gürkanlı, 2000).

2. The Space $S_w^\alpha\left(\mathbb{R}^d\right)$ and Some Properties

Let w be a weight function on $\mathbb{R}^d$. It is well known by Theorem 8 in Toksoy and Sandıkçı (2015), $L_w^1\left(\mathbb{R}^d\right)$ is a Banach algebra under Θ convolution. Then also the space $L^1(\mathbb{R}^d)$ is a Banach algebra under Θ convolution.

In this section, the space that we will denote by $S^\Theta(\mathbb{R}^d)$ under a norm $\|.\|_{S^\Theta}$ will be called Θ convolution Segal algebra if it satisfies conditions of Segal algebra under Θ convolution operation.

Now, we will give an example that is a $S^\Theta(\mathbb{R}^d)$ space. Let proof the following Lemma for this example.

Lemma 2.1. Let $1\le p<\infty$. The space $L^p(\mathbb{R}^d)$ is Banach module over $L^1(\mathbb{R}^d)$ under Θ convolution.

Proof. Let $f\in L^1(\mathbb{R}^d)$ and $g\in L^p(\mathbb{R}^d)$. Then we write

$$\|f\Theta g\|_p^p$$
$$= \int_{\mathbb{R}^d} \left|\left(f\Theta g\right)(x)\right|^p dx$$
$$= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) g(x-y) e^{\sum_{j=1}^{d} i y_j (y_j - x_j)\cot\alpha_j} dy \right|^p dx$$
$$\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)||g(x-y)| dy \right)^p dx.$$

Since the numbers p and $\frac{p}{p-1}$ are conjugate, by using Hölder's inequality we get

$$\|f\Theta g\|_p^p$$
$$\leq \int_{\mathbb{R}^d} \left(\left(\int_{\mathbb{R}^d} |f(y)||g(x-y)|^p dy \right)^{\frac{1}{p}} \right.$$
$$\left. \times \left(\int_{\mathbb{R}^d} |f(y)|\, dy \right)^{\frac{p-1}{p}} \right)^p dx$$
$$= \|f\|_1^{p-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)||g(x-y)|^p dy dx$$
$$= \|f\|_1^{p-1} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |g(x-y)|^p dx \right) |f(y)| dy$$
$$= \|f\|_1^p \|g\|_p^p.$$

Thus we have

$$\|f\Theta g\|_p \leq \|f\|_1 \|g\|_p. \qquad (1)$$

It is easy to see that the other conditions of the Banach module are satisfied.

Example 2.2. Let $1 < p < \infty$. It is well known by Reiter (1971) the space $L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ is a Segal algebra under $*$ convolution with the norm $\|.\|_S = \|.\|_1 + \|.\|_p$. If we take Θ convolution instead of $*$ convolution, then the space $L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ be a $S^\Theta(\mathbb{R}^d)$ Segal algebra under the norm $\|.\|_{S^\Theta} = \|.\|_1 + \|.\|_p$. It is enough to show that the space $L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ is a Banach algebra under Θ convolution. Let $f, g \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$. By using Lemma 2.1 and Theorem 8 in Toksoy and Sandıkçı (2015), we have

$$\|f\Theta g\|_{S^\Theta} = \|f\Theta g\|_1 + \|f\Theta g\|_p$$
$$\leq \|f\|_1 \|g\|_1 + \|f\|_1 \|g\|_p$$
$$\leq \|f\|_{S^\Theta} \|g\|_{S^\Theta}.$$

It is easy to show other conditions of being Banach algebra. Then $L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ is a $S^\Theta(\mathbb{R}^d)$ Segal algebra.

Let w be a weight function on $\mathbb{R}^d$. If we take Θ convolution instead of $*$ convolution in the definition of $S_w(\mathbb{R}^d)$, then we will denote the space which provides conditions of the space $S_w(\mathbb{R}^d)$ by $S_w^\Theta(\mathbb{R}^d)$. The norm of this space will also be denoted by $\|.\|_{S_w^\Theta}$.

Now, we will give an example of a $S_w^\Theta(\mathbb{R}^d)$ space. For this, we need the following lemma.

Lemma 2.3. Let $1 \leq p < \infty$ and w be a weight function on $\mathbb{R}^d$. The space $L_w^p(\mathbb{R}^d)$ is Banach module over $L_w^1(\mathbb{R}^d)$ under Θ convolution.

Proof. Let $f \in L_w^1(\mathbb{R}^d)$ and $g \in L_w^p(\mathbb{R}^d)$. Then $fw \in L^1(\mathbb{R}^d)$ and $gw \in L^p(\mathbb{R}^d)$. Hence, we write

$$\begin{aligned}
&\|f\Theta g\|_{p,w}\\
&=\left(\int_{\mathbb{R}^d}\left|\left(f\Theta g\right)(x)w(x)\right|^p dx\right)^{1/p}\\
&\le\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(y)w(y)g(x-y)w(x-y)\right.\right.\\
&\left.\left.\times e^{\sum_{j=1}^{d} iy_j(y_j-x_j)\cot\alpha_j}dy\right|^p dx\right)^{1/p}\\
&=\left\|(fw)\Theta(gw)\right\|_p .
\end{aligned}$$

By using (1) we obtain

$$\begin{aligned}
\|f\Theta g\|_{p,w} &\le \left\|(fw)\Theta(gw)\right\|_p\\
&\le \|fw\|_1\|gw\|_p \qquad (2)\\
&= \|f\|_{1,w}\|g\|_{p,w}.
\end{aligned}$$

It is easy to see that the other conditions of the Banach module are satisfied.

Example 2.4. Let $1<p<\infty$ and w be a weight function on $\mathbb{R}^d$. It is well known by Doğan and Gürkanlı (2000) the space $L_w^1(\mathbb{R}^d)\cap L_w^p(\mathbb{R}^d)$ is a $S_w(\mathbb{R}^d)$ space with the norm $\|\cdot\|_{S_w}=\|\cdot\|_{1,w}+\|\cdot\|_{p,w}$. It is easy to see that the space $L_w^1(\mathbb{R}^d)\cap L_w^p(\mathbb{R}^d)$ is dense in $L_w^1(\mathbb{R}^d)$ for any w weight. If we take Θ convolution instead of $*$ convolution, then the space $L_w^1(\mathbb{R}^d)\cap L_w^p(\mathbb{R}^d)$ be a $S_w^\Theta(\mathbb{R}^d)$ space under the norm $\|\cdot\|_{S_w^\Theta}=\|\cdot\|_{1,w}+\|\cdot\|_{p,w}$. It is enough to show that the space $L_w^1(\mathbb{R}^d)\cap L_w^p(\mathbb{R}^d)$ is a Banach algebra under Θ convolution. Let $f,g\in L_w^1(\mathbb{R}^d)\cap L_w^p(\mathbb{R}^d)$. By Lemma 2.3 and Theorem 8 in Toksoy and Sandıkçı (2015), we get

$$\begin{aligned}
\|f\Theta g\|_{S_w^\Theta} &= \|f\Theta g\|_{1,w}+\|f\Theta g\|_{p,w}\\
&\le \|f\|_{1,w}\|g\|_{1,w}+\|f\|_{1,w}\|g\|_{p,w}\\
&\le \|f\|_{S_w^\Theta}\|g\|_{S_w^\Theta}.
\end{aligned}$$

It is easy to show that the other conditions of the Banach algebra are satisfied. Hence, $L_w^1(\mathbb{R}^d)\cap L_w^p(\mathbb{R}^d)$ is a $S_w^\Theta(\mathbb{R}^d)$ space.

Now, we can define the space $S_w^\alpha(\mathbb{R}^d)$.

Definition 2.5. Given a weight w on $\mathbb{R}^d$, denote by $S_w^\alpha(\mathbb{R}^d)$ the space of all $f\in L_w^1(\mathbb{R}^d)$ such that $F_\alpha f\in S_w^\Theta(\mathbb{R}^d)$ with norm

$$\|f\|_{S_w^\alpha}=\|f\|_{1,w}+\|F_\alpha f\|_{S_w^\Theta}.$$

Theorem 2.6. $\left(S_w^\alpha(\mathbb{R}^d),\|\cdot\|_{S_w^\alpha}\right)$ is a Banach space.

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $S_w^\alpha(\mathbb{R}^d)$. Therefore $(g_n)_{n\in\mathbb{N}}$ and $(F_\alpha g_n)_{n\in\mathbb{N}}$ are Cauchy spaces in $L_w^1(\mathbb{R}^d)$ and $S_w^\Theta(\mathbb{R}^d)$, respectively. Since $L_w^1(\mathbb{R}^d)$ and $S_w^\Theta(\mathbb{R}^d)$ are Banach spaces, there exists $g\in L_w^1(\mathbb{R}^d)$ and $h\in S_w^\Theta(\mathbb{R}^d)$ where $\|g_n-g\|_{1,w}\to 0$, $\|F_\alpha g_n-h\|_{S_w^\Theta}\to 0$. By using the inequality $\|\cdot\|_1\le\|\cdot\|_{1,w}\le\|\cdot\|_{S_w^\Theta}$, we get $\|g_n-g\|_1\to 0$ and $\|F_\alpha g_n-h\|_1\to 0$. Thus $(F_\alpha g_n)_{n\in\mathbb{N}}$ has a subsequence $\left(F_\alpha g_{n_k}\right)_{n_k\in\mathbb{N}}$ that converges pointwise to h almost everywhere. Also it is clear that $\|g_{n_k}-g\|_1\to 0$. Then we write

$$
\begin{aligned}
&\left|F_\alpha g(u)-h(u)\right| \\
&\le \left|F_\alpha\left(g_{n_k}-g\right)(u)\right|+\left|F_\alpha g_{n_k}(u)-h(u)\right| \\
&\le \prod_{j=1}^{d}\left|\sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right| \\
&\times\int_{\mathbb{R}^d}\left|\left(g_{n_k}-g\right)(t)\right|\left|e^{\sum_{j=1}^{d}\frac{i}{2}\left(u_j^2+t_j^2\right)\cot\alpha_j-iu_jt_j\operatorname{cosec}\alpha_j}\right|dt \\
&+\left|F_\alpha g_{n_k}(u)-h(u)\right| \\
&=\prod_{j=1}^{d}\left|\sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right|\left\|g_{n_k}-g\right\|_1 \\
&+\left|F_\alpha g_{n_k}(u)-h(u)\right|.
\end{aligned}
$$

This inequality ensures $F_\alpha g=h$ almost everywhere. Hence $\|g_n-g\|_{S_w^\alpha}\to 0$ and $g\in S_w^\alpha(\mathbb{R}^d)$. Thus $\left(S_w^\alpha(\mathbb{R}^d),\|.\|_{S_w^\alpha}\right)$ is a Banach space.

Theorem 2.7. Let $S_w^\Theta(\mathbb{R}^d)$ be a solid space. Then the space $S_w^\alpha(\mathbb{R}^d)$ is a Banach algebra under Θ convolution operation.

Proof. It is shown that $S_w^\alpha(\mathbb{R}^d)$ is a Banach space by Theorem 2.6. Let $f,g\in S_w^\alpha(\mathbb{R}^d)$. By using Theorem 7 in Toksoy and Sandıkçı (2015), we write

$$
\begin{aligned}
&\left|F_\alpha\left(f\Theta g\right)(u)\right| \\
&=\left|\prod_{j=1}^{d}\sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right| \\
&\times\left|e^{\sum_{j=1}^{d}-\frac{i}{2}u_j^2\cot\alpha_j}\right|\left|F_\alpha f(u)\right|\left|F_\alpha g(u)\right| \qquad (3)\\
&\le\left|F_\alpha g(u)\right|\int_{\mathbb{R}^d}\left|f(t)\right|dt \\
&\le\left|F_\alpha g(u)\right|\|f\|_{1,w}.
\end{aligned}
$$

Since $F_\alpha\left(f\Theta g\right)$ is continuous, then it is also measurable. Since $S_w^\Theta(\mathbb{R}^d)$ is a solid space, by using (3) we get $F_\alpha\left(f\Theta g\right)\in S_w^\Theta(\mathbb{R}^d)$ and

$$
\begin{aligned}
\left\|F_\alpha\left(f\Theta g\right)\right\|_{S_w^\Theta}&\le\left\|F_\alpha g\,\|f\|_{1,w}\right\|_{S_w^\Theta} \\
&=\left\|F_\alpha g\right\|_{S_w^\Theta}\|f\|_{1,w}.
\end{aligned} \qquad (4)
$$

By Theorem 8 in Toksoy and Sandıkçı (2015) and (4) we obtain

$$
\begin{aligned}
\|f\Theta g\|_{S_w^\alpha}&=\|f\Theta g\|_{1,w}+\left\|F_\alpha\left(f\Theta g\right)\right\|_{S_w^\Theta} \\
&\le\|f\|_{1,w}\|g\|_{1,w}+\|F_\alpha g\|_{S_w^\Theta}\|f\|_{1,w} \\
&=\|f\|_{1,w}\|g\|_{S_w^\alpha} \\
&\le\|f\|_{S_w^\alpha}\|g\|_{S_w^\alpha}.
\end{aligned} \qquad (5)
$$

It is easy to see other conditions of being Banach algebra.

Theorem 2.8. Let $S_w^\Theta(\mathbb{R}^d)$ be a solid space. Then the space $S_w^\alpha(\mathbb{R}^d)$ is a Banach ideal on $L_w^1\left(\mathbb{R}^d\right)$ under Θ convolution operation.

Proof. Let $f\in L_w^1(\mathbb{R}^d)$ and $g\in S_w^\alpha(\mathbb{R}^d)$. Since $L_w^1(\mathbb{R}^d)$ is a Banach algebra under Θ convolution, then we have $f\Theta g\in L_w^1\left(\mathbb{R}^d\right)$. We get the result from the proof the previous Theorem.

Theorem 2.9. Let $S_w^\Theta(\mathbb{R}^d)$ be a solid space and $C_c\left(\mathbb{R}^d\right)\cap S_w^\Theta(\mathbb{R}^d)$ is dense in $S_w^\Theta(\mathbb{R}^d)$. Then the mapping $y\to M_vT_bg$ (where $M_xf(t)=e^{itx}f(t)$ for all $x,t\in\mathbb{R}^d$) from $\mathbb{R}^d$ into $S_w^\Theta(\mathbb{R}^d)$ is continuous for all $g\in S_w^\Theta(\mathbb{R}^d)$, where

$$b=\left(y_1\cos\alpha_1,\ldots,y_d\cos\alpha_d\right)$$

and

$$v=\left(-y_1\sin\alpha_1,...,-y_d\sin\alpha_d\right)$$

for all $y=\left(y_1,...,y_d\right)\in\mathbb{R}^d$.

Proof. It is easy to see that mappings $y\to b$ and $y\to v$ from $\mathbb{R}^d$ into $\mathbb{R}^d$ are continuous. Also mappings $y\to T_y g$ and $y\to M_y g$ from $\mathbb{R}^d$ into $S_w^\Theta(\mathbb{R}^d)$ are continuous by definition of $S_w^\Theta(\mathbb{R}^d)$ and Lemma 2.4 in (Doğan and Gürkanlı, 2000), respectively. Then the composition mappings $y\to T_b g$ and $y\to M_v g$ from $\mathbb{R}^d$ into $S_w^\Theta(\mathbb{R}^d)$ are continuous. Let $b^*=\left(y_1^*\cos\alpha_1,...,y_d^*\cos\alpha_d\right)$ and $v^*=\left(-y_1^*\sin\alpha_1,...,-y_d^*\sin\alpha_d\right)$ for $y^*=\left(y_1^*,...,y_d^*\right)\in\mathbb{R}^d$. Hence we write $T_{b^*}g\in S_w^\Theta(\mathbb{R}^d)$. Let $\varepsilon>0$ be given. There exists $\delta_1>0$ such that

$$\left\|M_v\left(T_{b^*}g\right)-M_{v^*}\left(T_{b^*}g\right)\right\|_{S_w^\Theta}<\frac{\varepsilon}{2} \tag{6}$$

whenever $\left\|y-y^*\right\|<\delta_1$ and there exists $\delta_2>0$ such that

$$\left\|T_b g-T_{b^*}g\right\|_{S_w^\Theta}<\frac{\varepsilon}{2} \tag{7}$$

whenever $\left\|y-y^*\right\|<\delta_2$. Since $S_w^\Theta(\mathbb{R}^d)$ is a solid space, then it is strongly character invaryant, i.e $M_y g\in S_w^\Theta(\mathbb{R}^d)$ and $\left\|M_y g\right\|_{S_w^\Theta}=\left\|g\right\|_{S_w^\Theta}$ for all $g\in S_w^\Theta(\mathbb{R}^d)$ and $y\in\mathbb{R}^d$, by Lemma 2.4 in (Doğan and Gürkanlı, 2000). Let $\delta_3=\min\left\{\delta_1,\delta_2\right\}$. Therefore, combining (6) and (7) we obtain

$$\begin{aligned}
&\left\|M_vT_bg-M_{v^*}T_{b^*}g\right\|_{S_w^\Theta}\\
&=\left\|M_vT_bg-M_vT_{b^*}g+M_vT_{b^*}g-M_{v^*}T_{b^*}g\right\|_{S_w^\Theta}\\
&\le\left\|M_v\left(T_bg-T_{b^*}g\right)\right\|_{S_w^\Theta}\\
&+\left\|M_v\left(T_{b^*}g\right)-M_{v^*}\left(T_{b^*}g\right)\right\|_{S_w^\Theta}\\
&=\left\|T_bg-T_{b^*}g\right\|_{S_w^\Theta}+\left\|M_v\left(T_{b^*}g\right)-M_{v^*}\left(T_{b^*}g\right)\right\|_{S_w^\Theta}\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}$$

whenever $\left\|y-y^*\right\|<\delta_3$. This is the desired result.

Theorem 2.10. Let $S_w^\Theta(\mathbb{R}^d)$ be a solid space.

i) The space $S_w^\alpha(\mathbb{R}^d)$ is translation invariant.

ii) Let $C_c\left(\mathbb{R}^d\right)\cap S_w^\Theta(\mathbb{R}^d)$ is dense in $S_w^\Theta(\mathbb{R}^d)$. Then the mapping $y\to T_yg$ from $\mathbb{R}^d$ into $S_w^\alpha(\mathbb{R}^d)$ is continuous.

Proof. i) Let $g\in S_w^\alpha(\mathbb{R}^d)$. Thus $g\in L_w^1(\mathbb{R}^d)$ and $F_\alpha g\in S_w^\Theta(\mathbb{R}^d)$. It is well known that the space $L_w^1(\mathbb{R}^d)$ is translation invariant and holds the inequality $\left\|T_yg\right\|_{1,w}\le w\left(y\right)\left\|g\right\|_{1,w}$ for all $y=\left(y_1,...,y_d\right)\in\mathbb{R}^d$ (Fischer vd., 1996). Let $b=\left(y_1\cos\alpha_1,...,y_d\cos\alpha_d\right)$ and $v=\left(-y_1\sin\alpha_1,...,-y_d\sin\alpha_d\right)$ for all $y=\left(y_1,...,y_d\right)\in\mathbb{R}^d$. By using Proposition 3(1) in Toksoy and Sandıkçı (2015), we may write

$$\begin{aligned}
&F_\alpha\left(T_yg\right)(u)\\
&=e^{\sum_{j=1}^{d}\frac{i}{2}y_j^2\sin\alpha_j\cos\alpha_j}e^{\sum_{j=1}^{d}-iu_jy_j\sin\alpha_j}\mathrm{T}_bF_\alpha g(u) \qquad (8)\\
&=e^{\sum_{j=1}^{d}\frac{i}{2}y_j^2\sin\alpha_j\cos\alpha_j}M_vT_bF_\alpha g\left(u\right)
\end{aligned}$$

Since $S_w^\Theta(\mathbb{R}^d)$ is a solid space then it is strongly character invaryant by Lemma 2.4 in (Doğan and Gürkanlı, 2000). Also the space $S_w^\Theta(\mathbb{R}^d)$ is translation invaryant. Then we have

$$e^{\sum_{j=1}^{d}\frac{i}{2}y_j^2\sin\alpha_j\cos\alpha_j} M_vT_bF_\alpha g\in S_w^\Theta(\mathbb{R}^d).$$

By using equality (8) we obtain $F_\alpha\left(T_y g\right)\in S_w^\Theta(\mathbb{R}^d)$ and

$$\begin{aligned}\left\|F_\alpha\left(T_y g\right)\right\|_{S_w^\Theta} &= \left\|e^{\sum_{j=1}^{d}\frac{i}{2}y_j^2\sin\alpha_j\cos\alpha_j} M_vT_bF_\alpha g\right\|_{S_w^\Theta} \\ &= \left|e^{\sum_{j=1}^{d}\frac{i}{2}y_j^2\sin\alpha_j\cos\alpha_j}\right| \left\|M_vT_bF_\alpha g\right\|_{S_w^\Theta} \\ &= \left\|T_bF_\alpha g\right\|_{S_w^\Theta} \\ &\le w(b)\left\|F_\alpha g\right\|_{S_w^\Theta}.\end{aligned}$$

This indicates that $S_w^\alpha(\mathbb{R}^d)$ is translation invariant. Also we may write the inequality

$$\left\|T_y g\right\|_{S_w^\alpha}\le w(y)\left\|g\right\|_{1,w}+w(b)\left\|F_\alpha g\right\|_{S_w^\Theta}. \qquad (9)$$

ii) Let $g\in S_w^\alpha(\mathbb{R}^d)$. It is enough to show that if $\lim\limits_{n\to\infty}y_n=0$ for $\left(y_n\right)_{n\in\mathbb{N}}\subset\mathbb{R}^d$, then $\lim\limits_{n\to\infty}T_{y_n}g=g$. It is known that the mapping $y\to T_y g$ from $\mathbb{R}^d$ into $L_w^1(\mathbb{R}^d)$ is continuous (Fischer vd., 1996). Therefore we may write

$$\left\|T_{y_n}g-g\right\|_{1,w}\to 0 \qquad (10)$$

as $n\to\infty$. Let we define sequences $\left(b_n\right)_{n\in\mathbb{N}}$ and $\left(v_n\right)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that j sequences of coordinates $b_{nj}=y_{nj}\cos\alpha_j$ and $v_{nj}=-y_{nj}\sin\alpha_j$, respectively. By using (8), we get

$$\begin{aligned}&\left\|F_\alpha\left(T_{y_n}g-g\right)\right\|_{S_w^\Theta} \\ &=\left\|F_\alpha\left(T_{y_n}g\right)-F_\alpha g\right\|_{S_w^\Theta} \\ &=\left\|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\sin\alpha_j\cos\alpha_j}M_{v_n}T_{b_n}F_\alpha g-e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\sin\alpha_j\cos\alpha_j}F_\alpha g\right. \\ &\left.\quad+e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\sin\alpha_j\cos\alpha_j}F_\alpha g-F_\alpha g\right\|_{S_w^\Theta} \\ &\le\left|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\sin\alpha_j\cos\alpha_j}\right|\left\|\left(M_{v_n}T_{b_n}F_\alpha g-F_\alpha g\right)\right\|_{S_w^\Theta} \\ &\quad+\left|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\sin\alpha_j\cos\alpha_j}-1\right|\left\|F_\alpha g\right\|_{S_w^\Theta} \\ &=\left\|\left(M_{v_n}T_{b_n}F_\alpha g-F_\alpha g\right)\right\|_{S_w^\Theta} \\ &\quad+\left|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\sin\alpha_j\cos\alpha_j}-1\right|\left\|F_\alpha g\right\|_{S_w^\Theta}.\end{aligned}$$

Let $b=\left(y_1\cos\alpha_1,...,y_d\cos\alpha_d\right)$ and $v=\left(-y_1\sin\alpha_1,...,-y_d\sin\alpha_d\right)$ for all $y=\left(y_1,...,y_d\right)\in\mathbb{R}^d$. The mapping $y\to M_vT_bg$ from $\mathbb{R}^d$ into $S_w^\Theta(\mathbb{R}^d)$ is continuous by the previous Theorem. Thus we have

$$\left\|M_{v_n}T_{b_n}F_\alpha g-F_\alpha g\right\|_{S_w^\Theta}\to 0 \qquad (11)$$

as $n\to\infty$. Let $h_n=-1+e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\sin\alpha_j\cos\alpha_j}$ for all $n\in\mathbb{N}$. It is obvious that $\left|h_n\right|\to 0$ as $n\to\infty$. Then by combining (10) and (11)

$$\begin{aligned}
&\left\|T_{y_n} g-g\right\|_{S_w^\alpha} \\
&=\left\|T_{y_n} g-g\right\|_{1,w}+\left\|F_\alpha\left(T_{y_n} g\right)-F_\alpha g\right\|_{S_w^\Theta} \\
&\le\left\|T_{y_n} g-g\right\|_{1,w}+\left\|\left(M_{v_n} T_{b_n} F_\alpha g-F_\alpha g\right)\right\|_{S_w^\Theta} \\
&+\left|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\sin\alpha_j\cos\alpha_j}-1\right|\left\|F_\alpha g\right\|_{S_w^\Theta}\to 0
\end{aligned}$$

as $n\to\infty$. This is the desired result.

Theorem 2.11. Let $S_w^\Theta(\mathbb{R}^d)$ be a solid space and $C_c\left(\mathbb{R}^d\right)\cap S_w^\Theta(\mathbb{R}^d)$ is dense in $S_w^\Theta(\mathbb{R}^d)$. Then the mapping $y\to M_a T_c g$ from $\mathbb{R}^d$ into $S_w^\Theta(\mathbb{R}^d)$ is continuous for all $g\in S_w^\Theta(\mathbb{R}^d)$, where

$$c=\left(z_1\sin\alpha_1,\dots,z_d\sin\alpha_d\right)$$

and

$$a=\left(z_1\cos\alpha_1,\dots,z_d\cos\alpha_d\right)$$

for all $z=\left(z_1,\dots,z_d\right)\in\mathbb{R}^d$.

Proof. The proof is the same as the proof of Theorem 2.9.

Theorem 2.12. Let $S_w^\Theta(\mathbb{R}^d)$ be a solid space.

i) The space $S_w^\alpha(\mathbb{R}^d)$ is character invaryant.

ii) Let $C_c\left(\mathbb{R}^d\right)\cap S_w^\Theta(\mathbb{R}^d)$ is dense in $S_w^\Theta(\mathbb{R}^d)$. Then the mapping $z\to M_z f$ from $\mathbb{R}^d$ into $S_w^\alpha(\mathbb{R}^d)$ is continuous.

Proof. i) Let $g\in S_w^\alpha(\mathbb{R}^d)$. Thus $g\in L_w^1(\mathbb{R}^d)$ and $F_\alpha g\in S_w^\Theta(\mathbb{R}^d)$. It is easy to see that $M_z g\in L_w^1(\mathbb{R}^d)$ and $\left\|M_z g\right\|_{1,w}=\left\|g\right\|_{1,w}$. Let $c=\left(z_1\sin\alpha_1,\dots,z_d\sin\alpha_d\right)$ and $a=\left(z_1\cos\alpha_1,\dots,z_d\cos\alpha_d\right)$ for all $z=\left(z_1,\dots,z_d\right)\in\mathbb{R}^d$. By using Proposition 3(2) in Toksoy and Sandıkçı (2015), we may write

$$\begin{aligned}
&F_\alpha\left(M_z g\right)(u) \\
&=e^{\sum_{j=1}^{d}-\frac{i}{2}z_j^2\sin\alpha_j\cos\alpha_j}e^{\sum_{j=1}^{d}iu_jz_j\cos\alpha_j}\mathrm{T}_c F_\alpha g(u) \qquad (12)\\
&=e^{\sum_{j=1}^{d}-\frac{i}{2}z_j^2\sin\alpha_j\cos\alpha_j}M_a T_c F_\alpha g(u).
\end{aligned}$$

Since $S_w^\Theta(\mathbb{R}^d)$ is a solid space then it is strongly character invaryant by Lemma 2.4 in (Doğan and Gürkanlı, 2000). Also the space $S_w^\Theta(\mathbb{R}^d)$ is translation invaryant. Then we have

$$e^{\sum_{j=1}^{d}-\frac{i}{2}z_j^2\sin\alpha_j\cos\alpha_j}M_a T_c F_\alpha g\in S_w^\Theta(\mathbb{R}^d).$$

By using equality (12) we get $F_\alpha\left(M_z g\right)\in S_w^\Theta(\mathbb{R}^d)$ and

$$\begin{aligned}
\left\|F_\alpha\left(M_z g\right)\right\|_{S_w^\Theta}&=\left\|e^{\sum_{j=1}^{d}-\frac{i}{2}z_j^2\sin\alpha_j\cos\alpha_j}M_a T_c F_\alpha g\right\|_{S_w^\Theta} \\
&=\left|e^{\sum_{j=1}^{d}-\frac{i}{2}z_j^2\sin\alpha_j\cos\alpha_j}\right|\left\|M_a T_c F_\alpha g\right\|_{S_w^\Theta} \\
&=\left\|T_c F_\alpha g\right\|_{S_w^\Theta} \\
&\le w(c)\left\|F_\alpha g\right\|_{S_w^\Theta}.
\end{aligned}$$

This means that $S_w^\alpha(\mathbb{R}^d)$ is character invariant. Also we may write the inequality

$$\left\|M_z g\right\|_{S_w^\alpha}\le\left\|g\right\|_{1,w}+w(c)\left\|F_\alpha g\right\|_{S_w^\Theta}.$$

ii) This is analogous the proof of Theorem 2.10 (ii).

Example 2.13. Let w and ω be weight functions on $\mathbb{R}^d$ and $1 \le p < \infty$. Let us take the space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ consist of all $f \in L_w^1(\mathbb{R}^d)$ such that $F_\alpha f \in L_\omega^p(\mathbb{R}^d)$ (Toksoy and Sandıkçı, 2015). If $\omega(x) = w(x)$ for all $x \in \mathbb{R}^d$ and $p = 1$, then denote by B the space of all $f \in L_w^1(\mathbb{R}^d)$ such that $F_\alpha f \in L_w^1(\mathbb{R}^d)$. This space is a $S_w^\alpha(\mathbb{R}^d)$ space with sum norm

$$\| f \|_B = \| f \|_{1,w} + \| F_\alpha f \|_{1,w}$$

for all $f \in B$. If we take $\alpha_i = \frac{\pi}{2}$ for all $1 \le i \le d$, then the space B is a $S_{w,w}(\mathbb{R}^d)$ space that define in (Doğan and Gürkanlı, 2000). Hence the space $S_w^\alpha(\mathbb{R}^d)$ is a generalization of the space $S_{w,w}(\mathbb{R}^d)$.

3. Inclusion Properties of The Space $S_w^\alpha(\mathbb{R}^d)$

Proposition 3.1. Let w be a weight functions on $\mathbb{R}^d$ and $b = (y_1 \cos\alpha_1, ..., y_d \cos\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. If $w(b) \le w(y)$, then there exists $c(g) > 0$ such that

$$c(g) w(y) \le \| T_y g \|_{S_w^\alpha} \le w(y) \| g \|_{S_w^\alpha}$$

for all $0 \ne g \in S_w^\alpha(\mathbb{R}^d)$.

Proof. Let $0 \ne g \in S_w^\alpha(\mathbb{R}^d)$. Thus there exists $c(g) > 0$ such that

$$c(g) w(y) \le \| T_y g \|_{1,w} \le w(y) \| g \|_{1,w} \tag{13}$$

by (Fischer vd., 1996). Let $w(b) \le w(y)$ such that $b = (y_1 \cos\alpha_1, ..., y_d \cos\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. By using (9), we get

$$\| T_y g \|_{S_w^\alpha} \le w(y) \| g \|_{S_w^\alpha}. \tag{14}$$

Combinning (13) and (14) we obtain

$$c(g) w(y) \le \| T_y g \|_{S_w^\alpha} \le w(y) \| g \|_{S_w^\alpha}.$$

Lemma 3.2. Let w_1 and w_2 be weight functions on $\mathbb{R}^d$. If $S_{w_1}^\alpha(\mathbb{R}^d) \subset S_{w_2}^\alpha(\mathbb{R}^d)$, then $S_{w_1}^\alpha(\mathbb{R}^d)$ is a Banach space under the norm $|||\cdot||| = \|\cdot\|_{S_{w_1}^\alpha} + \|\cdot\|_{S_{w_2}^\alpha}$.

Proof. Let $(g_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\left(S_{w_1}^\alpha(\mathbb{R}^d), |||\cdot|||\right)$. Thus $(g_n)_{n \in \mathbb{N}}$ is also a Cauchy sequence in $\left(S_{w_1}^\alpha(\mathbb{R}^d), \|\cdot\|_{S_{w_1}^\alpha}\right)$ and $\left(S_{w_2}^\alpha(\mathbb{R}^d), \|\cdot\|_{S_{w_2}^\alpha}\right)$. Since these spaces are Banach spaces, then there exists $g \in S_{w_1}^\alpha(\mathbb{R}^d)$ and $h \in S_{w_2}^\alpha(\mathbb{R}^d)$ such that $\| g_n - g \|_{S_{w_1}^\alpha} \to 0$ and $\| g_n - h \|_{S_{w_2}^\alpha} \to 0$ as $n \to \infty$. By using inequalities

$$\|\cdot\|_1 \le \|\cdot\|_{1,w_1} \le \|\cdot\|_{S_{w_1}^\alpha}$$

and

$$\|\cdot\|_1 \le \|\cdot\|_{1,w_2} \le \|\cdot\|_{S_{w_2}^\alpha}$$

We have $\| g_n - g \|_1 \to 0$ and $\| g_n - h \|_1 \to 0$ as $n \to \infty$. Besides, we may write

$$\| g - h \|_1 \le \| g - g_n \|_1 + \| g_n - h \|_1.$$

Hence we obtain $||| g_n - g ||| \to 0$ as $n \to \infty$. Therefore $\left(S_{w_1}^\alpha(\mathbb{R}^d), |||\cdot|||\right)$ is a Banach space.

Proposition 3.3. Let w_1 and w_2 be weight functions on $\mathbb{R}^d$ and $w_1(b) \le w_1(y)$, $w_2(b) \le w_2(y)$ where

$$b = (y_1 \cos\alpha_1, ..., y_d \cos\alpha_d)$$

for all

$$y = (y_1, ..., y_d) \in \mathbb{R}^d .$$

If $S_{w_1}^\alpha(\mathbb{R}^d) \subset S_{w_2}^\alpha(\mathbb{R}^d)$ then $w_2 \prec w_1$.

Proof. Assume that $S_{w_1}^\alpha(\mathbb{R}^d) \subset S_{w_2}^\alpha(\mathbb{R}^d)$. Then $g \in S_{w_2}^\alpha(\mathbb{R}^d)$ for all $g \in S_{w_1}^\alpha(\mathbb{R}^d)$. Let $w_1(b) \le w_1(y)$ and $w_2(b) \le w_2(y)$ where

$$b = (y_1 \cos\alpha_1, ..., y_d \cos\alpha_d)$$

for all

$$y = (y_1, ..., y_d) \in \mathbb{R}^d .$$

Thus, by Proposition 3.1, there exists constants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 w_1(y) \le \|T_y g\|_{S_{w_1}^\alpha} \le c_2 w_1(y) \tag{15}$$

and

$$c_3 w_2(y) \le \|T_y g\|_{S_{w_2}^\alpha} \le c_4 w_2(y). \tag{16}$$

It is shown that the space $\left(S_{w_1}^\alpha(\mathbb{R}^d), \|\!|\cdot|\!\|\right)$ is a Banach space by Lemma 3.2. Then by closed graph theorem, there exists $c > 0$ such that

$$\|T_y g\|_{S_{w_2}^\alpha} \le c \|T_y g\|_{S_{w_1}^\alpha} . \tag{17}$$

Hence, combining (15), (16) and (17) we obtain

$$c_3 w_2(y) \le \|T_y g\|_{S_{w_2}^\alpha} \le c \|T_y g\|_{S_{w_1}^\alpha} \le c c_2 w_1(y).$$

Therefore, $w_2(y) \le \frac{c c_2}{c_3} w_1(y)$. If we take $\frac{c c_2}{c_3} = k$, then we have

$$w_2(y) \le k w_1(y)$$

for all $y \in \mathbb{R}^d$. This means $w_2 \prec w_1$.

4. References

Almeida, L. B. 1994. “The fractional Fourier transform and time-frequency representations”, *IEEE Transactions on Signal Processing*, 42(11), 3084-3091.

Almeida, L. B. 1997. “Product and convolution theorems for the fractional Fourier transform”, *IEEE Signal Processing Letters*, 4(1), 15-17.

Bultheel, A. and Martinez, H. 2002. “A shattered survey of the fractional Fourier transform”, *Department of Computer Science*, K.U.Leuveven, Report TW337.

Cigler, J. 1969. “Normed ideals in $L^1(G)$”, *Indagationes Mathematicae*, 72(3), 273-282.

Doğan, M. and Gürkanlı, A. T. 2000. “On functions with Fourier transforms in S_ω”, *Bulletin of Calcutta Mathematical Society*, 92(2), 111-120.

Feichtinger, H. G. 1977. “On a class of convolution algebras of functions”, *Annales de l’institut Fourier*, 27(3), 135-162.

Feichtinger, H. G., Graham, C. and Lakien, E. 1979. “Nonfactorization in commutative, weakly selfadjoint Banach algebras”, *Pacific Journal of Mathematics*, 80(1), 117-125.

Feichtinger, H. G. ve Gürkanlı A. T. 1990. “On a family of weighted convolution algebras”, *International Journal of Mathematics and Mathematical Sciences*, 13(3), 517-525.

Fischer, R. H., Gürkanlı, A. T. and Liu, T. S. 1996. “On a family of weighted spaces”, *Mathematica Slovaca*, 46(1), 71-82.

Namias, V. 1980. “The fractional order of Fourier transform and its application in quantum mechanics”, *Journal of the Institute of Mathematics and its Applications,* 25, 241-265.

Ozaktas, H. M., Kutay, M. A. and Zalevsky, Z. 2001. “The fractional Fourier transform with applications in optics and signal processing”, *John Wiley and Sons*, England.

Reiter, H. and Stegeman, J. D. 2000. “Classical harmonic analysis and locally compact group”, *Clarendon Press*, Oxford.

Reiter, H. 1971. “L^1-Algebras and Segal Algebras”, *Springer-Verlag*, New York.

Toksoy, E. and Sandıkçı, A. 2015. “On function spaces with fractional Fourier transform in weighted Lebesgue spaces”, *Journal of Inequalities and Applications*, 2015(1), 87.

Wang, H. C. 1977. “Homogeneous Banach Algebras”, *Marcel Dekker Inc.*, New York and Basel.