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A Related Fixed Point Theorem in Two Menger Spaces

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ABSTRACT

In this paper, we prove a related fixed point theorem for single-valued mappings in two Menger spaces.

Key words: t-norm,Menger space, fixed point.

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1. INTRODUCTION

Professor Karl Menger introduced probabilistic metric spaces in his seminal paper [13] and studied their properties. The idea in his paper was that, instead of a single positive number, we should associate a distribution function with the point pairs. Since then the theory of PM-spaces has grown rapidly with the pioneering works of Schweizer and Sklar [17]. Sehgal and Bharucha-Reid [18] initiated the study of contraction mappings on PM- spaces (see also [5]). Fisher [7, 8] investigated the conditions for the existence of a relation connecting the fixed points of two mappings in two different metric spaces.

Subsequently several other authors have extensively studied various related fixed point theorems in metric spaces [1, 2, 4, 6, 9, 10-12, 19]. Recently Pant [15] generalized the results of Fisher [7, 8] in the framework of probabilistic settings. Pant and Kumar [16] further proved a related fixed point theorem in two complete Menger spaces. In 2009, Aliouche et al. [3] utilized a class of implicit functions and proved related fixed point theorem in two complete fuzzy metric spaces. The aim of this paper is to prove a related fixed point theorem for single-valued mappings in two Menger spaces. Our results generalize several comparable results in the existing literature.

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2. PRELIMINARIES

Let $T: X \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$.

Definition 2.1[17] A mapping $\Delta: [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (briefly, t-norm) if thefollowing conditions are satisfied: for all $a, b, c, d \in [0,1]$

(1) $\Delta(a, 1) = a$ for all $a \in [0,1]$,
(2) $\Delta(a, b) = \Delta(b, a)$,
(3) $\Delta(a, b) \leq \Delta(c, d)$ for $a \leq c$, $b \leq d$,
(4) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Examples of continuous t-norms are: $\Delta(a, b) = \min\{a, b\}$, $\Delta(a, b) = ab$ and $\Delta(a, b) = \max\{a + b - 1, 0\}$.

Definition 2.2[17] A mapping $F: \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing, left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

Let $\mathfrak{I}$ be the set of all distribution functions whereas H stands for the specific distribution function (also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, \text{if } t \leq 0; \\ 1, \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F}: X \times X \to \mathfrak{I}$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.3[17] The ordered pair $(X, \mathcal{F})$ is called a PM-space if X is a non-empty set and $\mathcal{F}$ is a probabilisticdistance satisfying the followingconditions: for all $x, y, z \in X$ and $t, s > 0$

(1) $F_{x,y}(t) = H(t) \Leftrightarrow x = y$,
(2) $F_{x,y}(t) = F_{y,x}(t)$,
(3) $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1 \Rightarrow F_{x,z}(t + s) = 1$.

Definition 2.4[17] A Menger space is a triplet $(\boldsymbol{X}, \boldsymbol{\mathcal{F}}, \Delta)$ where $(\boldsymbol{X}, \boldsymbol{\mathcal{F}})$ is a PM-space and t-normΔ is such that the inequality

$$\boldsymbol{F_{x,z}(t + s) \geq \Delta\left(F_{x,y}(t), F_{y,z}(s)\right)},$$

holds for all $\boldsymbol{x, y, z \in X}$ and $\boldsymbol{t, s > 0}$.

Every metric space $(\boldsymbol{X}, \boldsymbol{d})$ can be realized as a PM-space by taking $\boldsymbol{\mathcal{F}: X \times X \to \mathfrak{I}}$ defined by $\boldsymbol{F_{x,y}(t) = H(t - d(x, y))}$ for all $\boldsymbol{x, y \in X}$. So PM-spaces offer a wider framework (than that of the metric spaces) and are general enough to cover even wider statistical situations.

Definition 2.5[17] Let $(X, \mathcal{F}, \Delta)$ be a Menger space and Δ be a continuous t-norm. A sequence $\{x_n\}$ in X is said to be (i) convergent to a point x in X iff for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ for all $n \geq N(\varepsilon, \lambda)$; (ii) Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer $N(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ for all $n, m \geq N(\varepsilon, \lambda)$.

A Menger space in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.1 [12] Let $(X, \mathcal{F}, \Delta)$ be a Menger space. If there exists a constant $k \in (0,1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $t > 0$ with fixed $x, y \in X$ then $x = y$.

3. RESULTS

Theorem 3.1 Let $(X, \mathcal{F}, \Delta)$ and $(Y, \mathcal{G}, \Delta)$ be two complete Menger spaces, where Δ is a continuous t-norm (i.e., min. t-norm). Let A, B be mappings from X into Y and let S, T be mappings from Y into X satisfying inequalities

$$(3.1)\ F_{SAx,TBx'}(kt) \geq \min\begin{Bmatrix} F_{x,x'}(t), F_{x,SAx}(t), \\ F_{x',TBx'}(t), G_{Ax,Bx'}(t) \end{Bmatrix}$$

$$(3.2)\ G_{BSy,ATy'}(kt) \geq \min\begin{Bmatrix} G_{y,y'}(t), G_{y,BSy}(t), \\ G_{y',ATy'}(t), F_{Sy,Ty'}(t) \end{Bmatrix}$$

for all $x, x' \in X$, $y, y' \in Y$, $k \in (0,1)$ and $t > 0$. If one of the mappings A, B, S and T is continuous then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let x_0 be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively as follows:

$$Ax_0 = y_1, Sy_1 = x_1, Bx_1 = y_2, Ty_2 = x_2, Ax_2 = y_3,$$

and in general let

$$Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n},$$
$$Ty_{2n} = x_{2n}, Ax_{2n} = y_{2n+1},$$

For = 1,2, ... Using inequality (3.1), we get

$$\begin{aligned} F_{x_{2n+1},x_{2n}}(kt) &= F_{SAx_{2n},TBx_{2n-1}}(kt) \\ &\geq \min\begin{Bmatrix} F_{x_{2n},x_{2n-1}}(t), F_{x_{2n},SAx_{2n}}(t), \\ F_{x_{2n-1},TBx_{2n-1}}(t), G_{Ax_{2n},Bx_{2n-1}}(t) \end{Bmatrix} \\ &= \min\begin{Bmatrix} F_{x_{2n},x_{2n-1}}(t), F_{x_{2n},x_{2n+1}}(t), \\ F_{x_{2n-1},x_{2n}}(t),, G_{y_{2n+1},y_{2n}}(t) \end{Bmatrix} \\ &= \min\{F_{x_{2n},x_{2n-1}}(t), F_{x_{2n},x_{2n+1}}(t), G_{y_{2n+1},y_{2n}}(t)\} \end{aligned}$$

$$\geq \min\{F_{x_{2n},x_{2n-1}}(t), G_{y_{2n+1},y_{2n}}(t)\}. \quad (3.3)$$

Using inequality (3.1) again, it follows similarly that

$$F_{x_{2n},x_{2n-1}}(kt) \geq \min\{F_{x_{2n-1},x_{2n-2}}(t), G_{y_{2n},y_{2n-1}}(t)\}. \quad (3.4)$$

Similarly, using inequality (3.2), we have

$$G_{y_{2n},y_{2n+1}}(kt) \geq \min\{F_{x_{2n-1},x_{2n}}(t), G_{y_{2n-1},y_{2n}}(t)\}. \quad (3.5)$$

Again using inequality (3.2), we get

$$G_{y_{2n-1},y_{2n}}(kt) \geq \min\{F_{x_{2n-2},x_{2n-1}}(t), G_{y_{2n-2},y_{2n-1}}(t)\}. \quad (3.6)$$

Using inequalities (3.3) and (3.5), we have

$$F_{x_{2n+1},x_{2n}}(kt) \geq \min\{F_{x_{2n},x_{2n-1}}(t), G_{y_{2n+1},y_{2n}}(t)\}$$

$$\geq \min\begin{Bmatrix} F_{x_{2n},x_{2n-1}}(t), F_{x_{2n-1},x_{2n}}\left(\frac{t}{k}\right), \\ G_{y_{2n-1},y_{2n}}\left(\frac{t}{k}\right) \end{Bmatrix},$$

since, $F_{x_{2n-1},x_{2n}}\left(\frac{t}{k}\right) \geq F_{x_{2n-1},x_{2n}}(t)$ and $G_{y_{2n-1},y_{2n}}\left(\frac{t}{k}\right) \geq G_{y_{2n-1},y_{2n}}(t)$, hence

$$F_{x_{2n+1},x_{2n}}(kt) \geq \min\{F_{x_{2n},x_{2n-1}}(t), G_{y_{2n-1},y_{2n}}(t)\},$$

or

$$F_{x_{2n+1},x_{2n}}(t) \geq \min\left\{F_{x_{2n},x_{2n-1}}\left(\frac{t}{k}\right), G_{y_{2n-1},y_{2n}}\left(\frac{t}{k}\right)\right\} \quad (3.7)$$

Similarly, using inequalities (3.4) and (3.6), we have

$$F_{x_{2n},x_{2n-1}}(t) \geq \min\left\{F_{x_{2n-1},x_{2n-2}}\left(\frac{t}{k}\right), G_{y_{2n-2},y_{2n-1}}\left(\frac{t}{k}\right)\right\}. \quad (3.8)$$

It now follows from inequalities (3.5)-(3.8) that

$$F_{x_{n+1},x_n}(t) \geq \min\left\{F_{x_1,x_2}\left(\frac{t}{k^{n-1}}\right), G_{y_1,y_2}\left(\frac{t}{k^{n-1}}\right)\right\},$$

$$G_{y_{n+1},y_n}(t) \geq \min\left\{F_{x_1,x_2}\left(\frac{t}{k^{n-1}}\right), G_{y_1,y_2}\left(\frac{t}{k^{n-1}}\right)\right\},$$

for all $n = 1,2,\ldots$. Since $F_{x_1,x_2}\left(\frac{t}{k^{n-1}}\right) \to 1$ and $G_{y_1,y_2}\left(\frac{t}{k^{n-1}}\right) \to 1$ as $n \to \infty$, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X and w in Y.

Suppose that A is continuous. Then

$$\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w,$$

and so $Az = w$. Using inequality (3.1), we have

$$F_{Sw,x_{2n}}(kt) = F_{SAz,TBx_{2n-1}}(kt)$$

$$\geq \min\left\{\begin{matrix} F_{z,x_{2n-1}}(t), F_{z,SAz}(t), \\ F_{x_{2n-1},TBx_{2n-1}}(t), G_{Az,Bx_{2n-1}}(t)\end{matrix}\right\}.$$

Taking limit $n \to \infty$, we get

$$F_{Sw,z}(kt) \geq \min\left\{\begin{matrix} F_{z,z}(t), F_{z,Sw}(t), \\ F_{z,z}(t), G_{w,w}(t)\end{matrix}\right\},$$

and so

$$F_{Sw,z}(kt) \geq \min\{1, F_{z,Sw}(t), 1,1\} = F_{z,Sw}(t).$$

On employing Lemma 2.1, we have $z = Sw$. Now using inequality (3.2), we have

$$G_{Bz,y_{2n+1}}(kt) = G_{BSw,ATy_{2n}}(kt)$$

$$\geq \min\left\{\begin{matrix} G_{w,y_{2n}}(t), G_{w,BSw}(t), \\ G_{y_{2n},ATy_{2n}}(t), F_{Sw,Ty_{2n}}(t)\end{matrix}\right\}.$$

Taking limit $n \to \infty$, we get

$$G_{Bz,w}(kt) \geq \min\left\{\begin{matrix} G_{w,w}(t), G_{w,Bz}(t), \\ G_{w,w}(t), F_{z,z}(t)\end{matrix}\right\} = \min\{1, G_{w,Bz}(t), 1,1\} = G_{w,Bz}(t).$$

Appealing to Lemma 2.1, we have $Bz = w$. Using inequality (3.1), we have

$$F_{z,Tw}(kt) = F_{SAz,TBz}(kt)$$

$$\geq \min\{F_{z,z}(t), F_{z,SAz}(t), F_{z,TBz}(t), G_{Az,Bz}(t)\}$$

$$= \min\{1,1, F_{z,Tw}(t), 1\} = F_{z,Tw}(t).$$

Owing to Lemma 2.1, we have $z = Tw$. Therefore, $SA(z) = S(w) = z = Tw = TB(z)$ and $BS(w) = B(z) = w = A(z) = AT(w)$, which shows that SA and TB have a common fixed point $z \in X$ and BS and AT have a common fixed point $w \in Y$.

The proof is similar in case one of mappings B, S, T is continuous.

Uniqueness: Suppose that TB has another fixed point $z'(\neq z)$. then using inequalities (3.1) and (3.2), we have

$$\begin{aligned} F_{z,z'}(kt) &= F_{SAz,TBz'}(kt) \\ &\geq \min\{F_{z,z'}(t), F_{z,SAz}(t), F_{z',TBz'}(t), G_{Az,Bz'}(t)\} \\ &= \min\{F_{z,z'}(t), F_{z,z}(t), F_{z',z'}(t), G_{Az,Bz'}(t)\} \\ &= \min\{F_{z,z'}(t), 1,1, G_{Az,Bz'}(t)\} \end{aligned}$$

$$= \min\{F_{z,z'}(t), G_{Az,Bz'}(t)\}.$$

If we assume $F_{z,z'}(t)$ is minimum then by Lemma 2.1, the result follows. In case of $G_{Az,Bz'}(t)$, we have

$$\begin{aligned} F_{z,z'}(kt) &\geq G_{Az,Bz'}(t) = G_{BSw,ATBz'}(t) \\ &\geq \min\left\{\begin{matrix} G_{w,Bz'}\left(\frac{t}{k}\right), G_{w,BSw}\left(\frac{t}{k}\right), \\ G_{Bz',ATBz'}\left(\frac{t}{k}\right), F_{Sw,TBz'}\left(\frac{t}{k}\right)\end{matrix}\right\} \\ &= \min\left\{\begin{matrix} G_{Az,Bz'}\left(\frac{t}{k}\right), G_{BSw,BSw}\left(\frac{t}{k}\right), \\ G_{Bz',Bz'}\left(\frac{t}{k}\right), F_{z,z'}\left(\frac{t}{k}\right)\end{matrix}\right\} \\ &= \min\left\{G_{Az,Bz'}\left(\frac{t}{k}\right), 1,1, F_{z,z'}\left(\frac{t}{k}\right)\right\} \\ &= \min\left\{G_{Az,Bz'}\left(\frac{t}{k}\right), F_{z,z'}\left(\frac{t}{k}\right)\right\}. \end{aligned}$$

It implies

$$F_{z,z'}(t) \geq \min\left\{G_{Az,Bz'}\left(\frac{t}{k^2}\right), F_{z,z'}\left(\frac{t}{k^2}\right)\right\}$$

$$\begin{aligned} &\geq \min\left\{\min\left\{G_{Az,Bz'}\left(\frac{t}{k^4}\right), F_{z,z'}\left(\frac{t}{k^4}\right)\right\}, F_{z,z'}\left(\frac{t}{k^2}\right)\right\} \\ &= \min\left\{G_{Az,Bz'}\left(\frac{t}{k^4}\right), F_{z,z'}\left(\frac{t}{k^4}\right), F_{z,z'}\left(\frac{t}{k^2}\right)\right\} \\ &= \min\left\{G_{Az,Bz'}\left(\frac{t}{k^4}\right), F_{z,z'}\left(\frac{t}{k^2}\right)\right\}. \end{aligned}$$

By repeated application of above inequality, we get for each $m \in \{1,2,\ldots\}$

$$F_{z,z'}(t) \geq \min\left\{G_{Az,Bz'}\left(\frac{t}{k^{2m}}\right), F_{z,z'}\left(\frac{t}{k^2}\right)\right\}.$$

Thus since $G_{Az,Bz'}\left(\frac{t}{k^{2m}}\right) \to 1$ as $n \to \infty$, and so

$$F_{z,z'}(t) \geq F_{z,z'}\left(\frac{t}{k^2}\right),$$

Again repeating this inequality, we have

$$F_{z,z'}(t) \geq F_{z,z'}\left(\frac{t}{k^2}\right) \geq F_{z,z'}\left(\frac{t}{k^4}\right) \geq \cdots \geq F_{z,z'}\left(\frac{t}{k^{2m}}\right),$$

since $F_{z,z'}\left(\frac{t}{k^{2m}}\right) \to 1$ as $n \to \infty$, we get $F_{z,z'}(t) \geq 1$, for all $t > 0$. Hence $F_{z,z'}(t) = 1$, we have $z = z'$. Thus z is the unique fixed point of TB. It follows similarly that z is the unique fixed point of SA and w is the unique fixed point of $BS$and AT.

By setting $X = Y$ in Theorem 3.1, we deduce the following:

Corollary 3.1 Let $(X, \mathcal{F}, \Delta)$ be a complete Menger spaces, where Δ is a continuous t-norm (i.e., min. t-norm). Let A, B, S and T be mappings from X into itself satisfying inequalities

$$F_{SAx,TBy}(kt) \geq \min \begin{Bmatrix} F_{x,y}(t), F_{x,SAx}(t), \\ F_{y,TBy}(t), F_{Ax,By}(t) \end{Bmatrix} \quad (3.9)$$

$$F_{BSx,ATy}(kt) \geq \min \begin{Bmatrix} F_{x,y}(t), F_{x,BSx}(t), \\ F_{y,ATy}(t), F_{Sx,Ty}(t) \end{Bmatrix} \quad (3.10)$$

for all $x, y \in X$, $k \in (0,1)$ and $t > 0$. If one of the mappings A, B, S and T is continuous then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$and $Sw = Tw = z$.

Remark 3.1 Theorem 3.1 generalizes the result of Fisher and Murthy [9, Theorem 2] (as well as the references mentioned therein) in the framework of probabilistic settings.

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