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Matrix Representation of Dual Quaternions

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ABSTRACT

After a review of some properties of dual quaternions, De Moivre's and Euler's formulas for the matrices associated with these quaternions are studied. In special case, De Moivre's formula implies that there are uncountably many matrices of unit dual quaternions satisfying $A^n = I_4$ for $n \geq 3$. Also; we give the relation between the powers of matrices of dual quaternions.

Key words: De-Moiver's formula, Hamilton operator, Dual quaternion

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1. INTRODUCTION

Quatenion algebra has been playing a important role in several areas of science; namely, in differential geometry, in analysis, synthesis of mechanism and machines, simulation of particle motion in molecular physics and quaternionic formulation of equation of motion in theory of relativity [2]. Dual numbers and dual quaternions were introduced in the 19th century by W.K. Clifford [5], as a tool for his geometrical investigation. Study [12] and Kotel'nikov[10] systematically applied the dual number and dual vector in their studies of line geometry and kinematics and independently discovered the transfer principle. The dual quaternion algebra was applied in kinematics and statics analysis of space mechanism in [15]. The Euler's and De-Moivre's formulas for the complex numbers are generalized for quaternions in [4]. These formulas are also investigated for the cases of split and dual quaternions in [9,11]. Some algebraic properties of Hamilton operators are considered in [2] where real quaternions have been expressed in terms of 4×4

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matrices by means of these operators. These matrices have applications in kinematics, mechanics, quantum physics and computer-aided geometric design [1]. In addition to, Yayli has considered homothetic motions with aid of the Hamilton operators in four-dimensional Euclidean space E^4 [16]. The eigenvalues, eigenvectors and the others algebraic properties of these matrices are studied by several authors [7, 17]. Subsequently, in [6] eigenvalues and eigenvectors of the dual Hamilton operators are given and also a special type of dual quaternion equation by using these concepts are investigated. Recently, we have derived the De-Moivre's and Euler's formulas for matrices associated with real quaternion and every power of these matrices are immediately obtained [8]. Here, after review of some algebraic properties of the dual quaternions, we study De-Moivre's and Euler's formulas for the matrices associated with dual quaternions. With the aid of the De Moivre's formula, any powers of these matrices can be obtained. In special case, the nth roots of these matrices are derived. Also, we give some examples for more clarification.

2. PRELIMINARIES

In this section, we give a brief summary of the dual number and dual quaternion. For detailed information about these concepts, we refer the reader to [2] and [3].

Definition 2.1. Let a and a^* be two real numbers, the combination

$$A = a + \varepsilon a^*,$$

is called a dual number. Here ε is the dual unit. Dual numbers are considered as polynomials in ε, subject to the rules

$$\varepsilon \neq 0, \varepsilon^2 = 0, \varepsilon.1 = 1.\varepsilon = \varepsilon.$$

The set of dual numbers, D, forms a commutative ring having the εa^* (a^* real) as divisors of zero, not field.

Definition 2.2. A dual quaternion Q is written as

$$Q = A_0 1 + A_1 e_1 + A_2 e_2 + A_3 e_3$$

where A_0, A_1, A_2 and A_3 are dual numbers and e_1, e_2, e_3 are quaternionic units which satisfy the equalities;

$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1$$

$$e_1 e_2 = e_3 = -e_2 e_1, \; e_2 e_3 = e_1 = -e_3 e_2$$

and

$$e_3 e_1 = -e_2 = e_1 e_3.$$

As a consequence of this definition, a dual quaternion Q can also be written as;

$$Q = q + \varepsilon q^*,$$

where q and q^*, real and pure dual quaternion components, respectively. The dual quaternionic multiplication of two dual quaternions $Q = S_Q + \vec{V}_Q$ and $P = S_P + \vec{V}_P$ is defined;

$$\begin{aligned} QP = & \; S_Q S_P - < \vec{V}_Q, \vec{V}_P > + S_Q \vec{V}_P + S_P \vec{V}_Q + \vec{V}_Q \times \vec{V}_P \\ & = (A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3) + (A_0 B_1 + A_1 B_0 - A_2 B_3 + A_3 B_2) e_1 \\ & + (A_0 B_1 + A_1 B_0 + A_2 B_3 - A_3 B_2) e_2 + (A_0 B_3 + A_1 B_2 - A_2 B_1 + A_3 B_0) e_3 \\ & = qp + \varepsilon (q p^* + q^* p). \end{aligned}$$

Also, it could be written

$$QP = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

So, the multiplication of dual quaternions as matrix-by-vector product. The norm of a dual quaternion is given by the sum of the squares of its components:

$$N_Q = A_\circ^2 + A_1^2 + A_2^2 + A_3^2, \; N_Q \in D.$$

It can also be obtained by multiplying the quaternion by its conjugate, in either order since a dual quaternion and conjugated commute:

$$N_Q = \bar{Q} Q = Q \bar{Q}.$$

Every non-zero dual quaternion has a multiplicative inverse given by its conjugate divided by its norm:

$$Q^{-1} = \bar{Q} / N_Q.$$

Definition 2.3. Let S_D^3 be the set of all unit dual quaternions and S_D^2 the set of unit dual vector, that is,

$$S_D^3 = \{ Q \in H_D : N_Q = 1 \} \subset H_D,$$

$$S_D^2 = \{ \vec{V}_Q = (A_1, A_2, A_3) : A_1^2 + A_2^2 + A_3^2 = 1 \}.$$

Under quaternionic multiplication, S_D^3 is a group, and is isomorphic to the group of all 2×2 unitary dual matrices of determinant 1, namely to $SU(2, D)$ [3].

Definition 2.4. Every nonzero dual quaternion

$$Q = A_0 1 + A_1 e_1 + A_2 e_2 + A_3 e_3,$$

can be written in the polar form

$$Q = r(\cos\phi + \vec{W} \sin\phi),$$

with

$$r = \sqrt{N_Q} = \sqrt{A_\circ^2 + A_1^2 + A_2^2 + A_3^2},$$

$\cos\phi = \frac{A_0}{r}$ and $\sin\phi = \frac{\sqrt{A_1^2 + A_2^2 + A_3^2}}{r}$.

$\phi = \varphi + \varepsilon\varphi^*$ is a dual angle and the unit dual vector $\vec{W}$ is given by

$$\vec{W} = \frac{A_1e_1 + A_2e_2 + A_3e_3}{\sqrt{A_1^2 + A_2^2 + A_3^2}},$$

with $A_1^2 + A_2^2 + A_3^2 \neq 0$.

Since $\vec{W}^2 = -1$, we have a natural generalization of Euler's formula for dual quaternions

$$\begin{aligned} e^{\vec{W}\phi} &= 1 + \vec{W}\phi - \frac{\phi^2}{2!} - \vec{W}\frac{\phi^3}{3!} + \frac{\phi^4}{4!} - \ldots \\ &= (1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \ldots) + \vec{W}(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots) \\ &= \cos\phi + \vec{W}\sin\phi, \end{aligned}$$

for any dual number ϕ.

Theorem 2.1. (De-Moivre's formula) Let

$$Q = e^{\vec{W}\phi} = \cos\phi + \vec{W}\sin\phi \in S_D^3$$

where $\phi = \varphi + \varepsilon\varphi^*$ is dual angle and $\vec{W} \in S_D^2$. Then for every integer n;

$$Q^n = e^{\vec{W}n\phi} = \cos n\phi + \vec{W}\sin n\phi.$$

Proof: The proof of this theorem can be done using induction (see [9]).

The formula holds for all integers n since;

$$Q^{-1} = \cos\phi - \vec{W}\sin\phi,$$

$$\begin{aligned} Q^{-n} &= \cos(-n)\phi + \vec{W}\sin(-n)\phi \\ &= \cos n\phi - \vec{W}\sin n\phi. \end{aligned}$$

We investigate some properties of the dual quaternions by separating them to two cases:

1) Dual quaternions with dual angles; *i.e.*

$$Q = r(\cos\phi + \vec{W}\sin\phi).$$

2) Dual quaternions with real angles; *i.e.*

$$Q = r(\cos\varphi + \vec{W}\sin\varphi).$$

Theorem 2.2. Let $Q = \cos\varphi + \vec{W}\sin\varphi \in S_D^3$. De Moivre's formula implies that there are uncountably many unit dual quaternions satisfying $Q^n = 1$ for $n \geq 3$.

The proof can be found in [9].

□

Example 2.1. Let

$$Q_1 = \frac{1}{2} + \frac{1}{2}(1+\varepsilon)e_1 + \frac{1}{2}(1-\varepsilon)e_2 + \frac{1}{2}e_3 = \cos\frac{\pi}{3} + \vec{W}\sin\frac{\pi}{3}$$

is of order 6 and

$$Q_2 = \frac{\sqrt{2}}{2} + \frac{1}{2}(1+\varepsilon)e_1 + \frac{1}{2}(1-\varepsilon)e_2 + 0e_3$$

is of order 8.

Remark 2.1. The equation $Q^n = 1$ does not have any solution for a general unit dual quaternion.

Example2.2. Let

$$Q = (\frac{1}{2} - \varepsilon\frac{\sqrt{3}}{2}) + \frac{1}{2}e_1 + \varepsilon e_2 + (\frac{1}{\sqrt{2}} - \varepsilon\sqrt{\frac{3}{8}})e_3 =$$
$$\cos(\frac{\pi}{3} + \varepsilon) + \vec{W}\sin(\frac{\pi}{3} + \varepsilon)$$

be a unit dual quaternion. There is no n ($n > 0$) such that $Q^n = 1$.

Also, we find n-*th* root of $Q = \cos\varphi + \vec{W}\sin\varphi \in S_D^3$. The equation $X^n = Q$ has n roots, and they are

$$X_k = \cos(\frac{\varphi + 2k\pi}{n}) + \vec{W}\sin(\frac{\varphi + 2k\pi}{n}), \qquad k = 0,1,2,\ldots,n-1.$$

The relation between the powers of dual quaternion can be found in the following theorem.

Theorem 2.3. Let Q be a unit dual quaternion with the polar form $Q = \cos\varphi + \vec{W}\sin\varphi$. If $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$, then $n \equiv p\,(\text{mod } m)$ if and only if $Q^n = Q^p$.

Proof: Let $n \equiv p\,(\text{mod}\, m)$. Then we have $n = am + p$, where $a \in \mathbb{Z}$.

$$\begin{aligned} Q^n &= \cos n\varphi + \vec{W}\sin n\varphi \\ &= \cos(am+p)\varphi + \vec{W}\sin(am+p)\varphi \\ &= \cos(a\frac{2\pi}{\varphi} + p)\varphi + \vec{W}\sin(a\frac{2\pi}{\varphi} + p)\varphi \\ &= \cos(p\varphi + a2\pi) + \vec{W}\sin(\varphi\theta + a2\pi) \\ &= \cos p\varphi + \vec{W}\sin p\varphi \\ &= Q^p. \end{aligned}$$

Now suppose $Q^n = \cos n\varphi + \vec{W}\sin n\varphi$ and $Q^p = \cos p\varphi + \vec{W}\sin p\varphi$. If $Q^n = Q^p$ then we get $\cos n\varphi = \cos p\varphi$ and $\sin n\varphi = \sin p\varphi$, which means $n\varphi = p\varphi + 2\pi a$, $a \in \mathbb{Z}$. Thus $n = p + 2\pi/\theta \times a$, or $n \equiv p(\text{mod}\, m)$.

3. De-Moivre's Formula for Matrices of Dual Qauternions

In this section, we introduce the ℝ-linear transformations representing left multiplication in H_D and look for also the De-Moiver's formula for corresponding matrix representation. Let Q be a dual quaternion, then $\varphi_Q : H_D \to H_D$ defined as follows;

$$\varphi_Q(P) = QP, \qquad P \in H_D.$$

The Hamilton's operator φ_Q, could be represented as the matrix;

$$A_{\varphi_Q} = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}.$$

If Q be a unit dual quaternion, then φ_Q is orthogonal linear transformation. Properties of these matrices are found in [2,6].

Theorem 3.1. The map ψ defined as

$$\psi : (H_D, +, .) \to (\mathrm{M}_{(4,\mathrm{D})}, \oplus, \otimes)$$

$$\psi(A_0 + A_1 e_1 + A_2 e_2 + A_3 e_3) \mapsto \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix},$$

is an isomorphism of algebras.

Proof: See[13] for a similar proof. □

We can express the matrix A_{φ_Q} in polar form. Let $Q = A_0 1 + A_1 e_1 + A_2 e_2 + A_3 e_3$ be a unit dual quaternion. Since

$$\begin{aligned} Q &= \cos\phi + \vec{W}\sin\phi \\ &= \cos\phi + (w_1, w_2, w_3)\sin\phi, \end{aligned}$$

so we have

$$\begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -w_1\sin\phi & -w_2\sin\phi & -w_3\sin\phi \\ w_1\sin\phi & \cos\phi & -w_3\sin\phi & w_2\sin\phi \\ w_2\sin\phi & w_3\sin\phi & \cos\phi & -w_1\sin\phi \\ w_3\sin\phi & -w_2\sin\phi & w_1\sin\phi & \cos\phi \end{bmatrix}.$$

Theorem 3.2. (De Moivre's formula) For an integer n and matrix

$$A = \begin{bmatrix} \cos\phi & -w_1\sin\phi & -w_2\sin\phi & -w_3\sin\phi \\ w_1\sin\phi & \cos\phi & -w_3\sin\phi & w_2\sin\phi \\ w_2\sin\phi & w_3\sin\phi & \cos\phi & -w_1\sin\phi \\ w_3\sin\phi & -w_2\sin\phi & w_1\sin\phi & \cos\phi \end{bmatrix}, \quad (3.1)$$

the n-th power of the matrix A reads as

$$A^n = \begin{bmatrix} \cos n\phi & -w_1\sin n\phi & -w_2\sin n\phi & -w_3\sin n\phi \\ w_1\sin n\phi & \cos\phi & -w_3\sin\phi & w_2\sin n\phi \\ w_2\sin n\phi & w_3\sin n\phi & \cos n\phi & -w_1\sin n\phi \\ w_3\sin n\phi & -w_2\sin n\phi & w_1\sin n\phi & \cos n\phi \end{bmatrix}.$$

The proof follows easily from the induction.

Note that if ϕ, w_1, w_2 and w_3 are real numbers, then Theorem 3.2 holds for real quaternions (see [8]).

Example 3.1. Let

$$Q = (\frac{1}{2} - \varepsilon\frac{\sqrt{3}}{2}) + \frac{1}{2}e_1 + \varepsilon e_2 + (\frac{1}{\sqrt{2}} - \varepsilon\sqrt{\frac{3}{8}})e_3 = \cos(\frac{\pi}{3} + \varepsilon) + \vec{W}\sin(\frac{\pi}{3} + \varepsilon)$$

be a unit dual quaternion. The matrix corresponding to this dual quaternion is

$$A = \begin{bmatrix} \frac{1}{2} - \varepsilon\frac{\sqrt{3}}{2} & -\frac{1}{2} & -\varepsilon & -\frac{1}{\sqrt{2}} - \varepsilon\sqrt{\frac{3}{8}} \\ \frac{1}{2} & \frac{1}{2} - \varepsilon\frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} - \varepsilon\sqrt{\frac{3}{8}} & \varepsilon \\ \varepsilon & \frac{1}{\sqrt{2}} + \varepsilon\sqrt{\frac{3}{8}} & \frac{1}{2} - \varepsilon\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} + \varepsilon\sqrt{\frac{3}{8}} & -\varepsilon & \frac{1}{2} & \frac{1}{2} - \varepsilon\frac{\sqrt{3}}{2} \end{bmatrix},$$

every powers of this matrix are found to be with the aid of Theorem 3.2, for example, *2-th* and *12-th* powers are

$$A^2 = \begin{bmatrix} -\frac{1}{2} - \varepsilon\sqrt{3} & -\frac{1}{2} + \varepsilon\frac{\sqrt{3}}{2} & -\varepsilon & -\frac{1}{\sqrt{2}} - \varepsilon\frac{5}{2\sqrt{6}} \\ \frac{1}{2} - \varepsilon\frac{\sqrt{3}}{2} & -\frac{1}{2} - \varepsilon\sqrt{3} & -\frac{1}{\sqrt{2}} - \varepsilon\frac{5}{2\sqrt{6}} & \varepsilon \\ \varepsilon & \frac{1}{\sqrt{2}} + \varepsilon\frac{5}{2\sqrt{6}} & -\frac{1}{2} - \varepsilon\sqrt{3} & -\frac{1}{2} + \varepsilon\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} + \varepsilon\frac{5}{2\sqrt{6}} & -\varepsilon & \frac{1}{2} - \varepsilon\frac{\sqrt{3}}{2} & -\frac{1}{2} - \varepsilon\sqrt{3} \end{bmatrix},$$

$$A^{12} = \begin{bmatrix} 1 & -4\sqrt{3}\varepsilon & 0 & -4\sqrt{6}\varepsilon \\ 4\sqrt{3}\varepsilon & 1 & -4\sqrt{6}\varepsilon & 0 \\ 0 & 4\sqrt{6}\varepsilon & 1 & -4\sqrt{3}\varepsilon \\ 4\sqrt{6}\varepsilon & -4\sqrt{3}\varepsilon & 0 & 1 \end{bmatrix}.$$

Corollary 3.1. De Moivre's formula implies that there are uncountably many matrices of the unit dual quaternions with real angle satisfying $A^n = I_4$ for every integer $n \geq 3$.

4. Euler's Formula for Matrices Associated with Dual Quaternions

Let $A=\begin{bmatrix} 0 & -w_1 & -w_2 & -w_3 \\ w_1 & 0 & -w_3 & w_2 \\ w_2 & w_3 & 0 & -w_1 \\ w_3 & -w_2 & w_1 & 0 \end{bmatrix}$ be a dual matrix. One immediately finds $A^2=-I_4$. We have a natural generalization of Euler's formula for matrix A ;

$$\begin{aligned} e^{A\phi} &= I_4 + A\phi + \frac{(A\phi)^2}{2!}+\frac{(A\phi)^3}{3!}+\frac{(A\phi)^4}{4!}+... \\ &= I_4(1-\frac{\phi^2}{2!}+\frac{\phi 4}{4!}-...)+A(\phi-\frac{\phi^3}{3!}+\frac{\phi^5}{5!}-...) \\ &= I_4\cos\phi + A.\sin\phi \\ &= I_4\cos\phi + \begin{bmatrix} 0 & -w_1 & -w_2 & -w_3 \\ w_1 & 0 & -w_3 & w_2 \\ w_2 & w_3 & 0 & -w_1 \\ w_3 & -w_2 & w_1 & 0 \end{bmatrix}.\sin\phi \\ &= \begin{bmatrix} \cos\phi & -w_1\sin\phi & -w_2\sin\phi & -w_3\sin\phi \\ w_1\sin\phi & \cos\phi & -w_3\sin\phi & w_2\sin\phi \\ w_2\sin\phi & w_3\sin\phi & \cos\phi & -w_1\sin\phi \\ w_3\sin\phi & -w_2\sin\phi & w_1\sin\phi & \cos\phi \end{bmatrix}.. \end{aligned}$$

For detailed information about Euler's formula, see [14].

5. *n*th Roots of Matrices of Dual Quaternions

Let $Q=A_0 1 + A_1 e_1 + A_2 e_2 + A_3 e_3$ be a unit dual quaternion with real angle, *i.e.* $\phi=\varphi$ and $\varphi^*=0$. The matrix associated with the quaternion Q is of the form (3.1). In a more general case, we assume for the matrix of (3.1) by

$$A=\begin{bmatrix} \cos(\varphi+2k\pi) & -w_1\sin(\varphi+2k\pi) & -w_2\sin(\varphi+2k\pi) & -w_3\sin(\varphi+2k\pi) \\ w_1\sin(\varphi+2k\pi) & \cos(\varphi+2k\pi) & -w_3\sin(\varphi+2k\pi) & w_2\sin(\varphi+2k\pi) \\ w_2\sin(\varphi+2k\pi) & w_3\sin(\varphi+2k\pi) & \cos(\varphi+2k\pi) & -w_1\sin(\varphi+2k\pi) \\ w_3\sin(\varphi+2k\pi) & -w_2\sin(\varphi+2k\pi) & w_1\sin(\varphi+2k\pi) & \cos(\varphi+2k\pi) \end{bmatrix},$$

$k\in\mathbb{Z}$. The equation $X^n=A$ has n roots, and they are

$$A_k^{\frac{1}{n}}=\begin{bmatrix} \cos(\frac{\varphi+2k\pi}{n}) & -w_1\sin(\frac{\varphi+2k\pi}{n}) & -w_2\sin(\frac{\varphi+2k\pi}{n}) & -w_3\sin(\frac{\varphi+2k\pi}{n}) \\ w_1\sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) & -w_3\sin(\frac{\varphi+2k\pi}{n}) & w_2\sin(\frac{\varphi+2k\pi}{n}) \\ w_2\sin(\frac{\varphi+2k\pi}{n}) & w_3\sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) & -w_1\sin(\frac{\varphi+2k\pi}{n}) \\ w_3\sin(\frac{\varphi+2k\pi}{n}) & -w_2\sin(\frac{\varphi+2k\pi}{n}) & w_1\sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) \end{bmatrix}.$$

For $k=0$, the first root is

$$A_0^{\frac{1}{n}} = \begin{bmatrix} \cos(\frac{\varphi}{n}) & -w_1\sin(\frac{\varphi}{n}) & -w_2\sin(\frac{\varphi}{n}) & -w_3\sin(\frac{\varphi}{n}) \\ w_1\sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) & -w_3\sin(\frac{\varphi}{n}) & w_2\sin(\frac{\varphi}{n}) \\ w_2\sin(\frac{\varphi}{n}) & w_3\sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) & -w_1\sin(\frac{\varphi}{n}) \\ w_3\sin(\frac{\varphi}{n}) & -w_2\sin(\frac{\varphi}{n}) & w_1\sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) \end{bmatrix},$$

andfor $k = 1$, the second root is

$$A_1^{\frac{1}{n}} = \begin{bmatrix} \cos(\frac{\varphi+2\pi}{n}) & -w_1\sin(\frac{\varphi+2\pi}{n}) & -w_2\sin(\frac{\varphi+2\pi}{n}) & -w_3\sin(\frac{\varphi+2\pi}{n}) \\ w_1\sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) & -w_3\sin(\frac{\varphi+2\pi}{n}) & w_2\sin(\frac{\varphi+2\pi}{n}) \\ w_2\sin(\frac{\varphi+2\pi}{n}) & w_3\sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) & -w_1\sin(\frac{\varphi+2\pi}{n}) \\ w_3\sin(\frac{\varphi+2\pi}{n}) & -w_2\sin(\frac{\varphi+2\pi}{n}) & w_1\sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) \end{bmatrix}.$$

Similarly, for $k = n-1$, we obtain the *n-th* root.

3. Relations Between Powers of Matrices

The relations between the powers of matrice sassociated with a dual quaternion can be realized by the following theorem.

Theorem 5.1. Let Q be a unit dual quaternion with the polar form $Q = \cos\varphi + \vec{W}\sin\varphi.$ If $m = 2\pi/\varphi \in \mathbb{Z}^+ - \{1\}$, then $n \equiv p \pmod{m}$ if and only if $A^n = A^p$.

The proof of this theorem can be using induction, similarly to the proof of the Theorem 2.3.

Example 5.1. Let $Q_1 = \frac{1}{2} + \frac{1}{2}(1+\varepsilon)e_1 + \frac{1}{2}(1-\varepsilon)e_2 + \frac{1}{2}e_3$ be a unit dual quaternion. The matrix corresponding to this dual quaternion is

$$A = \frac{1}{2}\begin{bmatrix} 1 & -(1+\varepsilon) & -(1-\varepsilon) & -1 \\ 1+\varepsilon & 1 & -1 & 1-\varepsilon \\ 1-\varepsilon & 1 & 1 & -(1+\varepsilon) \\ 1 & -(1-\varepsilon) & 1+\varepsilon & 1 \end{bmatrix}.$$

From the Theorem 5.1, $m = \frac{2\pi}{\pi/3} = 6,$ we have

$$\begin{aligned} A^1 &= A^7 = A^{13} = \ldots \\ A^2 &= A^8 = A^{14} = \ldots \\ A^3 &= A^9 = A^{15} = \ldots \\ &\ldots \\ A^6 &= A^{12} = A^{18} = \ldots = I_4. \end{aligned}$$

The square roots of the matrix A can be calculated as;

$$A_k^{\frac{1}{2}} = \begin{bmatrix} \cos(\frac{2k\pi+60^\circ}{2}) & -w_1\sin(\frac{2k\pi+60^\circ}{2}) & -w_2\sin(\frac{2k\pi+60^\circ}{2}) & -w_3\sin(\frac{2k\pi+60^\circ}{2}) \\ w_1\sin(\frac{2k\pi+60^\circ}{2}) & \cos(\frac{2k\pi+60^\circ}{2}) & -w_3\sin(\frac{2k\pi+60^\circ}{2}) & w_2\sin(\frac{2k\pi+60^\circ}{2}) \\ w_2\sin(\frac{2k\pi+60^\circ}{2}) & w_3\sin(\frac{2k\pi+60^\circ}{2}) & \cos(\frac{2k\pi+60^\circ}{2}) & -w_1\sin(\frac{2k\pi+60^\circ}{2}) \\ w_3\sin(\frac{2k\pi+60^\circ}{2}) & -w_2\sin(\frac{2k\pi+60^\circ}{2}) & w_1\sin(\frac{2k\pi+60^\circ}{2}) & \cos(\frac{2k\pi+60^\circ}{2}) \end{bmatrix}.$$

Thefirstrootfor k=0 is

$$A_0^{\frac{1}{2}} = \frac{\sqrt{3}}{2}\begin{bmatrix} 1 & -\frac{1}{3}(1+\varepsilon) & -\frac{1}{3}(1-\varepsilon) & -\frac{1}{3} \\ \frac{1}{3}(1+\varepsilon) & 1 & -\frac{1}{3} & \frac{1}{3}(1-\varepsilon) \\ \frac{1}{3}(1-\varepsilon) & \frac{1}{3} & 1 & -\frac{1}{3}(1+\varepsilon) \\ \frac{1}{3} & -\frac{1}{3}(1-\varepsilon) & \frac{1}{3}(1+\varepsilon) & 1 \end{bmatrix},$$

and the second one for k=1 becomes

$$A_1^{\frac{1}{2}} = \frac{\sqrt{3}}{2}\begin{bmatrix} -1 & \frac{1}{3}(1+\varepsilon) & \frac{1}{3}(1-\varepsilon) & \frac{1}{3} \\ -\frac{1}{3}(1+\varepsilon) & -1 & \frac{1}{3} & -\frac{1}{3}(1-\varepsilon) \\ -\frac{1}{3}(1-\varepsilon) & -\frac{1}{3} & -1 & \frac{1}{3}(1+\varepsilon) \\ -\frac{1}{3} & \frac{1}{3}(1-\varepsilon) & -\frac{1}{3}(1+\varepsilon) & -1 \end{bmatrix}.$$

Also, it is easy to see that $A_0^{\frac{1}{2}} + A_1^{\frac{1}{2}} = 0$.

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