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A Chebyshev Series Approximation for Linear Second-Order Partial Differential Equations with Complicated Conditions

Gamze YUKSEL[1,♠], Mehmet SEZER[2]

[1]*Department of Mathematics, Faculty of Science, Mugla Sitki Kocman University, Mugla 48000, Turkey*
[2]*Department of Mathematics, Faculty of Science and Letters, Celal Bayar University, Manisa 45140,Turkey*

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ABSTRACT

The purpose of this study is to present a new collocation method for the solution of second-order, linear partial differential equations (PDEs) under the most general conditions. The method has improved from Chebyshev matrix method, which has been given for solving of ordinary differential, integral and integro-differential equations. The method is based on the approximation by the truncated bivariate Chebyshev series. PDEs and conditions are transformed into the matrix equations, which corresponds to a system of linear algebraic equations with the unknown Chebyshev coefficients, via Chebyshev collocation points. Combining these matrix equations and then solving the system yields the Chebyshev coefficients of the solution function. Finally, the effectiveness of the method is illustrated in several numerical experiments and error analysis is performed.

Key words: Partial differential equations; Chebyshev collocation method, Chebyshev polynomial solutions, Bivariate Chebyshev series..

1.INTRODUCTION

It is well known that the numerical methods have played an important role in solving (PDEs). Several applications have been developed for numerical solutions of PDEs. Some of the most known numerical methods are finite difference methods, finite element methods, polynomial In recent years, the Chebyshev matrix method has been used to find the approximate solutions of differential, integral and integro-differential equations by Sezer et al [11-14] and by Akyüz-Dascioglu [15-16]. Akyüz-Dascioglu has also used the Chebyshev matrix method which is based on the Chebyshev coefficients for high order partial differential equations with complicated conditions in [17]. There are several researchers in literature using Chebyshev series expansions or Chebyshev polynomial approximations to solve linear approximate methods, spectral methods, Galerkin and collocation methods [1-4]. Recently, various approximate methods are discussed in the literature such as differential transform method, Legendre-wavelet method , Chebyshev-tau method, Adomian decomposition method, Homotopy perturbation method, etc [5-10].

differential equations [18-21], nonlinear problems [22-25], integro-differential equations [23-27], differential difference equations [27] and partial differential equations [28-30]. Some researchers have also studied different problems with Chebyshev polynomials [31-32]. In this paper, we have improved a new matrix method based on the relations, between the Chebyshev polynomials and their derivatives. It is very effective method for direct solution of PDEs with complicated conditions and it is also useful to obtain the approximate

♠Corersponding author, e-mail: ngamze@mu.edu.tr

solution in irregular domains. In literature, there are not so much applications about the solution of variable coefficients PDEs. Present method may also solve this kind of problems easily.

1.1. Definition of the problem

Let Ω be a rectangular region, $\Omega=\{(x,y):-1\le x,y\le 1\}$, and $\partial\Omega$ be the boundary of Ω. In the case of $\Omega=\{(x,y):0\le x,y\le 1\}$, the approximate solution is obtained by shifted Chebyshev polynomials. In this study, we consider the the second-order linear partial differential equation

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+ D(x,y)\frac{\partial u}{\partial x}+E(x,y)\frac{\partial u}{\partial y}+F(x,y)u=G(x,y) \tag{1.1}$$

with the following conditions.

Case 1: Conditions defined at the points $x=\alpha_k$ and $y=\beta_k$, where $\alpha_k,\beta_k\in\partial\Omega$,

$$\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1}a_{i,j}^{k}u^{(i,j)}(\alpha_k,\beta_k)=\lambda_k \tag{1.2}$$

Case 2: Conditions defined at the points $y=\gamma_k$, where $\gamma_k\in\partial\Omega$,

$$\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1}b_{i,j}^{k}(x)u^{(i,j)}(x,\gamma_k)=g_k(x) \tag{1.3}$$

Case 3: Conditions defined at the points $x=\eta_k$, where $\eta_k\in\partial\Omega$,

$$\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}^{k}(y)u^{(i,j)}(\eta_k,y)=h_k(y) \tag{1.4}$$

Here $A(x,y),B(x,y),C(x,y),D(x,y),E(x,y),F(x,y)$ and $G(x,y)$ are functions defined in $\Omega=\{(x,y):-1\le x,y\le 1\}$. $g_k(x)$ and $b_{i,j}^{k}(x)$ are defined in $-1\le x\le 1$. $h_k(y)$ and $c_{i,j}^{k}(y)$ are defined in $-1\le y\le 1$. $a_{i,j}^{k}$ and λ_k are constants and $t,p,m\in Z^{+}$. Also $u^{(0,0)}(x,y)=u(x,y)$ and $u^{(i,j)}(x,y)=\dfrac{\partial^{i+j}u(x,y)}{\partial x^i\partial x^j}$ where $i,j=0,1,2.$

The conditions cover the initial and boundary conditions in most numerical methods. The conditions must be defined in $\partial\Omega$ due to domain of the first kind of Chebyshev polynomials or the first kind of shifted Chebyshev polynomials. Moreover, since all finite ranges can be transformed to the interval [-1,1] or [0,1], the present method can be used for solutions of problems that are defined in any finite ranges.

We will find the approximate solution of (1.1) via truncated Chebyshev series, such that

$$u(x,y)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}T_{r,s}(x,y) \tag{1.5}$$

where $T_{r,s}(x,y)=T_r(x)T_s(y)$ and $a_{r,s}$'s are unknown constants to be determined. Here, $T_r(x)$ and $T_s(y)$ denote the Chebyshev polynomials of degree r and s, respectively, defined by $T_r(x)=\cos(r\arccos(x))$ and $T_s(y)=\cos(s\arccos(y))$.

We choose the collocation points as the extremes of the Chebyshev polynomials $T_r(x)$ and $T_s(y)$ respectively.

$$x_n=\cos\left(\frac{N-n}{N}\right)\pi,\quad y_l=\cos\left(\frac{N-l}{N}\right)\pi; \quad n=0,1,\ldots,N,\quad l=0,1,\ldots N \tag{1.6}$$

2. PRELIMINARIES AND NOTATIONS

2.1 Bivariate Chebyshev Series Expansion

The Chebyshev polynomial $T_r(x)$ is defined by the relation $T_r(\cos\theta)=\cos r\theta$, where $x=\cos\theta$. In one dimension it is well known that functions which are continuous and of bounded variation on the interval $[-1-\delta,1+\delta]$, for some $\delta>0$, have uniformly convergent Chebyshev series expansions on $[-1,1]$. In two dimensions to show a function have uniformly convergent double Chebyshev series expansions in $\Omega:=\{-1\le x,y\le 1\}$, the version of 'bounded variation' and convergence theorem as follows

Definition 2.1: Let $f(x,y)$ be defined on $\Omega:=\{-1\le x,y\le 1\}$; let $\{x_r\}$ and $\{y_r\}$ denote monotone non-decreasing sequences of $n+1$ values with $x_0=y_0=-1$ and $x_n=y_n=+1$; let

$$\Sigma_1:=\sum_{r=1}^{n}\left|f(x_r,y_r)-f(x_{r-1}-y_{r-1})\right|,$$

$$\Sigma_2:=\sum_{r=1}^{n}\left|f(x_r,y_{n-r+1})-f(x_{r-1}-y_{n-r})\right|.$$

Then $f(x,y)$ is of bounded variation on Ω if $\sum_1$ and $\sum_2$ are bounded for all possible sequences $\{x_r\}$ and $\{y_r\}$ for every $n>0$ [33].

Theorem 2.1: If $f(x,y)$ is continuous and of bounded variation in Ω, for some $\delta>0$, and if one of its partial derivatives is bounded in Ω, then f has a double Chebyshev series expansion, uniformly convergent on Ω, of the form

$$f(x,y)=\sum_{r=0}^{\infty}{}'\sum_{s=0}^{\infty}{}' a_{r,s}T_{r,s}(x,y)=\sum_{r=0}^{\infty}{}'\sum_{s=0}^{\infty}{}' a_{r,s}T_r(x)T_s(y)$$

See ref. [33] for the proof of Theorem 2.1.

Definition 2.2: (Chebyshev norm) Let us denote by Ω the rectangle $\Omega=\{(x,y):-1\le x,y\le 1\}$ and let $E(\Omega)$ be the set of all continuous real-valued functions endowed with the inner product $\langle,\rangle_p$ for f in $E(\Omega)$, the Chebyshev norm is defined

$$\|f\|_\infty=\max_{-1\le x,y\le 1}|f(x,y)|,\ \forall f\in E(\Omega)$$

This norm is also called the uniform norm, minimax norm or supremum norm.

Definition 2.3: (Shifted Chebyshev polynomials) The shifted Chebyshev polynomials $T_r^*(x)$ are defined in terms of the Chebyshev polynomials $T_r(x)$ by the relation $T_r^*(x)=T_r(2x-1)$, $0\le x\le 1$.

Theorem 2.2: Fix $T>0$. The Cauchy problem for the one-dimensional homogeneous wave equation is given by

$$u_{yy}-c^2u_{xx}=0,\qquad -\infty<x<\infty,\ 0\le y<\infty$$
$$u(x,0)=f(x),\ \ u_y(x,0)=g(x),\ \ -\infty<x<\infty.$$

is well-posed for $f\in C^2(IR)$, $g\in C^1(IR)$.
See ref. [34] for the proof of Theorem 2.2.

3. FUNDAMENTAL RELATIONS

To find the numerical solutions of PDEs with Chebyshev method, it is necessary to evaluate the Chebyshev coefficients of the approximate solution. For convenience, the relation (1.5) can be written in the matrix form as follows:

Lemma 3.1. [35] The Chebyshev series solution of (1.1)

$$u(x,y)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}T_{r,s}(x,y)$$

can be written as a matrix form

$$u(x,y)=\boldsymbol{T}(x)Q(y)\overline{\mathbf{A}} \tag{3.1}$$

where

$$T(x)=\begin{bmatrix}T_0(x) & T_1(x) & \cdots & T_N(x)\end{bmatrix}_{1x(N+1)}$$

and

$$\mathbf{Q}(y)=\begin{bmatrix} T_0(y) \cdots T_N(y) & 0 \cdots 0 & \cdots & 0 \cdots 0\\ 0 \cdots 0 & T_0(y)\cdots T_N(y) & \cdots & 0\cdots 0\\ \vdots & & \ddots & \vdots\\ 0\cdots 0 & 0\cdots 0 & \cdots & T_0(y)\cdots T_N(y)\end{bmatrix}_{(N+1)x(N+1)^2}$$

and $\overline{\mathbf{A}}$ is the unknown Chebyshev coefficients matrix

$$\overline{\mathbf{A}}=\begin{bmatrix}a_{0,0} & a_{0,1} & \ldots & a_{0,N} & a_{1,0} & a_{1,1} & \ldots & a_{1,N} & \ldots & a_{N,0} & a_{N,1} & \ldots & a_{N,N}\end{bmatrix}^T_{(N+1)^2x1}$$

Proof: We can easily prove it from the matrix multiplication such that

$$\begin{aligned}u(x,y)=\boldsymbol{T}(x)Q(y)\overline{\mathbf{A}}&=a_{00}T_0(x)T_0(y)+a_{01}T_0(x)T_1(y)+\ldots+a_{0N}T_0(x)T_N(y)\\&+a_{10}T_1(x)T_0(y)+a_{11}T_1(x)T_1(y)+\ldots+a_{1N}T_1(x)T_N(y)\\&\ \vdots\\&+a_{N0}T_N(x)T_0(y)+a_{N1}T_N(x)T_1(y)+\ldots+a_{NN}T_N(x)T_N(y)\\&=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}T_r(x)T_s(y)\end{aligned}$$

To solve PDEs approximately, an explicit relationship is essentially needed between the solution function $u(x,y)$ and its partial derivatives. In the next part, we present these relations.

3.1 Matrix Relations of the Derivatives of the Approximate Function $u(x,y)$

The matrix form for derivatives of (3.1),

$$u^{(i,j)}(x,y)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}T_{r,s}^{(i,j)}(x,y)$$

can be written by Lemma 3.1 as

$$u^{(i,j)}(x,y)=\boldsymbol{T}^{(i)}(x)\boldsymbol{Q}^{(j)}(y)\overline{\boldsymbol{A}}. \tag{3.2}$$

We present the following lemma to show the relation between the matrices $\mathbf{T}(x)$ and $\boldsymbol{Q}(y)$ and theirs derivatives.

Lemma 3.2 [35] Let $u(x,y)$ and its $(i+j)th$-order partial derivatives be denoted by (3.1) and (3.2), respectively. Then there is a following relation

$$u^{(i,j)}(x,y)=\boldsymbol{T}(x)\left(\boldsymbol{J}^T\right)^{i}\boldsymbol{Q}(y)\left(\overline{\boldsymbol{J}}\right)^{j}\overline{\boldsymbol{A}},\qquad i,j=0,1,2 \tag{3.3}$$

Where

$$\mathbf{J}=\begin{bmatrix}0&0&0&0&0&\cdots&0&0\\1&0&0&0&0&\cdots&0&0\\0&2.2&0&0&0&\cdots&0&0\\3&0&2.3&0&0&\cdots&0&0\\0&2.4&0&2.4&0&\cdots&0&0\\5&0&2.5&0&2.5&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&2N&0&2N&0&\cdots&2N&0\\N&0&2N&0&2N&\cdots&2N&0\end{bmatrix}_{(N+1)x(N+1)}\begin{array}{l}\\\\\\\\\\\\\longrightarrow N \text{ even}\\\longrightarrow N \text{ odd}\\\\\end{array}$$

$$,\overline{\boldsymbol{J}}=\begin{bmatrix}\boldsymbol{J}^T&0&\cdots&0\\0&\boldsymbol{J}^T&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\boldsymbol{J}^T\end{bmatrix}_{(N+1)^2x(N+1)^2}$$

Proof: It is clearly seen that the relation between the matrix $\boldsymbol{T}(x)$ and its derivative $\boldsymbol{T}^{(1)}(x)$ as follows

$$\boldsymbol{T}^{(1)}(x)=\boldsymbol{T}(x)\boldsymbol{J}^T$$

The second derivative of $\mathbf{T}(x)$ as follows,

$$\mathbf{T}^{(2)}(x)=\mathbf{T}^{(1)}(x)\mathbf{J}^{\mathbf{T}}=\mathbf{T}(x)(\mathbf{J}^{\mathbf{T}})^{\mathbf{2}}$$

Hence, we obtain the following formula for $i\,th$ derivatives of $\mathbf{T}(x)$:

$$\mathbf{T}^{(i)}(x)=\mathbf{T}^{(i-1)}(x)(\mathbf{J}^{\mathbf{T}})=\mathbf{T}(x)(\mathbf{J}^{\mathbf{T}})^{i}.$$

Similarly, we can obtain a formula for $j\,th$ derivatives of $\boldsymbol{Q}(y)$ as follows,

$$\boldsymbol{Q}^{(1)}(y)=\boldsymbol{Q}(y)\overline{\boldsymbol{J}}$$

$$\boldsymbol{Q}^{(2)}(y)=\boldsymbol{Q}^{(1)}(y)\overline{\boldsymbol{J}}=\boldsymbol{Q}(y)\left(\overline{\boldsymbol{J}}\right)^{2}$$

$$\vdots$$

$$\boldsymbol{Q}^{(j)}(y)=\boldsymbol{Q}^{(j-1)}(y)(\overline{\boldsymbol{J}})=\boldsymbol{Q}(y)(\overline{\boldsymbol{J}})^{j}$$

Finally, substituting the matrices $\boldsymbol{T}^{(i)}(x)$ and $\boldsymbol{Q}^{(j)}(y)$ in (3.2), we obtain the fundamental matrix equation (3.3)

$$u^{(i,j)}(x,y)=\boldsymbol{T}(x)\left(\boldsymbol{J}^T\right)^{i}\boldsymbol{Q}(y)\left(\overline{\boldsymbol{J}}\right)^{j}\overline{\boldsymbol{A}},\qquad i,j=0,1,2$$

3.2 Matrix Forms of the Conditions

We consider the conditions in three parts. From Lemma 3.1 and Lemma 3.2, the fundamental matrix relation associated with the condition in Case 1,

$$\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1}a_{i,j}^{k}u^{(i,j)}(\alpha_k,\beta_k)=\left(\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1}a_{i,j}^{k}\boldsymbol{T}(\alpha_k)(\boldsymbol{J}^T)^{i}\boldsymbol{Q}(\beta_k)(\overline{\boldsymbol{J}})^{j}\right)\overline{\boldsymbol{A}}=\lambda_k$$

, (3.4)

Similarly the fundamental matrix relations for Case 2 and Case 3 are obtained respectively,

$$\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1}b_{i,j}^{k}(x)u^{(i,j)}(x,\gamma_k)=\left(\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1}b_{i,j}^{k}(x)\boldsymbol{T}(x)(\boldsymbol{J}^T)^{i}\boldsymbol{Q}(\gamma_k)(\overline{\boldsymbol{J}})^{j}\right)\overline{\boldsymbol{A}}=g_k(x),\tag{3.5}$$

and

$$\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}^{k}(y)u^{(i,j)}(\eta_k,y)=\left(\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}^{k}(y)\boldsymbol{T}(\eta_k)(\boldsymbol{J}^T)^{i}\boldsymbol{Q}(y)(\overline{\boldsymbol{J}})^{j}\right)\overline{\boldsymbol{A}}=h_k(y)$$

. (3.6)

Now, we shall obtain the system of matrix equations, substituting the collocation points into the fundamental equations in the following section.

4. METHOD OF THE SOLUTION

Each term in (1.1) can be given in the matrix equation via (3.3)

$$\begin{aligned}&A(x,y)\boldsymbol{T}(x)(\boldsymbol{J}^T)^2\boldsymbol{Q}(y)\overline{\boldsymbol{A}}+B(x,y)\boldsymbol{T}(x)\boldsymbol{J}^T\boldsymbol{Q}(y)(\overline{\boldsymbol{J}})\overline{\boldsymbol{A}}+\\&C(x,y)\boldsymbol{T}(x)\boldsymbol{Q}(y)(\overline{\boldsymbol{J}})^2\overline{\boldsymbol{A}}+D(x,y)\boldsymbol{T}(x)(\boldsymbol{J}^T)\boldsymbol{Q}(y)\overline{\boldsymbol{A}}+\\&E(x,y)\boldsymbol{T}(x)\boldsymbol{Q}(y)(\overline{\boldsymbol{J}})\overline{\boldsymbol{A}}+F(x,y)\boldsymbol{T}(x)\boldsymbol{Q}(y)\overline{\boldsymbol{A}}=\boldsymbol{G}(x,y).\end{aligned}\tag{4.1}$$

By substituting the collocation points (1.6) into (4.1), we obtain the system of matrix equations as follows

$$\begin{aligned}&A(x_n,y_l)\boldsymbol{T}(x_n)(\boldsymbol{J}^T)^2\boldsymbol{Q}(y_l)\overline{\boldsymbol{A}}+B(x_n,y_l)\boldsymbol{T}(x_n)\boldsymbol{J}^T\boldsymbol{Q}(y_l)(\overline{\boldsymbol{J}})\overline{\boldsymbol{A}}+\\&C(x_n,y_l)\boldsymbol{T}(x_n)\boldsymbol{Q}(y_l)(\overline{\boldsymbol{J}})^2\overline{\boldsymbol{A}}+D(x_n,y_l)\boldsymbol{T}(x_n)(\boldsymbol{J}^T)\boldsymbol{Q}(y_l)\overline{\boldsymbol{A}}+\\&E(x_n,y_l)\boldsymbol{T}(x_n)\boldsymbol{Q}(y_l)(\overline{\boldsymbol{J}})\overline{\boldsymbol{A}}+F(x_n,y_l)\boldsymbol{T}(x_n)\boldsymbol{Q}(y_l)\overline{\boldsymbol{A}}=\boldsymbol{G}(x_n,y_l)\\&n=0,1,\ldots,N,\ l=0,1,\ldots,N.\end{aligned}\tag{4.2}$$

Hence, the fundamental matrix equation of (4.2) is obtained as

$$\begin{aligned}&A\boldsymbol{T}(\boldsymbol{J}^T)^2\boldsymbol{Q}\overline{A}+B\boldsymbol{T}(\boldsymbol{J}^T)\boldsymbol{Q}(\overline{\boldsymbol{J}})\overline{A}+C\boldsymbol{T}\boldsymbol{Q}(\overline{\boldsymbol{J}})^2\overline{A}+\\&D\boldsymbol{T}(\boldsymbol{J}^T)\boldsymbol{Q}\overline{A}+E\boldsymbol{T}\boldsymbol{Q}(\overline{\boldsymbol{J}})\overline{A}+F\boldsymbol{T}\boldsymbol{Q}\overline{A}=\boldsymbol{G}.\end{aligned}\tag{4.3}$$

Here, (4.3) corresponds to a system of $(N+1)^2$ linear algebraic equations with unknown Chebyshev coefficients

$a_{0,0},a_{0,1},\ldots,a_{0,N},a_{1,0},a_{1,1},\ldots,a_{1,N},\ldots,a_{N,0},a_{N,1},\ldots,a_{N,N}$.

Also, we can write (4.3) in the form

$$\underbrace{\left\{\begin{aligned}&A\boldsymbol{T}(\boldsymbol{J}^T)^2\boldsymbol{Q}+B\boldsymbol{T}(\boldsymbol{J}^T)\boldsymbol{Q}(\overline{\boldsymbol{J}})+C\boldsymbol{T}\boldsymbol{Q}(\overline{\boldsymbol{J}})^2+\\&D\boldsymbol{T}(\boldsymbol{J}^T)\boldsymbol{Q}+E\boldsymbol{T}\boldsymbol{Q}(\overline{\boldsymbol{J}})+F\boldsymbol{T}\boldsymbol{Q}\end{aligned}\right\}}_{\boldsymbol{W}}\overline{\boldsymbol{A}}=\boldsymbol{G}$$

or shortly

$$\boldsymbol{W}\overline{\boldsymbol{A}}=\boldsymbol{G}.\tag{4.4}$$

Similarly, by substituting the collocation points (1.6) into (3.4), (3.5) and (3.6) for the complicated conditions, respectively, we obtain the systems of the matrix equations as follows,
for Case 1

$$\underbrace{\left(\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1}a_{i,j}^{k}\boldsymbol{T}(\alpha_k)(\boldsymbol{J}^T)^i\boldsymbol{Q}(\beta_k)(\overline{\boldsymbol{J}})^j\right)}_{V_{1,k}}\overline{\boldsymbol{A}}=\boldsymbol{\lambda_k}$$

or

$$\boldsymbol{V_{1,k}}\overline{\boldsymbol{A}}=\boldsymbol{\lambda_k}\ ; \tag{4.5}$$

for Case 2

$$\underbrace{\left(\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1}b_{i,j}^{k}(x_n)\boldsymbol{T}(x_n)(\boldsymbol{J}^T)^i\boldsymbol{Q}(\gamma_k)(\overline{\boldsymbol{J}})^j\right)}_{V_{2,k}}\overline{\boldsymbol{A}}=g_k(x_n),\quad n=0,1,...N$$

or

$$\boldsymbol{V_{2,k}}\overline{\boldsymbol{A}}=\boldsymbol{g_k} \tag{4.6}$$

and for Case 3

$$\underbrace{\left(\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}^{k}(y_l)\boldsymbol{T}(\eta_k)(\boldsymbol{J}^T)^i\boldsymbol{Q}(y_l)(\overline{\boldsymbol{J}})^j\right)}_{V_{3,k}}\overline{\boldsymbol{A}}=h_k(y_l),\ l=0,1,...,N$$

or

$$\boldsymbol{V_{3,k}}\overline{\boldsymbol{A}}=\boldsymbol{h_k}\ . \tag{4.7}$$

We notice that, the conditions associated with PDEs may be given by either of them, or both of them, or all of them. Now, combining the (4.5), (4.6) and (4.7), we can show the matrix equations of conditions in a new matrix form

$$\underbrace{\begin{bmatrix}\boldsymbol{V_{1,k}}\\ \boldsymbol{V_{2,k}}\\ \boldsymbol{V_{3,k}}\end{bmatrix}}_{V}\overline{\boldsymbol{A}}=\underbrace{\begin{bmatrix}\boldsymbol{\lambda_k}\\ \boldsymbol{g_k}\\ \boldsymbol{h_k}\end{bmatrix}}_{R}$$

or shortly

$$\boldsymbol{V}\overline{\boldsymbol{A}}=\boldsymbol{R}. \tag{4.8}$$

To obtain the solution of (1.1) under the conditions (1.2), (1.3) and (1.4), the augmented matrix is formed from (4.4) and (4.8), as follows;

$$\left[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{G}}\right]=\begin{bmatrix}\boldsymbol{V} & ; & \boldsymbol{R}\\ \boldsymbol{W} & ; & \boldsymbol{G}\end{bmatrix} \tag{4.9}$$

Therefore, the unknown bivariate Chebyshev coefficients are obtained

$$\mathbf{A}=\left(\tilde{\tilde{\mathbf{W}}}\right)^{-1}\tilde{\tilde{\mathbf{G}}} \tag{4.10}$$

where $\left[\tilde{\tilde{\mathbf{W}}};\tilde{\tilde{\mathbf{G}}}\right]$ is generated by using the Gauss elimination method and then removing zero rows of augmented matrix $\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]$. The reason of being used Gauss elimination method for this direct solution that beware of the non-invertible case of the matrix $\tilde{\tilde{\mathbf{W}}}$. When the conditions are added to linear algebraic system, some rows can be same because of the symmetry of Chebyshev collocation points. These terms can be eliminated by Gauss elimination method.

5. SHIFTED CHEBYSHEV POLYNOMIAL SOLUTIONS

The present method can be developed for the problems defined in $\Omega=\{(x,y):0\le x,y\le 1\}$. We will find the approximate solution of (1.1) via truncated shifted Chebyshev series, such that

$$u(x,y)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}T_{r,s}^{*}(x,y)\ ,\ \ 0\le x,y\le 1$$

where $T_{r,s}^{*}(x,y)=T_r(2x-1)T_s(2y-1)$.

Similarly, there is a relation between the approximate solution $u(x,y)$ and its partial derivatives which is based on shifted Chebyshev polynomials.

Lemma 5.1: [35] Let $u^{(i,j)}(x,y)$, $(i+j)th$ -order partial derivatives of $u(x,y)$. Then there is a following relation

$$u^{(i,j)}(x,y)=2^i\boldsymbol{T}^*(x)\left(\boldsymbol{J}^T\right)^i2^j\boldsymbol{I}\boldsymbol{Q}^*(y)\left(\overline{\boldsymbol{J}}\right)^j\overline{\boldsymbol{A}},\ i,j=0,1,2 \tag{5.1}$$

where $u^{(0,0)}(x,y)=u(x,y)$,

$$\boldsymbol{T}^*(x)=\begin{bmatrix}T_0^*(x) & T_1^*(x) & \cdots & T_N^*(x)\end{bmatrix}_{1x(N+1)},$$

$$\boldsymbol{I}=\begin{bmatrix}1 & \cdots & 0\\ \vdots & \ddots & 0\\ 0 & \cdots & 1\end{bmatrix}_{(N+1)\times(N+1)}$$

$$\boldsymbol{Q}^*(y)=\begin{bmatrix}T_0^*(y) & \cdots & T_N^*(y) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\ 0 & \cdots & 0 & T_0^*(y) & \cdots & T_N^*(y) & \cdots & 0 & \cdots & 0\\ \vdots & & & & & & \ddots & & & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & T_0^*(y) & \cdots & T_N^*(y)\end{bmatrix}_{(N+1)x(N+1)^2}$$

$\boldsymbol{J}$ and $\overline{\boldsymbol{J}}$ are defined in Lemma 3.2.

Proof: Let we consider derivatives of approximate function from Lemma 3.1 such that,

$$u^{(i,j)}(x,y)=\boldsymbol{T}^{*(i)}(x)\boldsymbol{Q}^{*(j)}(y)\overline{\boldsymbol{A}} \tag{5.2}$$

There is a relation between the $\boldsymbol{T}^*(x)$ and derivatives of it as follows,

$$\boldsymbol{T}^{*(1)}(x) = 2\boldsymbol{T}^*(x)\boldsymbol{J}^T$$
$$\boldsymbol{T}^{*(2)}(x) = 2\boldsymbol{T}^{*(1)}(x)\boldsymbol{J}^T = 2^2\boldsymbol{T}^*(x)(\boldsymbol{J}^T)^2$$
$$\vdots$$
$$\boldsymbol{T}^{*(i)}(x) = 2^i\boldsymbol{T}^*(x)(\boldsymbol{J}^T)^i$$

(5.3)

A similar relation between the $\boldsymbol{Q}^*(y)$ and its derivatives can be given as

$$\boldsymbol{Q}^{*(1)}(y) = 2\boldsymbol{I}\boldsymbol{Q}^*(y)\overline{\boldsymbol{J}}$$
$$\boldsymbol{Q}^{*(2)}(y) = \boldsymbol{Q}^{*(1)}(y)\overline{\boldsymbol{J}} = 2^2\boldsymbol{I}\boldsymbol{Q}(y)\left(\overline{\boldsymbol{J}}\right)^2$$
$$\vdots$$
$$\boldsymbol{Q}^{*(j)}(y) = 2^j\boldsymbol{I}\boldsymbol{Q}^*(y)(\overline{\boldsymbol{J}})^j$$

(5.4)

Now, substituting (5.3) and (5.4) into (5.2), the fundamental matrix equation of $u^{(i,j)}(x,y)$ is obtained. Since $\Omega = \{(x,y): 0 \le x, y \le 1\}$, the collocation points are defined as

$$x_n = \left(\left(\cos\left(\frac{N-n}{N}\right)\pi\right)+1\right)\Big/2,\quad y_l = \left(\left(\cos\left(\frac{N-l}{N}\right)\pi\right)+1\right)\Big/2;.$$
$$n,l = 0,1,\ldots,N$$

Similar process can be improved for the complicated conditions based on the shifted Chebyshev polynomials as in Section 3.2.

In the next part we shall give a formula to check the accuracy of the solution.

6. ACCURACY OF THE SOLUTION AND ERROR ANALYSIS

Since the Chebyshev polynomial (1.5) is an approximate solution of (1.1), this solution is substituted in (1.1). The resulting equation must be satisfied approximately, that is, for $x = x_r \in [-1,1],\ y = y_s \in [-1,1]$, or $x = x_r \in [0,1],\ y = y_s \in [0,1]$:

$$E_{N,N}(x_r, y_s) =$$
$$\left|A(x_r,y_s)u_{xx}(x_r,y_s) + B(x_r,y_s)u_{xy}(x_r,y_s) + C(x_r,y_s)u_{yy}(x_r,y_s) + D(x_r,y_s)u_x(x_r,y_s)\right.$$
$$\left.+E(x_r,y_s)u_y(x_r,y_s) + F(x_r,y_s)u(x_r,y_s) - G(x_r,y_s)\right| \cong 0$$

and

$E(x_r, y_s) \le 10^{-k_i}$ (k_i positive integer).

If max $(10^{k_i}) = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{N,N}(x_r, y_s)$ at each of the points becomes smaller than the prescribed 10^{-k}. We can also calculate the maximum errors of the method as follows

$$e_{N,N} = \left\|u_{N,N} - u^*\right\|_\infty = \max\left\{\left|u_{N,N}(x,y) - u^*(x,y)\right|, (x,y) \in \Omega\right\}$$

where, u^* is the exact solution of the problem and $u_{N,N}(x,y)$ is the computed results for N.

7. NUMERICAL EXAMPLES

Example 1. Let us consider a model problem that its solution exists and unique. The Cauchy problem for the one-dimensional homogeneous wave equation is given by

$$u_{yy} - c^2u_{xx} = 0, \quad -\infty < x < \infty,\ 0 \le y < \infty$$
$$u(x,0) = f(x),\ u_y(x,0) = g(x),\ -\infty < x < \infty. \tag{7.1}$$

A solution of this problem can be interpreted as the amplitude of a sound wave propagating in a very long and narrow pipe, which in practice can be considered as a one-dimensional infinite medium. This system also represents the vibration of an infinite (ideal) string. The initial conditions f, g are given functions that represent the amplitude u and the velocity u_y of the string at time $y = 0$. The analytical solution of (7.1) is given by

$$u(x,y) = \frac{1}{2}[f(x+cy) + f(x-cy)] + \frac{1}{2c}\int_{x-cy}^{x+cy} g(s)ds$$

which is called d'Alembert's formula. Thus, Cauchy problem (7.1) is well posed with this d'Alembert's formula (See Theorem 2.2) [34]. Now, we obtain the Chebyshev polynomial solution of this model problem.

Let we consider Cauchy problem in $\Omega = \{(x,y): 0 \le x, y \le 1\}$

$$u_{yy} = u_{xx}$$
$$u(x,0) = x^3 \tag{7.2}$$
$$u_y(x,0) = x.$$

The exact solution is $u(x,y) = x^3 + 3xy^2 + xy$ from the *d'Alembert formula*. If we solve (7.2) with the present method for N=4 by using Chebyshev collocation points, we obtain the approximate solution. In table 1, the

In table 1, the maximum errors of the approximate solution are listed according to different values of digits. Actually, it is expected that the maximum errors would be zero. However, because of the rounding errors it doesn't. If we use equal interval such as,

$$x_n = \frac{n}{N}, \quad y_l = \frac{l}{N}; \quad n = 0,1,...,4, \quad l = 0,1,...4$$

instead of Chebyshev collocation points, we obtain the approximate solution $u(x,y) = x^3 + 3xy^2 + xy$, which is the exact solution.

Table 1. Comparison of maximum errors for N=4 for different values of digits

Maximum errors for different values of digits		
D=10	**D=20**	**D=30**
6.0 E-9	1.4 E-18	6.0 E-28

Example 2. Let us consider the following variable coefficients linear elliptic equation with Dirichlet boundary conditions,

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} = 2x^2 y^2 e^{xy} \tag{7.3}$$

$$u(x,0) = 1,\ u(x,1) = e^x,\ u(0,y) = 1,\ u(1,y) = e^y \tag{7.4}$$

The exact solution of this problem is $u(x,y) = e^{xy}$. We obtain the approximate solution as follows

$$u(x,y) = \sum_{r=0}^{N} \sum_{s=0}^{N} a_{r,s} T^*_{r,s}$$

which is based on the truncated double shifted Chebyshev series, on rectangular domain $0 \le x, y \le 1$. We can write (7.3) from the relation (5.1) as follows;

$$\left\{ \boldsymbol{4x_i^2 T}(x_i)(\boldsymbol{J}^T)^2 \boldsymbol{Q}(y_j) + 4y_j^2 \boldsymbol{IT}(x_i)\boldsymbol{Q}(y_j)(\overline{\boldsymbol{J}})^2 \right\} \overline{\boldsymbol{A}} = 2x_i^2 y_j^2 e^{x_i y_j}$$

$i, j = 0,...,$N

where the collocation points are,

$$x_i = \left(\left(\cos\left(\frac{N-i}{N} \right) \pi \right) + 1 \right) \Big/ 2,$$

$$y_j = \left(\left(\cos\left(\frac{N-j}{N} \right) \pi \right) + 1 \right) \Big/ 2; \quad i, j = 0,1,...,N$$

The fundamental matrix equation of (7.3) is

$$\left\{ \boldsymbol{4AT(J^T)^2 Q + 4CITQ(\overline{J})^2} \right\} \overline{\boldsymbol{A}} = \boldsymbol{G}$$

If we denote

$$\boldsymbol{4AT(J^T)^2 Q + 4CITQ(\overline{J})^2} = \boldsymbol{W}$$

then we have

$$\boldsymbol{W\overline{A} = G}$$

Matrix forms for the boundary conditions are

$$u(x_i,0) = \boldsymbol{T}(x_i)\boldsymbol{Q}(0)\overline{\boldsymbol{A}} = 1, \quad u(x_i,1) = \boldsymbol{T}(x_i)\boldsymbol{Q}(1)\overline{\boldsymbol{A}} = e^{x_i}$$

$$u(0,y_j) = \boldsymbol{T}(0)\boldsymbol{Q}(y_j)\overline{\boldsymbol{A}} = 1, \quad u(0,y_j) = \boldsymbol{T}(1)\boldsymbol{Q}(y_j)\overline{\boldsymbol{A}} = e^{y_j}$$

So the fundamental matrix equations of the conditions are

$$\underbrace{\boldsymbol{TQ(0)}}_{V_1}\overline{\boldsymbol{A}} = \lambda_1 \ , \quad \underbrace{\boldsymbol{TQ(1)}}_{V_2}\overline{\boldsymbol{A}} = \lambda_2$$

$$\underbrace{\boldsymbol{T(0)Q}}_{V_3}\overline{\boldsymbol{A}} = \lambda_3 \ , \quad \underbrace{\boldsymbol{T(1)Q}}_{V_4}\overline{\boldsymbol{A}} = \lambda_4$$

To obtain solution of (7.3) under the conditions (7.4), it is formed the augmented matrix as follows

$$\left[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{G}}\right] = \begin{bmatrix} \boldsymbol{V_1} & ; & \lambda_1 \\ \boldsymbol{V_2} & ; & \lambda_2 \\ \boldsymbol{V_3} & ; & \lambda_3 \\ \boldsymbol{V_4} & ; & \lambda_4 \\ \boldsymbol{W} & ; & \boldsymbol{G} \end{bmatrix}$$

Therefore, the unknown shifted Chebyshev coefficients is obtained as

$$\overline{\mathbf{A}} = \left(\tilde{\tilde{\mathbf{W}}} \right)^{-1} \tilde{\tilde{\mathbf{G}}}$$

We have solved the above problem by means of the fundamental matrix equation for N=7, 10 and 12. The maximum errors associated with the present method for different values of N are compared in Table 2. Since collocation methods are not stable, the solution may not converge to the exact solution whenever $N \to \infty$. For this reasons, the truncation limit *N* should be chosen sufficiently large. Also, the error functions for N=7, 10 and 12 are plotted in Fig.1

Table 2. Comparison of the maximum errors for different values of N for Example 2

Present Method Maximum Errors		
N=7	**N=10**	**N=12**
1.28 E-8	1.2 E-13	6.9 E-13

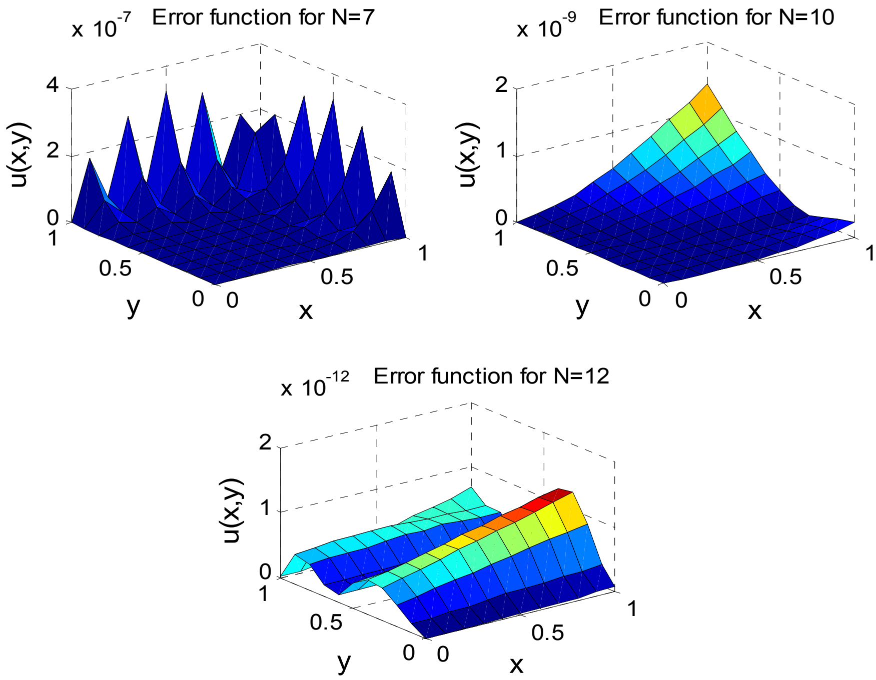

Figure 1. Comparison of the error functions for Example 2.

Example 3. [9] Let us consider the Poisson equation in

$$\Omega_0 \subset \Omega = \{(x, y) : -1 \le x, y \le 1\}$$

$$\Delta u = f(x, y)$$

where the Dirichlet boundary conditions are applied in Ω_0. The domain Ω_0 is also shown in Figure 2. The exact solution of this problem is $u(x, y) = \sin x \sin y$. We split the domain into four pieces such that

$$\Gamma_1 = \{(-1, y) | y \in [-1,1]\},\ \Gamma_2 = \{(x, -1) | x \in [-1,1]\},$$

$$\Gamma_3 = \{(\xi, -\xi) | \xi \in [0,1]\}, \Gamma_4 = \{(\xi, -\xi) | \xi \in [-1,0]\}$$

The comparison between the exact solution and the approximate solutions is made on test points $\{(x, y) \in \Omega | x = -1 : 0.01 : 1, y = -1 : 0.01 : 1\}$.

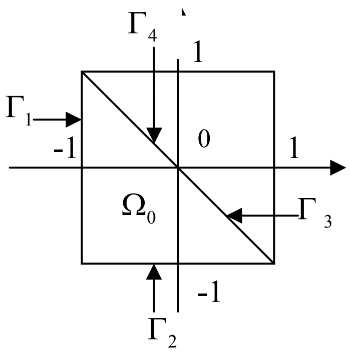

Figure 2. Physical domain of Example 3.

In Table 3, the maximum errors of the present method are compared with the Chebyshev-Tau matrix method for N=5,7,9 and 11. It is observed that the maximum errors of the present method are smaller than the Chebyshev-Tau matrix method. Also, the method is useful for obtaining the approximate solutions of some problems defined in irregular domain such as in this example.

Table 3. Comparison of the maximum errors with the Chebyshev-Tau matrix method

N	Chebyshev-Tau matrix method	Present Method
5	1.2275 E-2	2.2368 E-3
7	5.0757 E-5	7.2479 E-5
9	1.6734 E-5	2.8994 E-6
11	2.3147 E-7	1.3080 E-7

Example 4. We now consider the following problem on $\Omega = \{(x, y) : -1 \le x, y \le 1\}$ [5].

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = f(x, y)$$

with some boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=-1} = g(x, y) \quad , \quad u|_{\partial u} = k(x, y)$$

where $u = xe^y$ and $f(x, y) = 0$.

In Table 4, the maximum error of present method is compared with the Legendre-Wavelet Method for N=7. Also, the maximum errors of the present method are compared with each other for different values of N, in Table 4. It is clearly seen that, the present method has less error than the Legendre-Wavelet Method.

Table 4. Comparison of the maximum errors for different values of N of the present method with the Legendre-Wavelet Method

Legendre-Wavelet Method	Present Method			
N=7	**N=7**	**N=8**	**N=9**	**N=10**
1.44 E-6	3.32 Ê-7	2.72 E-8	1.16 E-9	6.0 E-10

Example 5. Consider the second-order hyperbolic partial differential equation with variable coefficients on domain $\Omega = \{(x,y): 0 \le x, y \le 1\}$

$$\frac{\partial^2 u}{\partial y^2} - \frac{1}{2}x^2 \frac{\partial^2 u}{\partial x^2} = 0$$

subject to the initial conditions

$$u(x,0) = x$$

$$u_y(x,0) = x^2$$

with the exact solution $u(x,t) = x + x^2 \sinh y$ [10]. The maximum errors for N=7, 8, 9 and 10 are given in Table 5. The error functions for N=7 and N=10 are plotted in Figure 3.

Table 5. Comparison of the maximum errors of the present method for different values of N.

Present Method for different values of N			
N=7	**N=8**	**N=9**	**N=10**
6.0 E-5	1.12 E-5	3.54 E-7	6.16 E-7

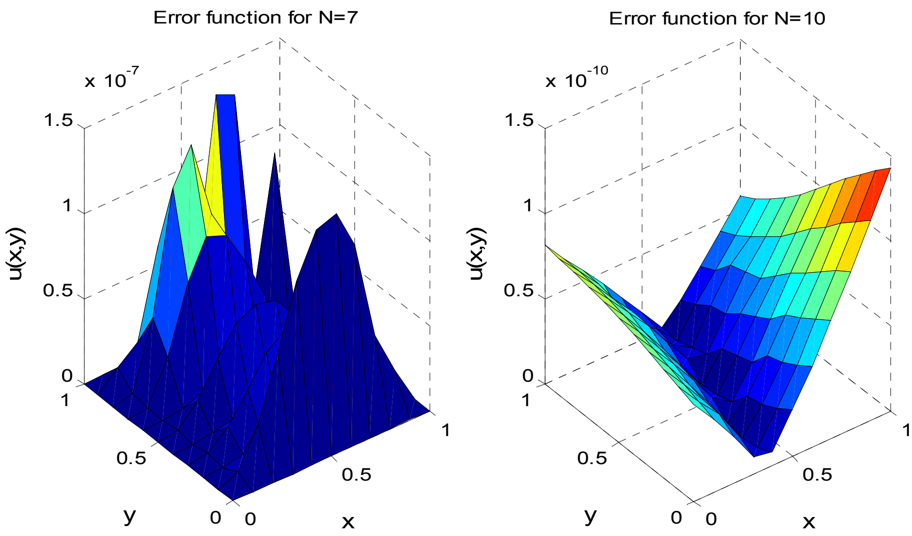

Figure 3. Error functions for N=7 and N=10

7. CONCLUSION

The purpose of our paper is to provide an approximation for linear second order PDEs with complicated conditions. We present a method based on bivariate Chebyshev series. The method finds the truncated Chebyshev series satisfying (1.1) with the conditions (1.2), (1.3) and (1.4) on the Chebyshev collocation nodes. The effectiveness of the method is illustrated in several numerical experiments. There are some advantages of this method.

1. This method is a direct method avoiding any iterative procedure. It doesn't need to run high computer algorithm.

2. It is observed that when the exact solution can be expanded to Chebyshev series, to get more accurate approximation, it should be taken more terms for Chebyshev approximate solutions. If N is chosen too large, more work than necessary will have been done. Also, there may be big computational errors. On the other hand, since collocation methods are not stable, the solution may not converge to the exact solution whenever $N \to \infty$. For this reasons, the truncation limit N should be chosen sufficiently large.

3. N th order approximation gives the exact solution when the solution is a polynomial and its degree equal to or less than N. If the solution is not a polynomial, it may get better result for sufficiently large N.

4. Since all finite ranges can be transformed to the interval [-1,1], this method can be applied to all finite ranges.

5. The method is also useful to obtain the approximate solutions on irregular domains.

For future work, the method can be applied to non-linear PDEs in different domains and it is also applied to higher order linear and nonlinear PDEs. We treat here twodimensional problems only. However, it is straightforward to extend the method to more dimensions.

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