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Generalized Well-Posedness and Multivalued Contraction

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ABSTRACT

In this paper, we establish the fixed point theorems for single and multivalued valued contraction in complete metric space. We investigate an iteration method involving projections which converges to a fixed point using multivalued contraction. Also we prove the generalized well-posedness of the fixed point problem and continuity of single valued contraction in metric space.

Key Words: *Fixed point; Multivalued mapping; Well-posedness.*

1. INTRODUCTION

Fixed point theory is an exciting branch of mathematics. In 1922, the Polish mathematician Stefan Banach proved a theorem on the existence and uniqueness of a fixed point in a complete metric space. Various authors have defined contractive mappings on a complete metric space which are generalizations of well-known banach contraction. Rhoades [5] compared and discussed the relation between all the contractions with appropriate examples. We introduce contraction mapping and establish fixed point theorems on complete metric space. The notion of set valued contraction was initiated by Nadler [2] in 1969. He proved that a set valued contraction possesses a fixed point in a complete metric space. Subsequently many authors generalized Nadlers fixed point theorem in different ways. Kunze et al. [3] have introduced an iteration method involving a projection which converges to a fixed point using multivalued nadler contraction.We investigate an iteration method involving projections which converges to a fixed point using multivalued contraction.

Let (X,d) be a metric space. Let $P(X)$ be the family of all non-empty subsets of X and let $\boldsymbol{T}: X \to P(X)$ be a multivalued mapping. A point is said to be a fixed point of the multi-valued mapping $\boldsymbol{T}$ if $x \in \boldsymbol{T}x$ or $F_{\boldsymbol{T}} = \{x \in X : x \in \boldsymbol{T}(x)\}$ and $SF_{\boldsymbol{T}} = \{x \in X : \{x\} = \boldsymbol{T}(x)\}$.

Definition 1.1 Let (X,d) be a metric space and $P(X)$ be the family of all non-empty closed and bounded subsets of X ,

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Define $d(x,B)=\inf_{y\in B} d(x,y)$,
$h(A,B)=\sup_{x\in A} d(x,B)$ for all $x\in X$
$A,B\in P(X)$ The Hausdorff metric or Hausdorff distance H_d is a function
$H_d : P(X)\times P(X)\to R$ defined by
$H_d(A,B)=\max\{h(A,B),h(B,A)\}$
$(P(X),H_d)$ is called a Hausdorff metric space.

Lemma 1.1 [3] Let (X,d) be a metric space, $x,y\in X$ and A,B,C are subsets of X. Then the following statements hold:

If $A\subseteq B$, then $d(A,C)\geq d(B,C)$ and $h(A,C)\leq h(B,C)$ and $h(C,A)\geq h(C,B)$.

$$d(x,A)\leq d(x,y)+d(y,A)$$

$$d(x,A)\leq d(x,y)+d(y,B)+h(B,A).$$

Definition 1.2 Let (X,d) be a metric space. A map $T:X\to X$ is called banach contraction if there exists $0\leq k<1$ such that $d(Tx,Ty)\leq kd(x,y)$, for all $x,y\in X$.

Definition 1.3 [2] Let (X,d) be a metric space. A map $\boldsymbol{T}:X\to P(X)$ is called multivalued contraction if there exists $0\leq k<1$ such that $H_d(Tx,Ty)\leq kd(x,y)$, for all $x,y\in X$.

Lemma 1.2 [2] If $A,B\in P(X)$ and $a\in A$ then for each $k>0$, there exists $b\in B$ such that

$$d(a,b)\leq H_d(A,B)+k.$$

2. MAIN RESULTS

Theorem 2.1 Let (X,d) be a complete metric space. Let $T:X\to X$ be a single valued map satisfying

$$d(Tx,Ty)\leq ad(x,y)+b[d(x,Tx)+d(Ty,Tx)]$$

there exists $a,b\in R^+$ with $a+2b<1$, $a+b<1$ then

(i) T has a unique fixed point i.e., $F_T=u$,

(ii) The picard iteration associated to T i.e., the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1}=Tx_n$ $n=0,1,2,\ldots.$ converges to the fixed point u.

Proof: To prove the existence of the fixed point, we show that for any $x_0\in X$ the picard iteration $\{x_n\}$ is a cauchy sequence.

$$d(x_1,x_2)=d(Tx_0,Tx_1)$$

$$\leq ad(x_0,x_1)+b[d(x_0,Tx_0)+d(Tx_0,Tx_1)]$$

$$\leq ad(x_0,x_1)+b[d(x_0,x_1)+d(x_1,x_2)]$$

$$\leq (a+b)d(x_0,x_1)+bd(x_1,x_2)$$

$$d(x_1,x_2)-bd(x_1,x_2)\leq (a+b)d(x_0,x_1)$$

$$(1-b)d(x_1,x_2)\leq (a+b)d(x_0,x_1)$$

$$d(x_1,x_2)\leq \frac{a+b}{1-b}d(x_0,x_1)$$

$$d(x_1,x_2)\leq kd(x_0,x_1)$$

$$k=\frac{a+b}{1-b},\ a+2b<1\ \ 0<k<1.$$

and by induction

$$d(x_n,x_{n+1})\leq k^n d(x_0,x_1)$$

$$d(x_n,x_m)\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots\ldots\ldots\ +d(x_{m-1},x_m)$$

$$\leq (k^n+k^{n+1}+\ldots\ldots+k^{m-1})d(x_0,x_1)$$

$$\leq \frac{k^n}{1-k}d(x_0,x_1)$$

Since $0<k<1$, it results that $k^n\to 0$ as $n\to\infty$ shows that $\{x_n\}$ is a cauchy sequence. Since (X,d) be a complete metric space, therefore $\{x_n\}$ converges to u.

$$u=\lim_{n\to\infty} x_n$$

Hence u is a fixed point of T.

$$d(u,Tu)\leq d(u,x_{n+1})+d(x_{n+1},Tu)$$

$$=d(u,x_{n+1})+d(Tx_n,Tu)$$

$$\leq d(u,x_{n+1})+ad(x_n,u)+b[d(x_n,Tx_n)+d(Tu,Tx_n)]$$

$$=d(u,x_{n+1})+ad(x_n,u)+bd(x_n,x_n)+bd(Tu,x_n)$$

$$\leq d(u,x_{n+1})+ad(x_n,u) + bk^n d(x_0,x_1)+d(Tu,x_{n+1}) \quad (1)$$

$n \to \infty$ in (1) we get

$$d(u,Tu) \leq bd(u,Tu)$$

$$(1-b)d(u,Tu) \leq 0$$

$b<1$, $d(u,Tu)=0 \quad \Rightarrow u = Tu$.

Uniqueness: On the contrary let u and v be two fixed points of T then $u = Tu$ and $v = Tv$.

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\leq ad(u,v)+b[d(u,Tu)+d(Tv,Tu)] \\ &\leq ad(u,v)+bd(u,u)+bd(v,u) \\ &\leq (a+b)d(u,v) \\ 1-(a+b)&d(u,v) \leq 0 \\ a+b<1,\ d(u,v)&=0 \quad \Rightarrow u=v. \end{aligned}$$

Example 2.1 $X=[0,1]$ endowed with Euclidean metric $d(x,y)=|x-y|$ and a map $T: X \to X$ defined as follows

$$Tx = \frac{x}{3} \quad \text{for } x \in [0,1)$$

$$T1 = \frac{1}{6}$$

Contraction condition is satisfied for $a = 1/3$ and $b = 1/2$.

Theorem 2.2 Let (X,d) be a complete metric space. Let $\boldsymbol{T}: X \to P(X)$ be a multivalued map satisfying

$$H_d(\boldsymbol{T}x,\boldsymbol{T}y) \leq ad(x,y)+b[d(x,\boldsymbol{T}x)+d(\boldsymbol{T}y,\boldsymbol{T}x)]$$

there exists $a,b \in R^+$ with $a+2b<1$, $a+b<1$ then

(i) $F_{\boldsymbol{T}} \neq \phi$

(ii) $\boldsymbol{T}$ has a unique fixed point u.

Proof: Fix any $x \in X$ and $0<k<1$, $x_1 = \boldsymbol{T}x_0$ if $H_d(\boldsymbol{T}x_0,\boldsymbol{T}x_1)=0$ then $\boldsymbol{T}x_0 = \boldsymbol{T}x_1$ i.e, $x_1 \in \boldsymbol{T}x_1$ which actually means that $F_{\boldsymbol{T}} \neq \phi$. Let $H_d(\boldsymbol{T}x_0,\boldsymbol{T}x_1) \neq 0$.

By lemma 1.2 there exist $x_2 \in \boldsymbol{T}x_1$ such that

$$d(x_1,x_2) \leq H_d(\boldsymbol{T}x_0,\boldsymbol{T}x_1)+k$$

$$\leq ad(x_0,x_1)+b[d(x_0,\boldsymbol{T}x_0)+d(\boldsymbol{T}x_1,\boldsymbol{T}x_0)]+k$$

$$\leq ad(x_0,x_1)+bd(x_0,x_1)+bd(x_2,x_1)+k$$

$$\leq (a+b)d(x_0,x_1)+bd(x_2,x_1)+k$$

$$d(x_1,x_2)-bd(x_2,x_1) \leq (a+b)d(x_0,x_1)+k$$

$$(1-b)d(x_1,x_2) \leq (a+b)d(x_0,x_1)+k$$

$$d(x_1,x_2) \leq \frac{a+b}{1-b}d(x_0,x_1)+k$$

$$d(x_1,x_2) \leq kd(x_0,x_1)+k$$

Where $k = \dfrac{a+b}{1-b}$, $a+2b<1$, $0<k<1$.

If $H_d(\boldsymbol{T}x_1,\boldsymbol{T}x_2)=0$ then $\boldsymbol{T}x_1 = \boldsymbol{T}x_2$ i.e. $x_2 \in \boldsymbol{T}x_2$. Let $H_d(\boldsymbol{T}x_1,\boldsymbol{T}x_2) \neq 0$. Again by lemma 1.2 there exists $x_3 \in \boldsymbol{T}x_2$.

$$d(x_2,x_3) \leq kd(x_1,x_2)+k^2$$

and by induction

$$\begin{aligned} d(x_n,x_{n+1}) &\leq kd(x_{n-1},x_n)+k \\ &\leq k\left(kd(x_{n-2},\, x_{n-1})+k^{n-1}\right)+k^n \\ &= k^2 d(x_{n-2},\, x_{n-1})+kk^{n-1}+k^n \\ &\leq \ldots\ldots \\ &\leq k^n d(x_0,x_1)+nk^n. \end{aligned}$$

Since $k<1$, $\sum k^n$ and $\sum nk^n$ have same radius of convergence, $\{x_n\}$ is a cauchy sequence.

Since (X,d) be a complete metric space, therefore $\{x_n\}$ converges to u.

$$\begin{aligned} d(u,\boldsymbol{T}u) &\leq d(u,x_{n+1})+d(x_{n+1},\boldsymbol{T}u) \\ &\leq d(u,x_{n+1})+H_d(\boldsymbol{T}x_n,\boldsymbol{T}u) \\ &\leq d(u,x_{n+1})+ad(x_n,u)+b\left[d(x_n,\boldsymbol{T}x_n)+d(\boldsymbol{T}u,\boldsymbol{T}x_n)\right] \\ &= d(u,x_n)+ad(x_n,u)+bd(x_n,x_n)+bd(\boldsymbol{T}u,x_n) \end{aligned}$$

$$\le d(u,x_n)+ad(x_n,u) + bk^n d(x_0,x_1)+d(\boldsymbol{T}u,x_{n+1}) \quad (2)$$

$n \to \infty$ in (2) we get

$$d(u,\boldsymbol{T}u)\le bd(u,\boldsymbol{T}u)$$

$$(1-b)d(u,\boldsymbol{T}u)\le 0$$

$$b<1,\ d(u,\boldsymbol{T}u)=0 \quad \Rightarrow\ u=\boldsymbol{T}u\,.$$

2.1 Projection on Multivalued Contraction

Given a point $x \in X$ and a compact set $A \subseteq X$ we know that the function $d(x,a)$ has at least one minimum point a^* when $a \in A$. So we have $d(x,a^*)\le d(x,a)$ for all $a \in A$. We call a^* the projection of the point x on the set A and denote it as $a^* = \pi_x A$. Obviously, a^* is not unique but we choose one of it. Let $\boldsymbol{T}: X \to P(X)$ be a multivalued mapping such that $\boldsymbol{T}(x)$ is a compact set for all $x \in X$. We define the following projection associated with a multivalued map $\boldsymbol{T}$ by $P(x)=\pi_x \boldsymbol{T}(x)$. For $x_0 \in X$ we define $x_{n+1} = P(x_n)$, $n = 0,1,2,3....$ and we call the sequence $\{x_n\}_{n=0}^{\infty}$ in this manner a picard projection iteration sequence of $\boldsymbol{T}$.

Theorem 2.3 Let (X,d) be a complete metric space. Let $\boldsymbol{T}: X \to P(X)$ be a multivalued map satisfying $H_d(Tx,Ty)\le ad(x,y)+b[d(x,Tx)+d(Ty,Tx)]$ there exists $a,b \in R^+$ with $a+2b<1$, $a+b<1$ then

(i) for all $x_0 \in X$ there exists a point $u \in X$ such that $x_{n+1} = P(x_n) \to u$ when $n \to \infty$,

(ii) u is a unique fixed point, i.e., $u \in \boldsymbol{T}u$.

Proof: Starting from the point $x_0 \in X$, take the projection $P(x_0)$ of the point on the set $\boldsymbol{T}x_0$ computing we have $d(x_0,\boldsymbol{T}x_0)=d(x_0,P(x_0))$.

Let $x_1 = P(x_0)$ and take the projection of x_1 on the set $\boldsymbol{T}x_1$, we have

$$d(x_2,x_1)=d(P(x_1),x_1)=d(x_1,\boldsymbol{T}x_1)$$
$$=d(P(x_0),\boldsymbol{T}x_1)$$
$$\le H_d(\boldsymbol{T}x_0,\boldsymbol{T}x_1)$$
$$\le ad(x_0,x_1)+b[d(x_0,\boldsymbol{T}x_0)+d(\boldsymbol{T}x_1,\boldsymbol{T}x_0)]$$
$$\le ad(x_0,x_1)+bd(x_0,x_1)+bd(x_2,x_1)$$
$$\le (a+b)d(x_0,x_1)+bd(x_2,x_1)$$
$$d(x_1,x_2)-bd(x_2,x_1)\le (a+b)d(x_0,x_1)$$
$$(1-b)d(x_1,x_2)\le (a+b)d(x_0,x_1)$$
$$d(x_1,x_2)\le \frac{a+b}{1-b}d(x_0,x_1)$$
$$d(x_1,x_2)\le kd(x_0,x_1)$$

We can use the same argument as in theorem 2.2. for the rest of the proof.

Regarding the generalized well-posedness of a fixed point problem, we have the subsequent result.

Theorem 2.4 Let (X,d) be a compact metric space and let $\boldsymbol{T}: X \to P(X)$ be a multivalued contraction then If $(x_n)_{n\in N}$ is such that $d(x_n,\boldsymbol{T}x_n)\to 0$ as $n \to \infty$, then there exists a sub sequence $(x_{n_i})_{i\in N}$ of $(x_n)_{n\in N}$ such that $x_{n_i} \xrightarrow{d} u \in F_{\boldsymbol{T}}$ as $i \to \infty$ generalized well-posedness of the fixed point problem with respect to d [4].

Proof: Let $(x_n)_{n\in N}$ be a sequence in X such that $d(x_n,\boldsymbol{T}x_n)\to 0$ as $n \to \infty$. Let $(x_{n_i})_{i\in N}$ be a subsequence of $(x_n)_{n\in N}$ such that $x_{n_i} \xrightarrow{d} u$ as $i \to \infty$. Then there exist $y_{n_i} \in \boldsymbol{T}(x_{n_i})$, such that $y_{n_i} \xrightarrow{d} u$ as $i \to \infty$. Then

$$d(u,\boldsymbol{T}u)\le d(u,\boldsymbol{T}x_{n_i})+d(y_{n_i},\boldsymbol{T}x_{n_i})+H_d(\boldsymbol{T}x_{n_i},\boldsymbol{T}u)$$
$$\le d(u,\boldsymbol{T}x_{n_i})+d(y_{n_i},\boldsymbol{T}x_{n_i})+ad(x_{n_i},u)+ b\left[d(x_{n_i},\boldsymbol{T}x_{n_i})+d(\boldsymbol{T}x_{n_i},\boldsymbol{T}u)\right]$$
$$\le d(u,\boldsymbol{T}x_{n_i})+ad(x_{n_i},u)+ bd(x_{n_i},\boldsymbol{T}x_{n_i})+bd(\boldsymbol{T}x_{n_i},\boldsymbol{T}u)\to 0$$

as $n \to \infty$. Hence $u \in F_{\boldsymbol{T}}$.

2.2 Continuity of Single Valued Contraction

Any contraction is continuous, while Kannan mapping [1] is not generally continuous on the whole space but continuous at the fixed points. Rhoades [6], [7] have found a large class of contractive type mapping which are continuous at their fixed points, but are not continuous on the whole space X. In this section we prove continuity of single valued contraction then results state that single valued contraction is continuous at its fixed point.

Theorem 2.5 Let (X,d) be a complete metric space. Let $T: X \to X$ be a single valued contraction. Then T is continuous at u, for any $u \in F_T$.

Proof: Since T is single valued contraction, there exists constants a, b such that $a+2b<1$ we know by theorem 2.1 that for any $x_0 \in X$ the picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$ $n = 0,1,2....$ converges to fixed point $u \in F_T$.

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X converging to u.

Then by taking $y = y_n$ and $x = u$ in the single valued contraction

$$d(Tx,Ty) \le ad(x,y) + b[d(x,Tx) + d(Ty,Tx)]$$

$$d(Tu,Ty_n) \le ad(u,y_n) + b[d(u,T(u)) + d(Ty_n,Tu)]$$

which in view of $Tu = u$ is equivalent to

$$d(Tu,Ty_n) \le ad(u,y_n) + bd(Ty_n,Tu)$$

$$d(Tu,Ty_n) - d(Tu,Ty_n) \le ad(u,y_n) \quad n = 0,1,2,...$$

$$(1-b)d(Tu,Ty_n) \le ad(u,y_n) \quad n = 0,1,2,.. \quad (3)$$

Now letting in (3) $n \to \infty$ $Ty_n \to Tu$ as $n \to \infty$ which shows that T is continuous at u.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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