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Some Common Fixed Point Theorems Using *CLRg* Property in Complex Valued Metric Spaces

Saurabh MANRO[1,♠], Krishnapada DAS[2]

[1] *School of Mathematics and Computer Applications, Thapar University, Patiala (Punjab), India*
[2] *Bengal Engineering and Science University, Shibpur, Botanic Garden, Howrah, India*

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ABSTRACT

The aim of this paper is to prove a common fixed point theorem for a pair of weakly compatible maps in a complex valued-metric space. As a consequence, a multitude of common fixed point theorems existing in the literature are sharpened and enriched.

Key Words: *Weak compatible mappings*, *property* (*E.A*), *CLRg property and complex valued metric space.*

1. INTRODUCTION

Fixed point theory is one of the most active branches of research in mathematics. It has broad set of applications. Banach contraction mapping principle gives us appropriate and simple conditions to establish the existence and uniqueness of solution of an operator equation $fx = x$. This principle is constructive in nature and is one of the most useful tools in the study of nonlinear equations. There are many generalizations of this principle in the literature. These generalizations were made either by using the contractive condition or by using some additional condition to the ambient space. One such generalization was made by Jungck in [4].

Recently fixed point theory for discontinuous and noncompatible maps has attracted much attention. Aamri *et al.* [1] generalized the concepts of non compatibility by defining property (*E.A.*) which allows replacing the completeness requirement of the space to a more natural condition of closedness of the range as well as relaxes the continuity of one or more maps and containment of the range of one map into the range of other which is utilized to construct the sequence of joint iterates. Sintunavarat *et al*. [7] introduced the notion of *CLRg* property. The concept of *CLRg* does not require a more natural condition of closeness of range.

The aim of this paper is to establish some new common fixed point theorems for weakly compatible maps in complex-valued metric space, introduced by Azam *et al.* [2] which is more general than classical metric space. Recently, Sastry *et al.* [6] proved that every complex-valued metric space is metrizable and hence is not real generalizations of metric spaces. But indeed it is a metric space and it is well known that complex numbers have many applications in Control theory, Fluid dynamics, Dynamic equations, Electromagnetism, Signal analysis, Quantum mechanics, Relativity, Geometry, Fractals, Analytic number theory, Algebraic number theory *etc*. Some of results in complex valued metric spaces may be seen in [3,5,8,9] and [10].

♠Corresponding author, e-mail: sauravmanro@hotmail.com

2. PRELIMINARIES

In this section we give some preliminary ideas and definitions which are needed for our discussion. The concept of complex valued metric space introduced by Azam *et al*. [2] is a generalization of the classical metric spaces. To discuss complex valued metric space they introduced a natural ordering relation $\precsim$ on the set of complex numbers as follows:

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$, recall a natural partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$,

$z_1 \prec z_2$ if and only if $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

Definition 2.1. [2] Let X be a nonempty set such that the map $d: X{\times}X \to \mathbb{C}$ satisfies the following conditions:

(C_1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C_3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex-valued metric on X, and (X, d) is called a complex-valued metric space.

Example 2.1. a) Define complex-valued metric

$$d: X{\times}X \to \mathbb{C} \text{ by } d(z_1, z_2) = \exp(3i|z_1 - z_2|).$$

Then (X, d) is a complex-valued metric space.

b) Let $X = [0, 1]$ and $d(x, y) = |x - y| + i|x - y|$ then it is easy to verify that (X, d) is a complex valued metric space.

Definition 2.2.[2] Let (X, d) complex-valued metric space and $x \in X$. Then sequence $\{x_n\}$ sequence is

(i) convergent if for every $0 \prec c \in \mathbb{C}$, there is a natural number N such that $d(x_n, x) \prec c$, for all $n > N$. We write it as $\lim_{n\to\infty} x_n = x$.

(ii) a Cauchy sequence, if for every $0 \prec c \in \mathbb{C}$, there is a natural number N such that $d(x_n, x_m) \prec c$, for all $n, m > N$.

Lemma 2.1.[2, 8] Let (X, d) be a complex valued metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to x if and only if and only if $|\, d(x_n, x)| \to 0$ as $n \to\infty$.

Definition 2.3.[4, 10] A pair of self-maps f and g of a complex-valued metric space are weakly compatible if $fgx = gfx$ for all $x \in X$ at which $fx = gx$.

Example 2.2.[10] Define complex-valued metric

$$d : X{\times}X \to \mathbb{C} \text{ by } d(z_1, z_2) = \exp(\, ia|z_1 - z_2|),$$

where a is any real constant. Then (X, d) is a complex-valued metric space. Suppose self maps f and g be defined as:

$fz = 2 \exp(i\pi/4)$ if $\mathrm{Re}(z) \ne 0$,

$fz = 3 \exp(i\pi/3)$ if $\mathrm{Re}(z) = 0$

and

$gz = 2 \exp(i\pi/4)$ if $\mathrm{Re}(z) \ne 0$,

$gz = 4 \exp(i\pi/6)$ if $\mathrm{Re}(z) = 0$.

Then maps f and g are weakly compatible at all $z \in \mathbb{C}$ with $\mathrm{Re}(z)\neq 0$.

Definition 2.4.[10] A pair of self maps f and g on a complex-valued metric space (X, d) satisfies the property $(E.A.)$ if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

The class of maps satisfying property $(E.A)$ contains the class of compatible

Definition 2.5.[7] A pair of self maps f and g on a complex-valued metric space (X, d) satisfies the common limit in the range of g property $(CLRg)$ if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gz$ for some $z \in X$.

3. MAIN RESULTS

Let us now go through our main theorem.

Theorem 3.1. Let (X, d) be a complex valued metric space. Suppose that the mappings $f, g : X\to X$ be weakly compatible self- mappings of X satisfying the contractive condition

(3.1)

$$d(fx, fy) \precsim k\,[d(fx, gy) + d(fy, gx) + d(fx, gx) + d(fy, gy)]$$

for all $x, y \in X$ where $k \in [0,1/4)$ is a constant. If f and g satisfy $CLRg$ property then f and g have a unique common fixed point.

Proof. Since f and g satisfy the $CLRg$ property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

First we claim that $gx = fx$. Suppose not, then from (3.1),

$$d(fx_n, fx) \precsim k\,[d(fx_n,gx) + d(fx,gx_n) + d(fx_n,gx_n) + d(fx,gx)].$$

By making $n\to\infty$, we have

$$d(gx, fx) \precsim k\,[d(gx,gx) + d(fx,gx) + d(gx,gx) + d(fx,gx)]$$
$$= 2k d(fx,gx)$$

which gives,

$|d(fx, gx)| \le 2k\,|d(fx, gx)| < |d(fx, gx)|$ as $k \in [0,1/4)$,

a contradiction, hence, $gx = fx$.

Now, let $w = fx = gx$. Since f and g are weakly compatible mappings, $fgx = gfx$ which implies that $fw = fgx = gfx = gw$.

Now, we claim that $fw = w$. Suppose not, then by (3.1), we have

$$d(fw,w) = d(fw,fx)$$
$$\precsim k\,[d(fw,gx) + d(fx,gw) + d(fw,gw) + d(fx,gx)]$$
$$= k\,[d(fw,gx) + d(fx,gw)]$$
$$= k\,[d(gw,fx) + d(fx,gw)]$$
$$= k\,[d(fw,w) + d(w,fw)] = 2k\, d(fw,w)$$

which gives,

$|d(fw, w)| \le 2k\,|d(fw, w)| < |d(fw, w)|$ as $k \in [0,1/4)$, a contradiction, hence, $fw = w = gw$.

Hence w is a common fixed point of f and g.

For uniqueness, we suppose that z is another common fixed point of f and g in X. Then, we have

$$d(z, w) = d(fz, fw)$$
$$\precsim k\,[d(fz, gw) + d(fw, gz) + d(fw, gw) + d(fz, gz)]$$
$$= k\,[d(z, w) + d(w, z) + d(w, w) + d(z, z)]$$
$$= 2k\, d(z, w)$$

which gives,

$|d(z, w)| \le 2k\,|d(z, w)| < |d(z, w)|$ as $k \in [0,1/4)$,

a contradiction, hence, $z = w$. Therefore, f and g have a unique common fixed point.

Theorem 3.2. Let (X, d) be a complex valued metric space and let $f, g : X \to X$ be mappings such that

(3.2)

$$d(fx, fy) \precsim a_1\, d(fx,gx) + a_2\, d(fy,gy) + a_3\, d(fy,gx) + a_4\, d(fx,gy) + a_5\, d(gy,gx)$$

for all $x, y \in X$ where $a_1, a_2, a_3, a_4, a_5 \in [0,1)$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Suppose f and g are weakly compatible and satisfy *CLRg* property then the mappings f and g have a unique common fixed point.

Proof. Since f and g satisfy the *CLRg* property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

First we claim that $gx = fx$. Suppose not, then from (3.2) and $|d(fx,gx)| > 0$ we have,

$$d(fx_n, fx) \precsim a_1\, d(fx_n, gx_n) + a_2\, d(fx, gx) + a_3\, d(fx, gx_n) + a_4\, d(fx_n, gx) + a_5\, d(gx, gx_n).$$

Making limit as $n\to\infty$, we have

$$d(gx, fx) \precsim a_1\, d(gx, gx) + a_2\, d(fx, gx) + a_3\, d(fx, gx) + a_4\, d(gx, gx) + a_5\, d(gx, gx)$$
$$= (a_2 + a_3)\, d(fx,gx),$$

which implies that, $[1 - (a_2 + a_3)]\, d(fx,gx) \precsim 0$,

that is, $d(fx,gx) \precsim 0$,

which gives us, $|d(fx, gx)| \le 0$, a contradiction.

Hence, $gx = fx$.

Now let $z = fx = gx$. Since f and g are weakly compatible mappings $fgx = gfx$ which implies that $fz = fgx = gfx = gz$.

We claim that $gz = z$. Suppose not, then by (3.2), we have

Here $d(gz,z) = d(fz,fx)$

$$\precsim a_1\, d(fz, gz) + a_2\, d(fx,gx) + a_3\, d(fx,gz) + a_4\, d(fz,gx) + a_5\, d(gx, gz)$$
$$= (a_3 + a_4 + a_5)\, d(fz, fx)$$
$$= (a_3 + a_4 + a_5)\, d(gz, z),$$

which implies that, $[1 - (a_3 + a_4 + a_5)]\, d(gz, z) \precsim 0$

that is, $d(gz, z) \precsim 0$, which gives us, $|d(gz, z)| \le 0$, a contradiction, hence, $gz = z = fz$.

Hence z is a common fixed point of f and g.

For uniqueness, let w is another common fixed point of f and g in X. Then, we have

$$d(w, z) = d(fw, fz)$$
$$\precsim a_1\, d(fw, gw) + a_2\, d(fz, gz) + a_3\, d(fz, gw) + a_4\, d(fw, gz) + a_5\, d(gz, gw)$$
$$= (a_3 + a_4 + a_5)\, d(w, z),$$

which implies that, $[1 - (a_3 + a_4 + a_5)]\, d(w, z) \precsim 0$,

that is, $d(w, z) \precsim 0$,

which gives us, $|d(z, w)| \le 0$, a contradiction, hence, $z = w$. Therefore, f and g have a unique common fixed point.

Example 3.1. Let $X = [0, 1]$ and $d(x, y) = |x - y| + i|x - y|$ and the mappings $f, g : X \to X$ be defined by

$$fx = (1 + x) / 5 \text{ and } gx = x \text{ for all } x \in X.$$

Then f and g satisfies all the condition of the Theorem 3.1 by taking $k = 1/8$ and $x = 1/4$ is the common fixed point theorem of f and g.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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