



Fixed Points of Expansion Mappings in Menger Spaces With Property (E.A.)

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ABSTRACT

The aim of this paper is to prove a common fixed point theorem for non-surjective expansion mappings in Menger space employing the property (E.A). Our results improve and generalize several known fixed point theorems existing in the literature.

Keywords: Menger space, non-surjective mappings, weakly compatible mappings, expansion mappings, property (E.A).

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1. INTRODUCTION

Fréchet [3] introduced the concept of metric space in which the notion of distance appears. The study of fixed point theorems satisfying certain contractive conditions has a wide range of applications in different areas such as, variational and linear inequality problems, optimization and parameterize estimation problems and many others. One of the simplest and most useful results in the fixed point theory is the Banach Caccioppoli contraction principle [1]. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Banach contraction principle has been generalized in different spaces by many mathematicians over the years. This natural theorem asserts that every contraction mapping defined on a complete metric space has a unique fixed point and that fixed point can be explicitly obtained as limit of repeated iteration of the mapping at any point of the underlying space. Evidently, every contraction mapping is a continuous but not conversely.

While carrying out measurements, assigning a fixed number to the distance between two points is an over idealized way of thinking. Practically, it will be more appropriate to assign the average of several measurements for the distance between two points. Inspired from this line of thinking, Menger [1] introduced the notion of probabilistic metric space (briefly, PM-space) as a generalization of metric space. Probabilistic contractions were firstly defined and studied by V. M. Sehgal [14]. The study of such spaces received an impetus with the pioneering works of Schweizer and Sklar [13]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis due to its extensive applications in random differential as well as random integral equations. Banach contraction principle [1] also yields a fixed point theorem for a diametrically opposite class of mappings, viz. expansion mappings. The study of metrical fixed point theorem for expansion mapping is initiated by Wang et al. [18]. Since then, Pant et al. [12] studied fixed point theorem for expansion mappings in framework of probabilistic metric spaces as follows:

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Let (X, \mathcal{F}, Δ) be a Menger space. A mapping $T: X \rightarrow X$ will be called an expansion mapping iff for a constant $k > 1$

$$(1.1) \quad F_{Tx, Ty}(kt) \leq F_{x,y}(t),$$

holds for all $x, y \in X$ and $t > 0$.

The interpretation of inequality (1.1) is as follows: The probability that the distance between the image points of Tu, Tv is less than kt is never greater than the probability that the distance between u, v is less than t .

2. PRELIMINARIES

Definition 2.1 [13] A mapping $\Delta: [0,1] \times [0,1] \rightarrow [0,1]$ is called a triangular norm (briefly, t-norm) if the following conditions are satisfied: for all $a, b, c, d \in [0,1]$

- (1) $\Delta(a, 1) = a$ for all $a \in [0,1]$,
- (2) $\Delta(a, b) = \Delta(b, a)$,
- (3) $\Delta(a, b) \leq \Delta(c, d)$ for $a \leq c, b \leq d$,
- (4) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Examples of continuous t-norms are: $\Delta(a, b) = \min\{a, b\}$, $\Delta(a, b) = ab$ and $\Delta(a, b) = \max\{a + b - 1, 0\}$.

Throughout this paper, Δ is considered by $\Delta(a, b) = \min\{a, b\}$, for all $a, b \in [0,1]$.

Definition 2.2 [13] A mapping $F: \mathbb{R} \rightarrow \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t \in \mathbb{R}} F(t) = 0$ and $\sup_{t \in \mathbb{R}} F(t) = 1$.

We shall denote by \mathfrak{F} the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F}: X \times X \rightarrow \mathfrak{F}$ is called a probabilistic distance on X and the value of \mathcal{F} at $(x, y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.3 [13] The ordered pair (X, \mathcal{F}) is called a PM-space if X is a non-empty set and \mathcal{F} is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$

- (1) $F_{x,y}(t) = H(t) \Leftrightarrow x = y$,
- (2) $F_{x,y}(t) = F_{y,x}(t)$,
- (3) $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1 \Rightarrow F_{x,z}(t + s) = 1$.

Definition 2.4 [13] A Menger space is a triplet (X, \mathcal{F}, Δ) where (X, \mathcal{F}) is a PM-space and t-norm Δ is such that the inequality

$$F_{x,z}(t + s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$$

holds for all $x, y, z \in X$ and $t, s > 0$.

Every metric space (X, d) can be realized as a PM-space by taking $\mathcal{F}: X \times X \rightarrow \mathfrak{F}$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$.

Definition 2.5 [13] Let (X, \mathcal{F}, Δ) be a Menger space with continuous t-norm Δ . A sequence $\{x_n\}$ in X is said to be convergent to a point x in X if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F_{x_n, x}(\epsilon) > 1 - \lambda$ for all $n \geq N(\epsilon, \lambda)$.

Definition 2.6 [11] A pair (A, S) of self mappings of a Menger space (X, \mathcal{F}, Δ) are said to be compatible if and only if $F_{ASx_n, SAx_n}(t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \rightarrow z$ for some $z \in X$ as $n \rightarrow \infty$.

Definition 2.7 [5] A pair (A, S) of self mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Sz$ for some $z \in X$, then $ASz = SAz$.

Two compatible self mappings are weakly compatible, but the converse is not true (see [15, Example 1]). Therefore the concept of weak compatibility is more general than compatibility.

Definition 2.8 [6] A pair (A, S) of self mappings of a Menger space (X, \mathcal{F}, Δ) is said to satisfy the property (E.A), if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$.

Lemma 2.1 [11] Let (X, \mathcal{F}, Δ) be a Menger space. If there exists a constant $k \in (0,1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $x, y \in X$ and $t > 0$ then $x = y$.

3. RESULTS

In [9], Kumar and Pant proved the following result:

Theorem 3.1 Let (X, \mathcal{F}, Δ) be a complete Menger space, where $\Delta(a, b) = \min\{a, b\}$ for all $a, b \in [0,1]$. Further, let A, B, S and T be self mappings of X satisfying the following conditions:

- (3.1) the mappings A and B are surjective,
- (3.2) one of A, B, S or T is continuous,
- (3.3) the pairs (A, S) and (B, T) are compatible,
- (3.4) there exists a constant $k > 1$ such that

$$F_{Ax, By}(kt) \leq F_{Sx, Ty}(t),$$

for all $x, y \in X$ and $t > 0$.

Then A, B, S and T have a unique common fixed point in X .

Now we prove our main result:

Theorem 3.2 Let A, B, S and T be four self mappings of a Menger space (X, \mathcal{F}, Δ) satisfying inequality (3.4) of Theorem 3.1. Suppose that

- (3.5) the (A, S) (or (B, T)) enjoys the property (E.A),
- (3.6) $T(X) \subseteq A(X), S(X) \subseteq B(X)$,

(3.7) one of the range of the mappings A, B, S or T is a closed subspace of X .

Then A, B, S and T have a unique common fixed point in X .

Proof. If the pair (B, T) satisfies the property (E.A), then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = z,$$

for some $z \in X$. Since $S(X) \subseteq B(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence, $\lim_{n \rightarrow \infty} Sy_n = z$. Also, since $T(X) \subseteq A(X)$, there exists a sequence $\{y'_n\}$ in X such that $Ay'_n = Tx_n$ and so $\lim_{n \rightarrow \infty} Ay'_n = z$.

Assume that $S(X)$ is a closed subspace of X , then there exists a point $u \in X$ such that $z = Su$. By inequality (3.4), we have

$$F_{Au, Bx_n}(kt) \leq F_{Su, Tx_n}(t).$$

On letting $n \rightarrow \infty$, we get

$$F_{Au, z}(kt) \leq F_{z, z}(t) = 1,$$

for all $t > 0$ and $k > 1$. By Lemma 2.1, we have $Au = z$ and hence $Au = Su = z$.

The weak compatibility of A and S implies that $Az = ASu = SAu = Sz$. Now, we assert that z is a common fixed point of A and S . From inequality (3.4), we have

$$F_{Az, Bx_n}(kt) \leq F_{Sx_n, Tx_n}(t).$$

Taking limit as $n \rightarrow \infty$, we get

$$F_{Az, z}(kt) \leq F_{Sx_n, z}(t).$$

Owing to Lemma 2.1, we have $Az = Sz = z$. On other hand, since $S(X) \subseteq B(X)$, there exists a point $v \in X$ such that $Bv = Su = Au = z$. On using inequality (3.4), we have

$$F_{Au, Bv}(kt) \leq F_{Su, Tv}(t),$$

or, equivalently,

$$F_{z, Bv}(kt) \leq F_{z, Tv}(t),$$

for all $t > 0$. In view of Lemma 2.1, we get $Bv = Tv = z$.

Similarly, the weak compatibility of B and T implies that $Bz = BTv = TBv = Tz$. By inequality (3.4), we have

$$F_{Au, Bz}(kt) \leq F_{Su, Tz}(t),$$

and so

$$F_{z, Bz}(kt) \leq F_{z, Tz}(t).$$

In view of Lemma 2.1, we have $Bz = Tz = z$. Thus in all, we have $Az = Bz = Sz = Tz = z$ which shows that z is a common fixed point of mappings A, B, S and T .

Finally, we prove the uniqueness of z . Let $w (\neq z)$ be another common fixed point of involved mappings A, B, S and T , using (3.4), we have

$$F_{Az, Bw}(kt) \leq F_{Sx_n, Tw}(t),$$

or, equivalently,

$$F_{z, w}(kt) \leq F_{z, w}(t).$$

Appealing to Lemma 2.1, it follows that $z = w$. This completes the proof.

The proof is similar if we assume that one of the subspace $B(X), S(X)$ or $T(X)$ is a closed subspace of X .

Remark 3.1 The conclusion of Theorem 3.2 remains true if we replace inequality (3.4) by one of the following: for all $k > 1, x, y > 0$ and $t > 0$

$$(3.8) F_{Ax, By}(kt) \leq \min\{F_{Sx, Ty}(t), F_{Ax, Sx}(t), F_{By, Ty}(t)\},$$

$$(3.9) (F_{Ax, By}(kt))^2 \leq F_{Ax, Sx}(t)F_{By, Ty}(t).$$

By setting $A = B$ and $S = T$ in Theorem 3.2, we can obtain a natural result for a pair of self mappings.

Corollary 3.1 Let A and S be two self mappings of a Menger space (X, \mathcal{F}, Δ) . Suppose that

$$(3.10) S(X) \subseteq A(X),$$

$$(3.11) \text{ the pair } (A, S) \text{ satisfies the property (E.A),}$$

$$(3.12) \text{ one of the range of the mappings } A \text{ or } S \text{ is a closed subspace of } X,$$

$$(3.13) \text{ there exists a constant } k > 1 \text{ such that}$$

$$F_{Ax, Ay}(kt) \leq F_{Sx, Sy}(t),$$

for all $x, y \in X$ and $t > 0$.

Then A and S have a unique common fixed point in X .

Remark 3.2 The results similar to Corollary 3.1 can also be outlined in view of conditions (3.8) and (3.9). The details of possible corollaries are not included here.

4. CONCLUSION

Theorem 3.2 is a generalization of Theorem 3.1 in the sense it is proved for non-surjective mappings under weak compatibility which is more general than compatibility. Theorem 3.2 extends the result of Kumar et al. [8, Theorem 3.2]. Theorem 3.2 (in view of Remark 3.1) improves and extends the results of Dimri et al. [2, Theorem 3.2] and Gujjetiya et al. [4, Theorem 3.1] without any requirement of completeness of the whole space and continuity of the involved mappings.

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