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On the Existence Solution of a Class Boundary Integral Equations

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ABSTRACT

This paper is devoted to investigating the existence and uniqueness solutions class boundary integral equations over a regular closed surface. This paper provides sufficient conditions for the existence and uniqueness solution in the space of continuous functions of class boundary integral equations.

Key words: Boundary integral equation, Boundary value problem, Helmholtz equation.

AMS Subject Classification: 45E05, 31B10.

1. INTRODUCTION

In this paper, has been studied a boundary integral equation (BIE) which is in the following form

$$\varphi(x)+\int_S \frac{\phi(x,y)}{|x-y|^2}\varphi(y)d\sigma_y = f(x), x\in S .$$

(1)

Here S is a closed, bounded and regular surface in đ^3, $\phi(x,y)$ and $f(x)$ are known continuous functions in their domains of definition, such that for every $x\in S$ $\phi(x,x)=0$. Function $\varphi(x)$ is an unknown function and $|x-y|$ denotes the distance between the points x and y.

It is well known that singular integral equation theory has broad applications to theoretical and practical investigations in mathematics, mathematical physics, mechanics, hydrodynamics and elasticity theory [1-3]. This fact has only given rise to multiple studies of singular integral equations, but also developed many effective approximate solution methods. There exist many studies that explore these methods [4-22].

In this paper, the reason of taking up the equations in the form (1) is that these kinds of equations have broad application field. It is well known that the solutions of a host of familiar boundary value problems have been reduced to solving equations of the form (1). For example, the solutions of many theoretical and applied problems of mathematics, mechanics, physics and engineering can be reduced to the equations in the form (1).

As it is known, the solution of mixed Dirichlet-Neumann boundary value problem for Helmholtz's equation, the solution of exterior boundary value problem for Helmholtz's equation with the Dirichlet boundary condition, the solution of interior Dirichlet and Neumann eigenvalue problems are being reduced to the solutions of the equations in form (1) and to the BIE's similar to (1) [23-25]. Similar BIE's have also been applied to a host of problems in engineering whenever the problems can be reformulated in terms of biharmonic equations with Dirichlet boundary conditions [26].

It is known that many engineering problems are being formulated by the biharmonic equations with Dirichlet boundary conditions and the solutions of these kinds of problems are being reduced to the solution of BIE's [26]. Furthermore, fundamental equations of viscous flows problems are being formulated by the help of Poisson-type BIE's [27]. In engineering analysis of singular, potential and biharmonic problems with

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BIE's are given by D.B. Ingham and M.A. Kelmanson (see [28]).

Many dispersion and radiation problems are related with finding the solution of the Helmholtz's equation in exterior region defined by an equation of the following form:

$$\Delta u + k^2 . u = 0, \ \operatorname{Im} k \ge 0.$$

As it is known, in the solutions of this kind equations with finite element and half finite difference methods there are difficulties in the general case. These difficulties have been overcome by using the integral equation method given by Jones instead (see [29-32]).

We consider the integral equation defined on a two-dimensional closed, bounded and regular surface in $\mathbb{R}^3$, which the kernel of the integral operator is a function of the form $\phi(x,y)/|x-y|^2$, with continuous $\phi(x,y)$.

In this paper, it is shown that if the modulus of continuity of $\phi(x,y)$ satisfies some minor restrictions, then in spite of the presence of a high level of singularity along the diagonal, the integral operator in question is well defined and compact in the space of continuous functions. In this case, the Fredholm theorem applies, so the integral equation has a solution for any continuous right-hand side if and only if the corresponding homogenous equation does not have a nontrivial solution.

In the second section of the paper, we introduce some concepts needed to prove their main results.

In the third section, we prove the existence and uniqueness of the solution of the class of BIE's under consideration.

2. PRELIMINARIES

In this section of the paper, we will introduce some necessary information for proving the main results.

We denote the real numbers by $\mathbb{R}$, the nonnegative real numbers by $\mathbb{R}_+$.

Let $\ell = \sup\{|x-y| : x, y \in S\}$, where S is a closed, bounded, regular surface in $\mathbb{R}^3$. We will denote the radius of the standard sphere associated to the surface S by d [33].

The symbol $C(X)$ denotes the space of continuous functions on X. In the space $C(X)$, we denote by $\|u\| = \max\{|u(x)| : x \in X\}$ the ϖ norm of the vector $u \in C(X)$.

Throughout this paper, the numbers c_i, $i = 1,2,\ldots$ will denote positive real numbers.

Definition 2.1. The function $\omega_\varphi : (0,\delta] \to \mathbb{R}_+$ which is defined by the following formula

$$\omega_\varphi(\delta) = \delta . \sup\{\overline{\omega}_\varphi(\tau) . \tau^{-1} : \tau \ge \delta\}, \ \delta \in (0,\ell]$$

is the modulus of continuity of the function $\varphi \in C(S)$.

Here,

$$\overline{\omega}_\varphi(\delta) = \ = \max\{|\varphi(x) - \varphi(y)| : |x-y| \le \delta, x, y \in S\}, \delta \in (0,\ell].$$

Definition 2.2. Let us define the following functions for $\phi \in C(S \times S)$ and $\delta \in (0,\ell]$:

$$\overline{\omega}^*_\phi(\delta) = \sup\{|\phi(x,y)| : |x-y| \le \delta, x, y \in S\},$$
$$\omega^*_\phi(\delta) = \delta . \sup\{\overline{\omega}^*_\phi(\tau) . \tau^{-1} : \tau \ge \delta\},$$
$$\omega^{1,0}_\phi(\delta) = \sup\{\max_{y \in S} |\phi(x_1,y) - \phi(x_2,y)| : |x_1 - x_2| \le \delta, x_1, x_2 \in S\},$$
$$\omega^{0,1}_\phi(\delta) = \sup\{\max_{x \in S} |\phi(x,y_1) - \phi(x,y_2)| : |y_1 - y_2| \le \delta, y_1, y_2 \in S\}.$$

Remark 2.1. If a real valued function $g : \mathbb{R} \to \mathbb{R}$ with real variable is increasing or decreasing in its domain, then we denote it by $g \uparrow$ and $g \downarrow$, respectively.

Definition 2.3. We define the following sets functions $g : (0,\ell] \to \mathbb{R}_+$:

$$E_1(0,\ell] = \{g : g \geq 0, g \uparrow, g(\delta).\delta^{-1} \downarrow, g(\delta_1 + \delta_2) \leq g(\delta_1) + g(\delta_2), \lim_{\delta \to 0} g(\delta) = 0\}$$
$$E_2(0,\ell] = \{g : g \geq 0, g \uparrow, \lim_{\delta \to 0} g(\delta) = 0\}$$

Proposition 2.1. *For* $\varphi \in C(S)$ *and* $\phi \in C(S \times S)$ *we have* $\omega_\varphi, \omega_\phi^* \in E_1(0,\ell]$ *and* $\omega_\phi^{1,0}, \omega_\phi^{0,1} \in E_2(0,\ell]$.

3. MAIN RESULTS

In this section, we shall prove the existence of solution of the BIE (1).

Before proving the existence and uniqueness of solution of the BIE (1) let us give the following lemma.

Lemma 3.1. *If (1)* $\int_0^\ell \omega_\phi^*(\tau).\tau^{-1} d\tau < +\infty$ *and (2) as* $\delta \to 0$, *we have* $\omega_\phi^{1,0}(\delta) = o(\ln^{-1}\delta)$, *then for* $\phi \in C(S \times S)$,

$$(K\varphi)(x) = \int_S \frac{\phi(x,y)}{|x-y|^2} \varphi(y) d\sigma_y, \quad x \in S \tag{2}$$

is a compact operator in the space $C(S)$.

Proof. For $\varphi \in C(S), \phi \in C(S \times S)$ and for every $x \in S$ the following inequality can be easily proven:

$$\left| \int_S \frac{\phi(x,y)}{|x-y|^2} \varphi(y) d\sigma_y \right| \leq c_4.\|\varphi\|.\left\{1 + \int_0^d \frac{\omega_\phi^*(\tau)}{\tau} d\tau \right\}. \tag{3}$$

From this inequality, according to the assumption (1) of the lemma, it can be seen that the integral at the right hand side of Equality (2) is converging.

Now we will show that the operator $K\varphi$ defined by (2) is continuous in $C(S)$.

Let $x_1, x_2 \in S, |x_1 - x_2| = \delta$ and $\delta \in (0, d/2]$. In that case, we can write:

$$(K\varphi)(x_1) - (K\varphi)(x_2) = \int_{S_1} \frac{\phi(x_1,y)}{|x_1-y|^2} \varphi(y) d\sigma_y - \int_{S_1} \frac{\phi(x_2,y)}{|x_2-y|^2} \varphi(y) d\sigma_y +$$
$$+ \int_{S_2} \left[\frac{\phi(x_1,y)}{|x_1-y|^2} - \frac{\phi(x_2,y)}{|x_2-y|^2} \right] \varphi(y) d\sigma_y,$$

where $S_1 = S_{\delta/2}(x_1) \cup S_{\delta/2}(x_2), S_2 = S \setminus S_1$, $S_\delta(x) = \{y \in S : |x-y| \leq \delta\}$. Let us denote the integrals on the right hand side of the above equation by I_1, I_2, I_3, respectively.

It is obvious that

$$I_1 = \int_{S_1^1} \frac{\phi(x_1,y)}{|x_1-y|^2} \varphi(y) d\sigma_y + \int_{S_1^2} \frac{\phi(x_1,y)}{|x_1-y|^2} \varphi(y) d\sigma_y,$$

where $S_1^i = S_{\delta/2}(x_i), i = 1,2$. In a manner similar to the proof of Inequality (3), we can prove the following inequality:

$$\left|I_1^1\right| = \left|\int_{S_1^1} \frac{\phi(x_1,y)}{\left|x_1-y\right|^2}\varphi(y)d\sigma_y\right| \le c_5.\|\varphi\|.\int_0^\delta \frac{\omega_\phi^*(\tau)}{\tau}d\tau . \quad (4)$$

Since $\delta/2 \le |x_1-y| \le 3\delta/2$ for $y \in S_1^2$, we have

$$\left|I_1^2\right| = \left|\int_{S_1^2} \frac{\phi(x_1,y)}{\left|x_1-y\right|^2}\varphi(y)d\sigma_y\right| \le \|\varphi\|.\int_{S_1^2} \frac{\omega_\phi^*(|x_1-y|)}{|x_1-y|}d\sigma_y \le c_6.\|\varphi\|.\omega_\phi^*(\delta). \quad (5)$$

Use the inequalities (4) and (5), we obtain

$$|I_1| \le c_7.\|\varphi\|.\left[\omega_\phi^*(\delta) + \int_0^\delta \frac{\omega_\phi^*(\tau)}{\tau}d\tau\right]. \quad (6)$$

Can be proved in a manner similar:

$$|I_2| \le c_8.\|\varphi\|.\left[\omega_\phi^*(\delta) + \int_0^\delta \frac{\omega_\phi^*(\tau)}{\tau}d\tau\right]. \quad (7)$$

Now, we will evaluate the I_3 integral. We can write

$$I_3 = \int_{S_2} \frac{\phi(x_1,y)-\phi(x_2,y)}{|x_1-y|^2}\varphi(y)d\sigma_y + \int_{S_2}\phi(x_2,y)\left[\frac{1}{|x_1-y|^2} - \frac{1}{|x_2-y|^2}\right]\varphi(y)d\sigma_y .$$

Let us denote the integrals, which are on the right hand side of the above equality by I_3^1, I_3^2, respectively. For I_3^1 we get:

$$\left|I_3^1\right| \le \|\varphi\|.\omega_\phi^{1,0}(\delta).\int_{S_2}\frac{d\sigma_y}{|x_1-y|^2} \le c_9.\|\varphi\|.\omega_\phi^{1,0}(\delta).|\ln\delta| . \quad (8)$$

Since $\frac{1}{3}|x_1-y| \le |x_1-y| \le 3.|x_2-y|$ for $y \in S_2$ we obtain:

$$\left|I_3^2\right| \le \|\varphi\|.\int_{S_2}\frac{\omega_\phi^*(|x_2-y|)|x_1-x_2|[|x_1-y|+|x_2-y|]}{|x_1-y|^2|x_2-y|^2}d\sigma_y \le c_{10}.\|\varphi\|.\delta.\int_\delta^\ell \frac{\omega_\phi^*(\tau)}{\tau}d\tau . \quad (9)$$

From (8) and (9), we are writing:

$$|I_3| \le c_{11}.\|\varphi\|.\left[\omega_\phi^{1,0}(\delta).|\ln\delta| + \delta.\int_\delta^\ell \frac{\omega_\phi^*(\tau)}{\tau}d\tau\right]. \quad (10)$$

Use the Inequalities (6), (7), (10), we obtain

$$\left|(K\varphi)(x_1)-(K\varphi)(x_2)\right|\le c_{12}.\|\varphi\|.\left[\omega_\phi^*(\delta)+\omega_\phi^{1,0}(\delta).|\ln\delta|+\int_0^\delta\frac{\omega_\phi^*(\tau)}{\tau}d\tau+\delta.\int_\delta^\ell\frac{\omega_\phi^*(\tau)}{\tau}d\tau\right].$$

From this inequality, according to the assumptions of the lemma and Proposition 2.1, we can see that $K: C(S)\to C(S)$.

Now, we will show that the operator K is compact. For any $\varphi\in C(S)$ let us define the following operators:

$$(G_n\varphi)=\int_S g_n(x,y)\varphi(y)d\sigma_y,\ x\in S,$$

where

$$g_n(x,y)=\begin{cases}0 & ,\quad if\ |x-y|\le\frac{1}{2n},\\ \frac{\left[2n.|x-y|-1\right].\varphi(x,y)}{|x-y|^2} & ,\quad if\ \frac{1}{2n}<|x-y|\le\frac{1}{n},\\ \frac{\varphi(x,y)}{|x-y|^2} & ,\quad if\ \frac{1}{n}<|x-y|,\end{cases}$$

$n=1,2,\ldots$.

It is obvious that the operators $G_n: C(S)\to C(S), n=1,2,\ldots$ are compact. Furthermore, from the inequality

$$\left|(K\varphi)(x)-(G_n\varphi)(x)\right|\le c_{13}.\|\varphi\|.\int_0^{1/n}\frac{\omega_\phi^*(\tau)}{\tau}d\tau$$

it can be seen that the compact operator sequence $\{G_n\}, n=1,2,\ldots$ is converging to the operator K. Therefore, the operator K is also compact (see [34, p. 241, Theorem 1]).
This completes the proof of the lemma. ■

We can write the BIE (1) in the form of an operator equation as follows:

$$(I+K)\varphi(x)=f(x). \tag{11}$$

Here, I is the identity operator on $C(S)$, and K is the operator that is defined by (2).

Now, we present a theorem on the existence and uniqueness of the solution of the operator equation (11).

Theorem 3.1. *If (1)* $\int_0^\ell\frac{\omega_\phi^*(\tau)}{\tau}d\tau<+\infty$ *; (2) as* $\delta\to 0$, *we have* $\omega_\phi^{1,0}(\delta)=o(\ln^{-1}\delta)$*; (3)* $Ker(I+K)=\{0\}$ *and (4) the operator* K *is onto from* $C(\mathrm{S})$ *to* $C(\mathrm{S})$, *then, for every* $f\in C(S)$ *and* $\phi\in C(S\times S)$, *the equation* (11) *has unique solution in the space* $C(S)$.

Proof. To prove the theorem, it is sufficient to show that the linear operator $I+K: C(S)\to C(S)$ satisfies of the hypotheses of the Banach theorem on the existence of a bounded inverse operator.

The operator K (according to Lemma 3.1) is compact. Since every compact operator is bounded, the operator K is bounded. Thus, the operator $I+K$ is also bounded. Furthermore, by condition (3) of the theorem, the operator $I+K$ is a one-to-one operator. Thus, $I+K$ is an invertible bounded linear operator. Furthermore, from the hypothesis (4) of the theorem the operator K is surjective.

Whit this, the conditions of the Banach Theorem for the existence of bounded linear inverse operator of $I+K$ are satisfied (see [34, pp. 225, Theorem 3]). Therefore, invertible bounded linear operator $I+K$ has a bounded inverse. Thus, for every $f \in C(S)$ the operator equation (11) has unique solution in the space $C(S)$.
With this the theorem is proven. ■

Remark 3.1. Cubature formula for integral (2) may be given and singular integral equation (1) can be solved by approximation methods.

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