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Integral Representations for Bessel Matrix Functions

Bayram ÇEKİM[1,♠], Esra ERKUŞ–DUMAN[1]

[1] *Gazi University, Faculty of Science, Department of Mathematics, Ankara, TURKEY*

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ABSTRACT

Jódar et. al. [Util. Math. 46 (1994) 129-141] introduced the concept of Bessel matrix functions of the first kind. In this paper, we derive integral representations for these functions.

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1. INTRODUCTION

The mathematicians have interested in some properties of the special matrix functions and polynomials [5, 6, 8, 9]. For example, in the 1990s, Bessel matrix functions are introduced and defined by Jódar et. al. in [2, 3, 4, 6]. The authors consider Bessel type differential equation

$$t^2 X''(t) + tX'(t) + (t^2 I - A^2)X(t) = \theta, 0 < t < \infty$$

where A is a matrix in $\mathbb{C}^{rxr}$ and $X(t)$ is a $\mathbb{C}^{rx1}$-valued function. They obtain different solutions of Bessel type differential equations according to matrix A. They also define Bessel matrix functions of the first kind and the second kind and give the general solution of this equation. In this paper, we obtain some integral representations for these Bessel matrix functions.

We first recall some concepts and properties of the matrix functional calculus. Throughout the paper, H represents the r -dimensional Jordan block defined by

$$H = \begin{bmatrix} \nu & 1 & 0 & \dots & 0 \\ 0 & \nu & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & \nu \end{bmatrix} \in \mathbb{C}^{r\times r}. \tag{1.1}$$

As usual, I and θ denote the identity matrix and the null matrix in $\mathbb{C}^{rxr}$, respectively. In [1], if $f(z)$ and $g(z)$ are holomorphic functions in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{rxr}$ for which $\sigma(A) \subset \Omega$, where $\sigma(A)$ denotes the spectrum of A, then

$$f(A)g(A) = g(A)f(A).$$

The reciprocal scalar Gamma function, $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$, is an entire function of the complex variable z. Thus, for any $A \in \mathbb{C}^{rxr}$, the Riesz-Dunford functional calculus [1] shows that $\Gamma^{-1}(A)$ is well defined and is, indeed, the inverse of $\Gamma(A)$. Hence: if $A \in \mathbb{C}^{rxr}$, is such that $z \neq 0$ and z is not a negative integer for every $z \in \sigma(A)$, it follows that

♠Corresponding author, e-mail: bayramcekim@gazi.edu.tr

$$\Gamma^{-1}(A) = A(A+I)...(A+kI)\Gamma^{-1}(A+(k+1)I). \quad (1.2)$$

Let us consider now the Bessel function of the first kind of order ν defined by

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(\nu+m+1)\left(\frac{t}{2}\right)^{2m+\nu}, 0<t<\infty.$$

From [7], $J_\nu(t)$ is an entire function of paremeter ν. Thus, if H is a Jordan block of the form (1.1), $t>0$, we can write the image by means of the matrix functional calculus acting on the matrix H and the function of ν, $J_\nu(t)$, one has Bessel matrix function of the first kind of order H as follows:

$$J_H(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+1)I)\left(\frac{t}{2}\right)^{2mI+H} \quad (1.3)$$

where λ is not a negative integer for every $\lambda \in \sigma(H)$ [3]. Let us take $A \in \mathbb{C}^{rxr}$ satisfying that

λ is not a negative integer for every $\lambda \in \sigma(A)$. (1.4)

In [3], the Bessel matrix function of the first kind of order A was defined as follows:

$$J_A(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(A+(m+1)I)\left(\frac{t}{2}\right)^{2mI+A}. \quad (1.5)$$

We now consider the general case. Let A be a matrix satisfying condition (1.4) and $H = diag(H_1, ..., H_k)$ be the Jordan canonical form of A, where H_i is a Jordan block defined in the following form, for $p_i \geq 1$,

$$H_i = \begin{bmatrix} \nu_i & 1 & 0 & \dots & 0 \\ 0 & \nu_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & \nu_i \end{bmatrix} \in \mathbb{C}^{p_i \times p_i}, p_1 + p_2 + \cdots + p_k = r \quad (1.6)$$

and $H_i = (\nu_i)$ if H_i is a Jordan block of size 1×1 for ν_i is not a negative integer for $1 \leq i \leq k$. Bessel matrix function of the first kind can be written

$$J(t,H) = [diag_{1\leq i\leq k}(J_{H_i}(t))].$$

Furthermore, if P is a invertible matrix in $\mathbb{C}^{rxr}$ such that

$$H = diag(H_1, ..., H_k) = PAP^{-1},$$

then we have the same Bessel matrix function of the first kind of order A in (1.5) (see [3]).

Definition 1. Let P be a positive stable matrix in $\mathbb{C}^{rxr}$, that is, $Re(\alpha) > 0$ for $\forall\alpha \in \sigma(P)$. Then Gamma matrix function in [5] is defined by

$$\Gamma(P) = \int_0^\infty e^{-t} t^{P-I} dt, \; t^{P-I} = \exp[(P-I)\ln t]. \quad (1.7)$$

Definition 2. Let X and Y be positive stable matrices in $\mathbb{C}^{rxr}$. Then Beta matrix function in [5] is defined by

$$B(X,Y) = \int_0^1 t^{X-I}(1-t)^{Y-I} dt. \quad (1.8)$$

Lemma 1. Let $X, Y, X+Y$ be positive stable matrices in $\mathbb{C}^{rxr}$ and $XY = YX$. Then we have

$B(X,Y) = \Gamma(X)\Gamma(Y)\Gamma^{-1}(X+Y)$, see [5].

Lemma 2. Let X and Y be matrices in $\mathbb{C}^{rxr}$ satisfying that $XY = YX$ and $X+nI, Y+nI, X+Y+nI$ are invertible for $\forall n \in \mathbb{N}$. Then

$$B(X,Y) = \Gamma(X)\Gamma(Y)\Gamma^{-1}(X+Y) \quad (1.9)$$

holds (see [8]).

2. SOME NEW INTEGRAL REPRESENTATIONS FOR BESSEL MATRIX FUNCTIONS

Let H be a Jordan block in (1.1) satisfying the condition

$Re(\nu) > -\frac{1}{2}$ for $\forall\nu \in \sigma(H)$.

Consider the integral

$$S = \int_{-1}^{1} (1-t^2)^{H-\frac{1}{2}I} e^{ixt} dt, x>0. \quad (2.1)$$

Using Taylor series for e^{ixt}, (2.1) can be written as follows:

$$S = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \int_{-1}^{1} (1-t^2)^{H-\frac{1}{2}I} t^{kI} dt. \quad (2.2)$$

If k is odd, integrand in (2.2) is θ since the integrand is an odd function. For $k = 2m$, we have

$$S = \sum_{m=0}^{\infty} \frac{(ix)^{2m}}{(2m)!} \int_{-1}^{1} (1-t^2)^{H-\frac{1}{2}I} t^{2mI} dt$$

$$= 2\sum_{m=0}^{\infty} \frac{(ix)^{2m}}{(2m)!} \int_{0}^{1} (1-t^2)^{H-\frac{1}{2}I} t^{2mI} dt.$$

Then, taking $u = t^2$, we get

$$S = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \int_0^1 (1-u)^{H-\frac{1}{2}I} u^{\left(m-\frac{1}{2}\right)I} du.$$

From the Beta matrix function in Definition 2 and Lemma 2, we obtain that

$$\begin{aligned} S &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} B\left(H+\frac{1}{2}I, \left(m+\frac{1}{2}\right)I\right) \\ &= \Gamma\left(H+\frac{1}{2}I\right) \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \Gamma\left(\left(m+\frac{1}{2}\right)I\right) \Gamma^{-1}\left(H+(m+1)I\right). \end{aligned}$$

We also get

$$(2m)! = \frac{1}{\sqrt{\pi}} 2^{2m} m! \Gamma\left(m+\frac{1}{2}\right).$$

Thus, we can write

$$\begin{aligned} S &= \sqrt{\pi}\Gamma\left(H+\frac{1}{2}I\right)\left(\frac{x}{2}\right)^{-H} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma^{-1}\left(H+(m+1)I\right)\left(\frac{x}{2}\right)^{H+2mI} \\ &= \sqrt{\pi}\Gamma\left(H+\frac{1}{2}I\right)\left(\frac{x}{2}\right)^{-H} J_H(x). \end{aligned}$$

Now, we ready to give the first integral representation for the Bessel matrix functions.

Theorem 1. Let H be a Jordan block in (1.1) satisfying the condition

$Re(\nu) > -\frac{1}{2}$ for $\forall \nu \in \sigma(H)$.

For $x > 0$, the Bessel matrix function holds the following representation:

$$J_H(x) = \frac{1}{\sqrt{\pi}} \Gamma^{-1}\left(H+\frac{1}{2}I\right)\left(\frac{x}{2}\right)^{H} \int_{-1}^{1} (1-t^2)^{H-\frac{1}{2}I} e^{ixt} dt. \quad (2.3)$$

Now, we generalize for this theorem. Let H_i be the same as in (1.1) and $H = diag(H_1, \ldots, H_k)$ be a matrix in $\mathbb{C}^{rxr}$. Here ν_i satisfies the condition $Re(\nu_i) > -\frac{1}{2}$ for $\forall \nu_i \in \sigma(H_i), 1 \le i \le k$. For the matrix $H = diag(H_1, \ldots, H_k)$, (2.3) can be provided, easily.

Now let H and M be Jordan blocks in (1.1) satisfying condition

$$\left.\begin{array}{l} Re(\nu) \notin \mathbb{Z}^- \text{ for } \forall \nu \in \sigma(H) \\ Re(\beta) \notin \mathbb{Z}^- \text{ for } \forall \beta \in \sigma(M) \\ \text{and } Re(\nu) > Re(\beta) \end{array}\right\}. \quad (2.4)$$

Then, consider the integral

$$S = \int_0^1 (1-t^2)^{H-M-I} t^{M+I} J_M(xt) dt, x > 0. \quad (2.5)$$

Using (1.3), we can write (2.5) as follows:

$$\begin{aligned} S = \sum_{m \ge 0} & \frac{(-1)^m \Gamma^{-1}(M+(m+1)I)}{m!} \left(\frac{x}{2}\right)^{M+2mI} \\ & \times \int_0^1 (1-t^2)^{H-M-I} t^{2M+(2m+1)I} dt. \end{aligned} \quad (2.6)$$

Then, taking $u = t^2$, we get

$$S = \frac{1}{2} \sum_{m \ge 0} \frac{(-1)^m}{m!} \Gamma^{-1}(M+(m+1)I) \left(\frac{x}{2}\right)^{M+2mI} \int_0^1 (1-u)^{H-M-I} u^{M+mI} du.$$

From the Beta matrix function in Definition 2 and Lemma 2, we obtain that

$$\begin{aligned} S &= \frac{1}{2} \sum_{m \ge 0} \frac{(-1)^m}{m!} \Gamma^{-1}(M+(m+1)I) \left(\frac{x}{2}\right)^{M+2mI} B(H-M, M+(m+1)I) \\ &= \frac{1}{2} \Gamma(H-M) \sum_{m \ge 0} \frac{(-1)^m}{m!} \Gamma^{-1}(H+(m+1)I) \left(\frac{x}{2}\right)^{M+2mI} \\ &= \frac{1}{2} \Gamma(H-M) \left(\frac{x}{2}\right)^{M-H} \sum_{m \ge 0} \frac{(-1)^m}{m!} \Gamma^{-1}(H+(m+1)I) \left(\frac{x}{2}\right)^{H+2mI} \\ &= \frac{1}{2} \Gamma(H-M) \left(\frac{x}{2}\right)^{M-H} J_H(x). \end{aligned}$$

Then, we get the next theorem.

Theorem 2. Let H and M be Jordan blocks in (1.1) satsifying the conditions in (2.4). Then for $x > 0$, the Bessel matrix function satisfies the following representations:

$$J_H(x) = 2\Gamma^{-1}(H-M) \left(\frac{x}{2}\right)^{H-M} \int_0^1 (1-t^2)^{H-M-I} t^{M+I} J_M(xt) dt. \quad (2.7)$$

One can also generalize the above result as follows.

Theorem 3. If A and M are matrices in $\mathbb{C}^{rxr}$ satisfy condition

$$\left.\begin{array}{l} Re(\nu) \notin \mathbb{Z}^- \text{ for } \forall \nu \in \sigma(A), \\ Re(\beta) \notin \mathbb{Z}^- \text{ for } \forall \beta \in \sigma(M), \\ Re(\alpha) \notin \mathbb{Z}^- \cup \{0\} \text{ for } \forall \alpha \in \sigma(A-M) \\ \text{and } AM = MA \end{array}\right\},$$

then we obtain

$$J_A(x) = 2\Gamma^{-1}(A-M) \left(\frac{x}{2}\right)^{A-M} \int_0^1 (1-t^2)^{A-M-I} t^{M+I} J_M(xt) dt, x > 0.$$

Now, we obtain integrals involving Bessel matrix functions.

Theorem 4. Let H be a Jordan block in (1.1) satisfying the condition

$Re(\nu) > 0$ for $\forall \nu \in \sigma(H)$.

Then we get

$$(i) \int_0^\infty J_H(bx)x^H e^{-ax}dx = \frac{2^H \Gamma\left(H+\frac{1}{2}I\right)}{\sqrt{\pi}} b^H (a^2+b^2)^{-H-\frac{1}{2}I}$$

$$(ii) \int_0^\infty J_H(bx)x^{H+I} e^{-ax}dx = a\frac{2^{H+I} \Gamma\left(H+\frac{3I}{2}\right)}{\sqrt{\pi}} b^H (a^2+b^2)^{-H-\frac{3}{2}I}$$

where a and b are arbitrary positive real numbers.

Proof. From (1.3), the left-hand side of (i) can be written

$$\int_0^\infty J_H(bx)x^H e^{-ax}dx$$
$$= \sum_{m\geq 0} \frac{(-1)^m}{m!} \Gamma^{-1}(H+(m+1)I)\left(\frac{b}{2}\right)^{H+2mI} \int_0^\infty x^{2H+2mI}e^{-ax}dx.$$

Taking $u = ax$ and using the Gamma matrix function, we have

$$\int_0^\infty J_H(bx)x^H e^{-ax}dx$$
$$= \sum_{m\geq 0} \frac{(-1)^m}{m!} \Gamma^{-1}(H+(m+1)I)\left(\frac{b}{2}\right)^{H+2mI} \Gamma(2H+(2m+1)I)a^{-(2H+(2m+1)I)}$$
$$= 2\sum_{m\geq 0} \frac{(-1)^m}{m!} \Gamma^{-1}(H+mI)\left(\frac{b}{2}\right)^{H+2mI} \Gamma(2H+2mI)a^{-(2H+(2m+1)I)}.$$

On the other hand, with the help of matrix fuctional calculus in [1], we get

$$\Gamma(2H+2mI) = \frac{1}{\sqrt{\pi}}\Gamma(H+mI)\Gamma\left(H+\left(m+\frac{1}{2}\right)I\right)2^{2H+(2m-1)I}.$$

Using the above equation, we write

$$\int_0^\infty J_H(bx)x^H e^{-ax}dx = \frac{2}{\sqrt{\pi}}\sum_{m\geq 0}\left\{\frac{(-1)^m}{m!}\left(\frac{b}{2}\right)^{H+2mI}\right.$$
$$\left.\times\Gamma\left(H+\left(m+\frac{1}{2}\right)I\right)2^{2H+(2m-1)I}a^{-(2H+(2m+1)I)}\right\}. \quad (2.8)$$

We also obtain that

$$(a^2+b^2)^{-H-\frac{1}{2}I}$$
$$= a^{-2H-I}\sum_{m\geq 0}(-1)^m \frac{\left(H+\frac{1}{2}I\right)_m}{m!}\left(\frac{b^2}{a^2}\right)^m$$
$$= a^{-2H-I}\Gamma^{-1}\left(H+\frac{1}{2}I\right)\sum_{m\geq 0}(-1)^m \frac{\Gamma\left(H+\left(m+\frac{1}{2}\right)I\right)}{m!}\left(\frac{b^2}{a^2}\right)^m. \quad (2.9)$$

Hence, the proof is completed from (2.8) and (2.9).

(ii) For the proof of (ii), it is enough to differentiate both sides with respect to a in (i).

Now, let us give a generalization for this theorem. Let H_i be the same as in (1.1) and $H = diag(H_1, \ldots, H_k)$ be a matrix in $\mathbb{C}^{rxr}$. Here ν_i satisfies the condition $Re(\nu_i) > 0$ for $\forall \nu_i \in \sigma(H_i), 1 \leq i \leq k$. For the matrix $H = diag(H_1, \ldots, H_k)$, Theorem 4 can be easily provided.

Furthermore, if P is an invertible matrix in $\mathbb{C}^{rxr}$ such that

$$H = diag(H_1, \ldots, H_k) = PAP^{-1},$$

then one can get the following theorem for the matrix A.

Theorem 5. If A is a matrix in $\mathbb{C}^{rxr}$ satisfying the condition

$Re(\nu) > 0$ for $\forall \nu \in \sigma(A)$,

then we obtain

$$(i) \int_0^\infty J_A(bx)x^A e^{-ax}dx = \frac{2^A \Gamma\left(A+\frac{1}{2}I\right)}{\sqrt{\pi}} b^A (a^2+b^2)^{-A-\frac{1}{2}I}$$

$$(ii) \int_0^\infty J_A(bx)x^{A+I} e^{-ax}dx = a\frac{2^{A+I} \Gamma\left(A+\frac{3I}{2}\right)}{\sqrt{\pi}} b^A (a^2+b^2)^{-A-\frac{3}{2}I}$$

where a and b are arbitrary positive real numbers.

Theorem 6. Let H be a Jordan block in (1.1) satisfying the condition

$Re(\nu) > -1$ for $\forall \nu \in \sigma(H)$.

Then we obtain that

$$(i)\quad \int_0^\infty J_H(bx)x^{H+I}e^{-ax^2}dx = b^H(2a)^{-H-I}\exp\left(-\frac{b^2}{4a}\right)$$

$$(ii)\quad \int_0^\infty J_H(bx)x^{H+3I}e^{-ax^2}dx = 2b^H(2a)^{-H-2I}\left(H+\left(1-\frac{b^2}{4a}\right)I\right)\exp\left(-\frac{b^2}{4a}\right),$$

where a and b are arbitrary positive real numbers.

Proof. The proof of the theorem is very similar to Theorem 4.

Now, let H_i, v_i $(i = 1,2,\dots,k)$ and H, P be as stated before. Then, we get the final result.

Theorem 7. If A is a matrix in $\mathbb{C}^{rxr}$ satisfying the condition

$Re(v) > -1$ for $\forall v \in \sigma(A)$,

then we obtain

$$(i)\quad \int_0^\infty J_A(bx)x^{A+I}e^{-ax^2}dx = b^A(2a)^{-A-I}\exp\left(-\frac{b^2}{4a}\right)$$

$$(ii)\quad \int_0^\infty J_A(bx)x^{A+3I}e^{-ax^2}dx = 2b^A(2a)^{-A-2I}\left(A+\left(1-\frac{b^2}{4a}\right)I\right)\exp\left(-\frac{b^2}{4a}\right),$$

where a and b are arbitrary positive real numbers.

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