Gazi University Journal of Science
GU J Sci
25(3):661-664 (2012)

ORIGINAL ARTICLE

Complete Lifts of Structures on Manifold to Vector Subbundle

Mustafa ÖZKAN[1,♠]

[1]*Gazi University, Faculty of Sciences, Department of Mathematics, 06500 Teknikokullar, Ankara, Turkey*

Recieived: 10.04.2012 Accepted: 21.06.2012

ABSTRACT

In this study, we studied the complete lift of almost product, almost complex, almost contact and almost paracontact structures on a given manifold M to vector subbundle $\tilde{E}$ of vector bundle (E,π,M). Moreover, we studied a correspondence between the integrability conditions of these structures on M and $\tilde{E}$.

Key words: Complete lift, Almost product structure, Almost complex structure, Almost contact structure, Almost paracontact structure, Integrability.

1. INTRODUCTION

The lift of geometrical objects, functions, vector fields, $1-$forms etc., defined on any manifold has an important role in differential geometry. Thus, we can generalize to differentiable structures on any manifold to its extensions. Recently, vertical and complete lifts of functions, vector fields, $1-$forms, connection and other tensor fields defined on any manifold M to vector subbundle $\tilde{E}$ of vector bundle (E,π,M) has been obtained by Özkan [2], Özkan and Esin [3], Yıldırım and Esin [11]. Lifts of almost product, almost complex, almost contact and almost paracontact structures on any manifold M to its tangent bundle TM are studied by Omran, Sharffuddin and Husain [1], Sasaki [5], Yano and Ishihara [8, 9], Yano and Kobayashi [10].

The complete lift of $F-$structure on any manifold M to vector subbundle $\tilde{E}$ is studied by Özkan and Keçilioğlu [4]. In the present paper, we have studied the complete lifts of almost product, almost complex, almost contact and almost paracontact structure to vector subbundle $\tilde{E}$ of vector bundle (E,π,M). Also, we obtained integrability conditions in the vector subbundle $\tilde{E}$.

♠Corresponding author, e-mail: ozkanm@gazi.edu.tr

2. PRELIMINARIES

Let M be an $m-$dimensional differentiable manifold of class C^∞, (TM,π_M,M) its tangent bundle and (E,π,M) vector bundle with the base space M. Let (U,x^i) be a local coordinate system on M and $(\pi^{-1}(U),x^i,v^a)$ be induced local coordinate system on E defined by $x^i(u)=x^i(\pi(u))$ and $u=v^a(u)\rho_a$ for all $u\in\pi^{-1}(U)$, where $1\le i\le m$, $1\le a\le n$ and $\rho_1,\dots,\rho_n$ are adapted sections. In particular for $E=TM$, we consider induced local coordinate system $(\pi_M^{-1}(U),x^i,y^i)$ on TM as $x^i=x^i(\pi_M(u))$ and $y^i(u)=dx^i(u)$ for all $u\in\pi^{-1}(U)$.

Let $\dim E=m+n$ and $n\ge m$. We denote a surjective vector bundle morphism ${}^0I:E\to TM$ where $\pi_M\circ{}^0I=\pi$. Then there are the components $B_a^i(p)$ such that ${}^0I(u)=B_a^i(p)u^a\frac{\partial}{\partial x^i}\Big|_p$ (if $u\in E_p=\pi^{-1}(p)$, $p\in M$) for any $u\in E$. Thus ${}^0I(x^i,v^a)=(x^i,B_a^iv^a)$ and its clear that $rank[B_a^i(p)]=m$ [11].

Let $\tilde{E}$ be a vector subbundle of vector bundle E such that $T\tilde{E}=Sp\{\delta_i,P_s:1\le i\le m,1\le s\le m\}\subset TE$ and $T^*\tilde{E}=Sp\{dx^i,P_s^*:1\le i\le m,1\le s\le m\}\subset T^*E$ where $\delta_i=\frac{\partial}{\partial x^i}-q_j^s\Gamma_i^jP_s$, $P_s=B_a^s\frac{\partial}{\partial v^a}$, $P_s^*=q_j^sB_a^jdv^a$, $\Gamma_i^j=\frac{\partial B_a^j}{\partial x^i}v^a$, $[q_i^s]=(BB^T)^{-1}$ and $[\tilde{q}_i^t]=BB^T$ [4].

Let X, ω and F be a vector field, a 1-form and a tensor field of type $(1,1)$ in M, respectively. We denote respectively by X^v, ω^v and F^v their vertical lifts to vector bundle $\tilde{E}$ and by X^c, ω^c and F^c their complete lifts to vector bundle $\tilde{E}$. Then we know from [2, 3, 4, 11] that:

$$\begin{aligned}&\omega^v(X^v)=0,\omega^v(X^c)=(\omega(X))^v,\\&\omega^cX^v=(\omega X)^v,\omega^cX^c=(\omega X)^c,\\&F^vX^v=0,F^vX^c=(FX)^v,\\&F^cX^v=(FX)^v,F^cX^c=(FX)^c,\\&[X^v,Y^v]=0,[X^v,Y^c]=[X,Y]^v,\\&[X^c,Y^v]=[X,Y]^v,[X^c,Y^c]=[X,Y]^c\end{aligned}\tag{1}$$

and

$$\begin{aligned}&(F+G)^v=F^v+G^v,\\&(F\otimes G)^v=F^v\otimes G^v,\\&(F+G)^c=F^c+G^c,\\&(F\otimes G)^c=F^c\otimes G^v+F^v\otimes G^c.\end{aligned}\tag{2}$$

where $X\in\Im_0^1(M)$, $\omega\in\Im_1^0(M)$, $F,G\in\Im_1^1(M)$.

Theorem 2.1. ([4]) *If $P(t)$ is a polynomial in one variable t, then*

$$P(F^c)=(P(F))^c\tag{3}$$

for any $F\in\Im_1^1(M)$.

Let $F,G\in\Im_1^1(M)$. Then the torsion tensor $N_{F,G}$ of F and G is, by definition, a tensor field of type $(1,2)$ given by

$$\begin{aligned}2N_{F,G}(X,Y)=&[FX,GY]+[GX,FY]\\&+(FG+GF)[X,Y]\\&-F[GX,Y]-F[X,GY]\\&-G[FX,Y]-G[X,FY]\end{aligned}$$

where X and Y are arbitrary vector fields in M [9]. For this tensor we have

Theorem 2.2. ([4]) *Let $F,G\in\Im_1^1(M)$. Then* $(N_{F,G})^c=N_{F^c,G^c}$.

Let $F\in\Im_1^1(M)$. Then the Nijenhuis tensor N_F of F is a tensor field of type $(1,2)$ defined by

$N_F=N_{F,F}$

that is,

$$\begin{aligned}N_F(X,Y)=&[FX,FY]-F[FX,Y]\\&-F[X,FY]+F^2[X,Y]\end{aligned}$$

for any $X,Y\in\Im_0^1(M)$. As a corollary to Theorem 2.2, we have

Theorem 2.3. *For any $F\in\Im_1^1(M)$*

$$(N_F)^c=N_{F^c}\tag{4}$$

3. COMPLETE LIFT OF ALMOST PRODUCT STRUCTURE

Let M be an m-dimensional differentiable manifold of class C^∞. $F\in\Im_1^1(M)$ satisfies the condition $F^2-I=0$ where I denotes the unit tensor, we say that F defines an almost product structure on M. We know that the integrability of F is equivalent to the vanishing of the Nijenhuis tensor N_F [7].

From (3), we have

$(F^2-I)^c=(F^c)^2-I$.

Thus $F^2-I=0$ if and only if $(F^c)^2-I=0$. It shows that if F is an almost product structure in M then F^c is an almost product structure in $\tilde{E}$. Moreover, by (4), we get that F^c is integrable in $\tilde{E}$ if and only if F is integrable in M. Thus we have

Theorem 3.1. *For $F\in\Im_1^1(M)$, F^c defines an almost product structure on $\tilde{E}$, if and only if so does F on M. Moreover, F^c is integrable in $\tilde{E}$, if and only if so is F in M.*

4. COMPLETE LIFT OF ALMOST COMPLEX STRUCTURE

Let F be a tensor field of type $(1,1)$ on an m-dimensional differentiable manifold M such that

$$F^2+I=0\tag{5}$$

where I denotes the unit tensor. Such a structure on M is called an almost complex structure on M which defined by F. We say that F is integrable if the Nijenhuis tensor of F is identically equal to zero [9].

Taking the complete lifts of both sides of (5), by (3), we get

$(F^c)^2+I=0$.

If F is an almost complex structure in M, then F^c is also an almost complex structure in $\tilde{E}$. Also, by (4), we get that F^C is integrable in $\tilde{E}$ if and only if F is integrable in M. Thus we have the following:

Theorem 4.1. *For $F \in \Im_1^1(M)$, F^c defines an almost complex structure on $\tilde{E}$, if and only if so does F on M. F^c is integrable in $\tilde{E}$, if and only if so is F in M.*

Let there be given a distribution T of dimension r in M. Suppose that T is determined by a projection tensor t, i.e., t is an element of $\Im_1^1(M)$ such that $t^2 = t$, $t(TM) = T$ and t is of rank r [9].

Taking account of $t^2 = t$ and (3), we obtain $(t^c)^2 = t^c$, i.e., the complete lift t^c of the projection tensor t is a projection tensor in $\tilde{E}$. By [4], the t^c is of rank $2r$, since t is of rank r. The $2r$-dimensional distribution T^c in $\tilde{E}$ determined by the projection tensor t^c is called the complete lift of the distribution T.

The distribution T is integrable if and only if we get

$$s[tX, tY] = 0 \qquad (6)$$

for any $X, Y \in \Im_0^1(M)$, where $s = I - t$ is the projection tensor complementary to t [9]. Hence from (1) and (6) we obtain

$$s^c[t^cX^c, t^cY^c] = 0 \qquad (7)$$

for any $X, Y \in \Im_0^1(M)$, where $s^c = (I-t)^c = I - t^c$ is the projection tensor complementary to t^c. Thus the conditions (6) and (7) are equivalent to each other. Consequently we get

Theorem 4.2. *The complete lift T^c of a distribution T in $\tilde{E}$ is integrable if and only if so is T in M.*

5. COMPLETE LIFT OF ALMOST CONTACT STRUCTURE

Let M be an m-dimensional differentiable manifold of class C^∞ equipped with a tensor field F of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$F^2 = -I + \eta \otimes \xi \quad , \quad F\xi = 0, \qquad \eta \circ F = 0 \quad , \quad \eta(\xi) = 1. \qquad (8)$$

where I denotes the unit tensor. Then (F, ξ, η) is called an almost contact structure on M and m is necessarily odd [5, 6, 9].

Taking the complete and vertical lifts of (8), from (1) and (2) we get

$$\begin{aligned}
&(F^c)^2 = -I + \eta^v \otimes \xi^c + \eta^c \otimes \xi^v \\
&F^c\xi^v = 0, F^v\xi^c = 0, F^c\xi^c = 0, \\
&\eta^v \circ F^c = 0, \eta^c \circ F^v = 0, \\
&\eta^c \circ F^c = 0, \eta^v(\xi^c) = 1, \\
&\eta^c(\xi^v) = 1, \eta^c(\xi^c) = 0.
\end{aligned} \qquad (9)$$

We now define a $(1,1)$ tensor field $\tilde{F}$ on $\tilde{E}$ by

$$\tilde{F} = F^c + \eta^v \otimes \xi^v - \eta^c \otimes \xi^c. \qquad (10)$$

From (9) and (10), we have

$$\tilde{F}^2 X^v = -X^v \quad , \quad \tilde{F}^2 X^c = -X^c.$$

Thus $\tilde{F}$ is an almost complex structure in $\tilde{E}$.

Theorem 5.1. *If an almost contact structure (F, ξ, η) is given in M defined by (8), then there exist in $\tilde{E}$ an almost complex structure $\tilde{F}$ defined by (10).*

We obtained from (10)

$$\tilde{F}X^v = (FX)^v - (\eta X)^v \xi^c$$
$$\tilde{F}X^c = (FX)^c + (\eta X)^v \xi^v - (\eta X)^c \xi^c.$$

for any $X \in \Im_0^1(M)$.

6. COMPLETE LIFT OF ALMOST PARACONTACT STRUCTURE

Let M be an m-dimensional differentiable manifold of class C^∞ equipped with a tensor field F of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$F^2 = I - \eta \otimes \xi \quad , \quad F\xi = 0, \qquad \eta \circ F = 0 \quad , \quad \eta(\xi) = 1 \qquad (11)$$

where I denotes the unit tensor. Then (F, ξ, η) is called an almost paracontact structure on M [1].

Taking the complete and vertical lifts of both sides of (11), by (1) and (2) we obtain

$$\begin{aligned}
&(F^c)^2 = I - \eta^v \otimes \xi^c - \eta^c \otimes \xi^v \\
&F^c\xi^v = 0, F^v\xi^c = 0, F^c\xi^c = 0, \\
&\eta^v \circ F^c = 0, \eta^c \circ F^v = 0, \\
&\eta^c \circ F^c = 0, \eta^v(\xi^c) = 1, \\
&\eta^c(\xi^v) = 1, \eta^c(\xi^c) = 0.
\end{aligned} \qquad (12)$$

We now define a $(1,1)$ tensor field $\tilde{F}$ on $\tilde{E}$ by

$$\tilde{F} = F^c + \eta^v \otimes \xi^v + \eta^c \otimes \xi^c. \qquad (13)$$

From (12) and (13), we have

$$\tilde{F}^2 X^v = X^v \quad , \quad \tilde{F}^2 X^c = X^c.$$

Thus $\tilde{F}$ is an almost product structure in $\tilde{E}$.

Theorem 6.1. *If an almost paracontact structure (F, ξ, η) is given in M defined by (11), then there exist in $\tilde{E}$ an almost product structure $\tilde{F}$ defined by (13).*

We obtained from (13)

$$\tilde{F}X^v = (FX)^v + (\eta X)^v \xi^c$$
$$\tilde{F}X^c = (FX)^c + (\eta X)^v \xi^v + (\eta X)^c \xi^c.$$

for any $X \in \Im_0^1(M)$.

REFERENCES

[1] Omran, T., Sharffuddin, A., Husain, S.I., "Lifts of structures on manifolds", ***Publications De L'institut Math***., 36 (50), 93-97, (1984).

[2] Özkan, M., "Complete lifts of tensor fields to vector bundle" ***Int.J.Math.Sci.App.***, 2 (2), 593-599, (2012).

[3] Özkan, M., Esin, E., "Vertical lifts of tensor fields to vector bundle", ***Algebras Groups Geom***., 25(1), 21-30, (2008)

[4] Özkan, M., Keçilioğlu, O., "On complete lift of a tensor field of type (1,1) to vector bundle" (submitted for publication).

[5] Sasaki, S., "On differentiable manifolds with certain structures which are closely related to almost contact structures I", ***Tohoku Math. Jour***., 12, 459-476, (1960).

[6] Sasaki, S., Hatakeyama, Y., "On differentiable manifolds with certain structures which are closely related to almost contact structures II", ***Tohoku Math. Jour***., 13, 281-294, (1961).

[7] Yano, K., Differential geometry on complex and almost complex spaces, New York, ***Pergamon Press***, (1965).

[8] Yano, K., Ishihara, S., "Almost complex structures induced in tangent bundles", ***Kodai Math. Sem. Rep.***, 19, 1-27, (1967).

[9] Yano, K., Ishihara, S., Tangent and cotangent bundle, ***Marcel Dekker Inc***., New York, (1973).

[10] Yano, K., Kobayashi, S., "Prolongations of tensor fields and connections to tangent bundles I. General theory", ***J. Math. Soc. Japan***, 18, 194-210, (1966)

[11] Yıldırım, M., Esin, E., "Vertical and complete lifts to vector bundle", ***Algebras Groups Geom***., 24(1), 75-84, (2007).