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Common Fixed Point Theorem for Three Mappings in Banach Valued Norm Spaces

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ABSTRACT

In this paper, we give a generalized theorem on point of coincidence and common fixed point for three weakly compatible mappings in Banach valued norm spaces. We give a new method for construction of the sequence, which is convergence to the common fixed point of these three mappings.

Key words: Banach valued normed spaces, common fixed point, weakly compatible mappings

1. INTRODUCTION

Fixed point theorems are irrevocable in the theory of nonlinear analysis. In this direction, one of the initial and crucial results is the Banach Contraction Mapping Principle [1]. Banach [1] proved that every contraction in complete metric space has a unique fixed point. After this pivotal result, theory of fixed point theorems has been studied by many authors in many directions. In some papers, authors define new contractions and discuss the existence and uniqueness of fixed point for such mappings. Instead of metric spaces, some authors investigate fixed point theorems on various spaces, such as, Banach valued metric spaces, quasi-metric spaces, fuzzy metric spaces, cat(0)-spaces, hyperconvex spaces etc.

The notion of Banach valued metric space, re-introduced by Huang and Zhang [2], is a generalization of a metric space defined by Fréchet [3] in 1906. Banach valued metric space is obtained by replacing the real positive line with an ordered Banach space in the definition of metric. As a matter of fact this spaces re-founded by many authors in the literature (see e.g.[4]–[9]). The concept of Banach valued metric spaces, known also cone metric spaces, have attracted attention of a number of authors after the remarkable paper of Huang and Zhang [2]. In this paper, the authors defined some properties of convergence of sequences and completeness in Banach valued metric spaces,and proved a number of fixed point theorems in the context of Banach valued metric spaces. During the recent years, cone normed spaces and properties of these spaces have been extensively investigated by a number of authors. (see e.g. [10]–[26]).

In this paper, we proved a generalized theorem on point of coincidence and common fixed point for three weakly compatible mappings on a Banach valued normed spaces.

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2. PRELIMINARIES

Definition 2.1 *[2] Let E be a real Banach space with norm $\|\cdot\|$ and P be a subset of E. P is called a cone if and only if the following conditions are satisfied:*

- P is closed, nonempty and $P \neq \{0\}$
- $a, b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P$
- $x \in P$ and $-x \in P \Rightarrow x = 0$

Let $P \subset E$ be a cone, we define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. we write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$ (interior of P). The cone $P \subset E$ is called normal if there is a positive real number k such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq k \leq \|y\|.$$

Throughout this paper, we assume that E is a real Banach space and P is a cone suth that $intP \neq \phi$.

Definition 2.2 *[17] Let X be a vector space over $\mathbb{R}$. Suppose that the mapping $\|\,.\,\|_P : X \to E$ satisfies*

- $\|x\|_P > 0$ for all $x \in X$,
- $\|x\|_P = 0$ if and only if $x = 0$,
- $\|x + y\|_P \leq \|x\|_P + \|y\|_P$ for all $x, y \in X$,
- $\|kx\|_P = |k|\,\|x\|_P$ for all $k \in \mathbb{R}$,

then $\|\,.\,\|_P$ is called cone norm on X, and the pair $(X, \|\,.\,\|_P)$ is called a cone normed space (in brief CNS). Note that each CNS is cone metric space (in brief CMS). Indeed, $d(x, y) = \|x - y\|_P$.

Definition 2.3 *[17] Let $(X, \|\,.\,\|_P)$ be a CNS, let $\{x_n\}$ be a sequence in X and $x \in X$. Then*

- $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x\|_P \ll c$ for all $n > N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;
- $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x_m\|_P \ll c$ for all $n, m > N$;
- $(X, \|\,.\,\|_P)$ is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called *cone Banach spaces*.

Lemma 2.4 *[17] Let $(X, \|\,.\,\|_P)$ be a CNS, P a* normal cone with normal constant M*, and $\{x_n\}$ a sequence in X. Then,*

- the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_P \to 0$, as $n \to \infty$;
- the sequence $\{x_n\}$ is Cauchy if and only if $\|x_n - x_m\|_P \to 0$ as $n, m \to \infty$;
- the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y then

$$\|x_n - y_n\|_P \to \|x - y\|_P.$$

From now on, we assume that P is a normal cone with $intP \neq \phi$.

Lemma 2.5 *Suppose that $(X, \|\,.\,\|_P)$ is a cone Banach space and $d : X \times X \to E$ is such that $d(x, y) = \|x - y\|_P$. Let $\{y_n\}$ be a sequence in $(X, \|\,.\,\|_P)$ such that $d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)$*

for some $0 < \lambda < 1$ and all $n \in \mathbb{N}$. Then $\{y_n\}$ is a Cauchy sequence in $(X, \|\,.\,\|_P)$.

Definition 2.6 *[14] Let S, T be self-mappings on a cone metric space (X,d). A point $z \in X$ is called a coincidence point of S, T if $Sz = Tz$, and it is called a common fixed point of S, T if $Sz = z = Tz$. Moreover, a pair of self-mappings (S,T) is called weakly compatible on X if they commute at their coincidence points, in other words*

$$z \in X, \quad Sz = Tz \Rightarrow STz = TSz$$

3. MAIN RESULTS

Theorem 3.1 *Let C be a subset of a cone Banach space $(X, \|.\|_P)$ and $d : X \times X \to E$ is such that $d(x,y) = \|x-y\|_P$. Suppose that $F,G,T : C \to C$ be three mappings which satisfy the following conditions*

- FC is closed and convex and $TC \cup GC \subseteq FC$,

$$d(Gx,Ty) \le pd(Fx,Fy) + \frac{q}{4}(d(Fx,Gx)+d(Fy,Ty)) + \frac{r}{4}(d(Fx,Ty)+d(Fy,Gx)) \quad (1)$$

- The pairs (F,G) and (F,T) are weakly compatible.

for all $x,y \in C$, where $p,q,r \ge 0$ and $p+q+r<1$. Then F, G and T have a unique common fixed point.

Proof. Let $x_0 \in C$ be arbitrary. we define a sequence $\{y_n\}$ in the following relation:

$$y_{2n} = Fx_{2n} := \frac{Fx_{2n-1}+Gx_{2n-1}}{2}, \quad n = 1,2,\cdots \quad (2)$$

and

$$y_{2n-1} = Fx_{2n-1} := \frac{Fx_{2n-2}+Tx_{2n-2}}{2}, \quad n = 1,2,\cdots \quad (3)$$

We see that

$$Fx_{2n} - Fx_{2n+1} = \frac{Fx_{2n-1}-Fx_{2n}}{2} + \frac{Gx_{2n-1}-Tx_{2n}}{2},$$

which implies that

$$\|Fx_{2n} - Fx_{2n+1}\|_P \le \left\|\frac{Fx_{2n-1}-Fx_{2n}}{2}\right\|_P + \left\|\frac{Gx_{2n-1}-Tx_{2n}}{2}\right\|_P,$$

then we get

$$2d(y_{2n},y_{2n+1}) - d(y_{2n-1},y_{2n}) \le d(Gx_{2n-1},Tx_{2n}). \quad (4)$$

Also we have

$$Fx_{2n-1} - Gx_{2n-1} = 2(Fx_{2n-1} - (\frac{Fx_{2n-1}+Gx_{2n-1}}{2})) = 2(Fx_{2n-1} - Fx_{2n}).$$

Then

$$d(Fx_{2n-1},Gx_{2n-1}) = 2d(y_{2n-1},y_{2n}). \quad (5)$$

Also

$$Fx_{2n} - Gx_{2n-1} = \frac{Fx_{2n-1}+Gx_{2n-1}}{2} - Gx_{n-1} = \frac{1}{2}(Fx_{2n-1} - Gx_{2n-1}),$$

so we have

$$d(Fx_{2n},Gx_{2n-1}) = \frac{1}{2}d(Fx_{2n-1},Gx_{2n-1}). \quad (6)$$

It follows from (5) and (6) that

$$d(Fx_{2n},Gx_{2n-1}) = d(y_{2n-1},y_{2n}). \quad (7)$$

Also we observe that

$$Fx_{2n} - Tx_{2n} = 2(Fx_{2n} - (\frac{Fx_{2n}+Tx_{2n}}{2})) = 2(Fx_{2n} - Fx_{2n+1}),$$

hence

$$d(Fx_{2n},Tx_{2n}) = 2d(y_{2n},y_{2n+1}). \quad (8)$$

By substituting $x = x_{2n-1}$ and $y = x_{2n}$ in (1) and using triangle inequality, we obtain that

$$d(Gx_{2n-1}, Tx_{2n}) \leq pd(Fx_{2n-1}, Fx_{2n})$$
$$+\frac{q}{4}(d(Fx_{2n-1}, Gx_{2n-1}) + d(Fx_{2n}, Tx_{2n})) \quad (9)$$
$$+\frac{r}{4}(d(Fx_{2n-1}, Fx_{2n}) + d(Fx_{2n}, Tx_{2n})$$
$$+d(Fx_{2n}, Gx_{2n-1})).$$

It follows from (4), (5), (7), (8) and (9) that

$$2d(y_{2n}, y_{2n+1}) - d(y_{2n-1}, y_{2n}) \leq pd(y_{2n-1}, y_{2n})$$
$$+\frac{q}{4}(2d(y_{2n-1}, y_{2n}) + 2d(y_{2n}, y_{2n+1}))$$
$$+\frac{r}{4}(d(y_{2n-1}, y_{2n}) + 2d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})),$$

which implies that

$$d(y_{2n}, y_{2n+1}) \leq \frac{1 + p + \frac{q}{2} + \frac{r}{2}}{2 - \frac{q}{2} - \frac{r}{2}} d(y_{2n-1}, y_{2n}). \quad (10)$$

On the other hand,

$$Gx_{2n+1} = 2Fx_{2n+2} - Fx_{2n+1},$$

so

$$Gx_{2n+1} - Fx_{2n} = (Fx_{2n+2} - Fx_{2n+1}) + (Fx_{2n+2} - Fx_{2n+1}) + (Fx_{2n+1} - Fx_{2n}),$$

then

$$\|Gx_{2n+1} - Fx_{2n}\|_P \leq \|Fx_{2n+2} - Fx_{2n+1}\|_P + \|Fx_{2n+2} - Fx_{2n+1}\|_P + \|Fx_{2n+1} - Fx_{2n}\|_P$$

which implies that

$$d(Gx_{2n+1}, Fx_{2n}) \leq 2d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n}) \quad (11)$$

By the Similar method as the proof of (4), it can be shown that

$$2d(y_{2n+2}, y_{2n+1}) - d(y_{2n+1}, y_{2n}) \leq d(Gx_{2n+1}, Tx_{2n}), \quad (12)$$

Also, it is easy to show that,

$$d(Fx_{2n+1}, Tx_{2n}) = \frac{1}{2} d(Fx_{2n}, Tx_{2n}). \quad (13)$$

By using (8) and (13) we can conclude that

$$d(Fx_{2n+1}, Tx_{2n}) = d(y_{2n}, y_{2n+1}).$$

Now, replacing x and y by x_{2n+1} and x_{2n} in (1), respectively, we get

$$d(Gx_{2n+1}, Tx_{2n}) \leq pd(Fx_{2n+1}, Fx_{2n})$$
$$+\frac{q}{4}(d(Fx_{2n+1}, Gx_{2n+1}) + d(Fx_{2n}, Tx_{2n})) \quad (15)$$
$$+\frac{r}{4}(d(Fx_{2n+1}, Tx_{2n}) + d(Gx_{2n+1}, Fx_{2n}))$$

then by using (5), (8), (11), (12), (14) and (15), we have

$$2d(y_{2n+2}, y_{2n+1}) - d(y_{2n+1}, y_{2n}) \leq pd(y_{2n}, y_{2n+1})$$
$$+\frac{q}{4}(2d(y_{2n+1}, y_{2n+2}) + 2d(y_{2n}, y_{2n+1}))$$
$$+\frac{r}{4}(d(y_{2n}, y_{2n+1}) + 2d(y_{2n+2}, y_{2n+1}) + d(y_{2n}, y_{2n+1})).$$

Hence

$$d(y_{2n+2}, y_{2n+1}) \leq \frac{1 + p + \frac{q}{2} + \frac{r}{2}}{2 - \frac{q}{2} - \frac{r}{2}} d(y_{2n}, y_{2n+1}). \quad (16)$$

Therefore, it follows from (10) and (16) that

$$d(y_{n+2}, y_{n+1}) \leq \frac{1 + p + \frac{q}{2} + \frac{r}{2}}{2 - \frac{q}{2} - \frac{r}{2}} d(y_{n+1}, y_n), \qquad \forall n \in \mathbb{N}$$

where, $\frac{1 + p + \frac{q}{2} + \frac{r}{2}}{2 - \frac{q}{2} - \frac{r}{2}} < 1$. Hence by Lemma 2.5, $\{y_n\}$ is a Cauchy sequence in FC. Since FC is closed, there exist $z \in C$ such that $y_n \to Fz$. Suppose that $w = Fz$, then by (2) and (3) we obtain, $Gx_{2n-1} \to w$ and $Tx_{2n} \to w$.

Now, by using (1), we get

$$\begin{aligned} d(Gz,w) &\le d(Gz,Tx_{2n}) + d(Tx_{2n},w) \\ &\le pd(Fz,Fx_{2n}) + \frac{q}{4}(d(Fz,Gz) \\ &+d(Fx_{2n},Tx_{2n})) + \frac{r}{4}(d(Fz,Tx_{2n}) \\ &+d(Fx_{2n},Gz)) + d(Tx_{2n},w). \end{aligned} \tag{17}$$

Therefore by taking the limit as $n \to \infty$ in (??), we obtain

$$d(Gz,w) \le pd(w,w) + \frac{q}{4}(d(w,Gz)+d(w,w)) + \frac{r}{4}(d(w,w)+d(w,Gz)) + d(w,w).$$

Thus, $(1-\frac{q}{4}-\frac{r}{4})d(Gz,w) \le 0$. So $Gz = Fz = w$

Similarly, by using (1), we have

$$\begin{aligned} d(w,Tz) &\le d(Gx_{2n-1},w) + d(Gx_{2n-1},Tz) \\ &\le pd(Fx_{2n-1},Fz) + \frac{q}{4}(d(Fx_{2n-1},Gx_{2n-1}) \\ &+d(Fz,Tz)) + \frac{r}{4}(d(Fx_{2n-1},Tz) \\ &+d(Fz,Gx_{2n-1})) + d(Gx_{2n-1},w), \end{aligned} \tag{18}$$

Then, taking the limit as $n \to \infty$ in (18), we obtain

$$d(w,Tz) \le pd(w,w) + \frac{q}{4}(d(w,w)+d(w,Tz)) + \frac{r}{4}(d(w,Tz)+d(w,w)) + d(w,w)$$

Therefore $(1-\frac{q}{4}-\frac{r}{4})d(w,Tz) \le 0$. Hence $Tz = Gz = Fz = w$. So we conclude that z is a point of coincidence of F, G and T.

Since F and G are weakly compatible mappings, then $FGz = GFz$, so $Fw = Gw$.

Now we show that w is a fixed point of G and F. By using (1), we get

$$\begin{aligned} d(Gw,w) = d(Gw,Tz) &\le pd(Fw,Fz) \\ &+\frac{q}{4}(d(Fw,Gw)+d(Fz,Tz)) \\ &+\frac{r}{4}(d(Fw,Tz)+d(Fz,Gw)) \end{aligned}$$

then

$$d(Gw,w) \le pd(Fw,w) + \frac{q}{4}(d(Fw,Gw) + d(w,w)) + \frac{r}{4}(d(Fw,w)+d(w,Gw)).$$

Therefore

$$\begin{aligned} &d(Gw,w) \\ &\le \frac{r}{4}d(w,Gw) + (p+\frac{r}{4})d(Fw,w) \\ &\le \frac{r}{4}d(w,Gw) + (p+\frac{r}{4})(d(Fw,Gw)+d(Gw,w)), \end{aligned}$$

which implies that

$$(1-p-\frac{r}{2})d(Gw,w) \le 0.$$

Then $w = Gw$, Hence, $w = Gw = Fw$.

On the other hand, scince F and T are weakly compatible mappings, we have $Fw = Tw$. So we conclude that $w = Gw = Fw = Tw$.

To prove the uniqueness of w, suppose that w_1 is another common fixed point of G, F and T. Replacing x and y by w_1 and w in (1), respectively, we get

$$\begin{aligned} d(w,w_1) = d(Gw,Tw_1) &\le pd(Fw,Fw_1) \\ &+\frac{q}{4}(d(Fw,Gw)+d(Fw_1,Tw_1)) \\ &+\frac{r}{4}(d(Fw,Tw_1)+d(Fw_1,Gw)) \end{aligned}$$

Therefore

$$(1-p-\frac{r}{2})d(w,w_1)\le 0.$$

So $w=w_1$. Then w is the unique common fixed point of F, G and T.

Corollary 3.2 *Let C be a closed and convex subset of a cone Banach space $(X,\|.\|_P)$ and $d:X\times X\to E$ be such that $d(x,y)=\|x-y\|_P$. Suppose that $T,G:C\to C$ are two mappings which satisfy the condition*

$$d(Gx,Ty)\le pd(x,y)+\frac{q}{4}(d(x,Gx)+d(y,Ty)) + \frac{r}{4}(d(x,Ty)+d(y,Gx)) \qquad (19)$$

for all $x,y\in C$, where $p,q,r\ge 0$ and $p+q+r<1$. Then G and T have a unique common fixed point.

Corollary 3.3 *Let C be a closed and convex subset of a cone Banach space $(X,\|.\|_P)$ and $d:X\times X\to E$ be such that $d(x,y)=\|x-y\|_P$. Suppose that $T:C\to C$ is a mapping which satisfies the condition*

$$d(Tx,Ty)\le pd(x,y)+\frac{q}{4}(d(x,Tx)+d(y,Ty)) + \frac{r}{4}(d(x,Ty)+d(y,Tx)) \qquad (20)$$

for all $x,y\in C$, where $p,q,r\ge 0$ and $p+q+r<1$. Then T has a unique fixed point.

REFERENCES

[1] Banach, S., "Sur les operations dans les ensembles abstraits et leur application aux equations integrales", ***Fund. Math.***, **3**(1922)133–181.

[2] Huang, L. G., Zhang, X., "Cone metric spaces and fixed point theorems of contractive mappings", ***J. Math. Anal. Appl.*** **332** (2007), 1468–1476.

[3] Fréchet, M., "Sur quelques points du calcul fonctionnel", ***Rend. Circ. Mat. Palermo*** 22 (1906) 1--74.

[4] Zabrejko, P. P., "K-metric and K-normed linear spaces: survey", ***Collect. Math***. 48 (1997), 825–859.

[5] Kurepa, D. R., "Tableaux ramifiés d'ensembles. Espace pseudo-distanciés", **Comptes Rendus l'Académie des Sciences**, 198,(1934) 1563–1565.

[6] Krein, M. G., Rutman, M. A., "Linear operators leaving invariant a cone in a Banach space", ***Uspekhi Matematicheskikh Nauk***,3(1),(1948) 3–95.

[7] Kantorovic, L. V., "The principle of the majorant and Newton's method", ***Doklady Akademii Nauk SSSR***, 76(1951) 17–20.

[8] Rzepecki, B., "On fixed point theorems of Maia type", ***Publications De L'institut Mathématique*** **28** (1980), 179--186.

[9] Lin, S. D., "A common fixed point theorem in abstract spaces", ***Indian J. Pure Appl. Math.*** **18** (8) (1987) 685–690.

[10] Abdeljawad, T., Karapnar, E., "Quasi-Cone Metric Spaces and Generalizations of Caristi Kirk's Theorem", ***Fixed Point Theory Appl.*** doi:10.1155/2009/574387 (2009).

[11] Abdeljawad, T., "Completion of cone metric spaces", ***Hacet. J. Math. Stat***., (to appear).

[12] Abdeljawad, T., Karapınar, E., "A Common Fixed Point Theorem of a Gregus Type on Convex Cone Metric Spaces", ***J. Comput. Anal. Appl.*** ,13(4), 609-621(2011).

[13] Abdeljawad, T., Turkoglu, D., Abuloha, M., "Some theorems and examples of cone Banach spaces", ***J. Comput. Anal. Appl.***, 12(4), 739-753 (2010).

[14] Abdeljawad, T., Karapınar, E., Tas, K., "Common fixed point theorems in cone banach spaces", ***Hacet. J. Math. Stat***. **40** (2) (2011), 211–217.

[15] Cvetkovic, A. S., Stanic, M. P., Dimitrijevic, S., Simic, S., "Common Fixed Point Theorems for Four Mappings on Cone Metric Type Space", ***Fixed point Theory Appl.*** 15 pages (2011) Article ID 589725.

[16] Ding, H.-S., Kadelburg, Z., Karapınar, E., Radenovic, S., "Common Fixed Points of Weak Contractions in Cone Metric Spaces", ***Abstr. Appl. Anal.***, 2012 2012 Article id: 793862

[17] Karapinar, E., "Fixed point theorems in cone Banach spaces", ***Fixed Point Theory and Appl***. 9 pages (2009) Article ID 60928.

[18] Karapinar, E., Yuksel, U., "On Common Fixed Point Theorems Without Commuting Conditions in tvs-Cone Metric Spaces", ***J. Comput. Anal. Appl*** ,13(6), 1115-1122 (2011)

[19] E. Karapınar, Some Nonunique Fixed Point Theorems of Ciric type on Cone Metric Spaces, ***Abstr. Appl. Anal.***, vol. 2010, Article ID 123094, 14 pages (2010).

[20] Karapınar, E., "Couple Fixed Point Theorems For Nonlinear Contractions in Cone Metric Spaces", ***Comput. Math. Appl.*** , 59(12), , Pages 3656–3668 (2010).

[21] Karapınar, E., Türkoğlu, D. A., "Best Approximations Theorem For a Couple in Cone Banach Space", ***Fixed Point Theory Appl.***, Volume 2010 (2010), Article ID 784578, 9 pages.

[22] Rezapour, Sh., Hamlbarani, R., "Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"", ***J. Math. Anal. Appl.*** **347** (2008), 719–724.

[23] Sonmez, A., "On paracompactness in cone metric spaces", ***Applied Mathematics Letters*** **23** (2010) 494, 497.

[24] Türkoğlu, D., Abuloha, "Cone metric spaces and fixed point theorems in diametrically contractive mappings", ***Acta. Math. Sin.*** **26** (3) (2010), 489–496.

[25] Türkoğlu, D., Abuloha, M., "Cone Metric Spaces and Fixed Point Theorems in Diametrically Contractive Mappings", ***Acta Mathematica Sinica***, English Series, 26(3), 489-496 (2010).

[26] Turkoglu, D., Abuloha, M., Abdeljawad, T., "KKM mappings in cone metric spaces and some fixed point theorems", ***Nonlinear Analysis: Theory, Methods & Applications,*** **72**(1), 348-353 (2010).