Gazi University Journal of Science
GU J Sci
27(2):739-745 (2014)

Common Coupled Fixed Point Theorems in Fuzzy Metric Spaces

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Received: 26.10.2013, Accepted: 29.11.2013

ABSTRACT

In this paper, we obtain two general common coupled fixed point theorems for maps in fuzzy metric spaces.

Key Words: *Fuzzy metric space, common fixed points, weakly compatible maps, Coupled fixed point*

1. INTRODUCTION

The theory of fuzzy sets was introduced by L. Zadeh [13] in 1965. George and Veeramani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [11]. Grabiec[15] proved the contraction principle in the setting of fuzzy metric spaces introduced in [1]. For fixed point theorems in fuzzy metric spaces some of the interesting references are [1,3-12,15-21, 25, 26]. In the sequel, we need the following.

Definition 1.1 ([2]). *A binary operation* $*:[0,1]\times[0,1]\to[0,1]$ *is a continuous t-norm if it satisfies the following conditions :*

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. a*1 = a for all a ∈ [0,1],
4. $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for each $a,b,c,d\in[0,1]$.

Two typical examples of continuous t-norm are $a*b=ab$ and $a*b=min\{a,b\}$.

Definition 1.2 ([1]). *A 3-tuple* $(X,M,*)$ *is called a fuzzy metric space if* X *is an orbitrary non-empty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2\times(0,\infty)$ *satisfying the following conditions for each* $x,y,z\in X$ *and each* t *and* $s>0$,

1. $M(x,y,t)>0$,
2. $M(x,y,t)=1$ if and only if $x=y$,
3. $M(x,y,t)=M(y,x,t)$,
4. $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,
5. $M(x,y,.):(0,\infty)\to[0,1]$ is continuous.

Let $(X,M,*)$ be a fuzzy metric space. For $t>0$, the open ball $B(x,r,t)$ with centre $x\in X$ and radius $0<r<1$ is defined by

$B(x,r,t)=\{y\in X: M(x,y,t)>1-r\}$.

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A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable. A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x,y,t) > 1 - r$ for all $x, y \in A$.

Lemma 1.3 ([15]). *Let $(X,M,*)$ be a fuzzy metric space. Then $M(x,y,t)$ is non-decreasing with respect to t, for all x,y in X.*

Definition 1.4 *Let $(X,M,*)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0,\infty)$ if*

$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x,y,t)$ whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0,\infty)$ converges to a point $(x,y,t) \in X^2 \times (0,\infty)$, i.e., whenever

$$\lim_{n\to\infty} M(x_n,x,t) = \lim_{n\to\infty} M(y_n,y,t) = 1 \text{ and } \lim_{n\to\infty} M(x,y,t_n) = M(x,y,t).$$

Lemma 1.5 ([12]). *Let $(X,M,*)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0,\infty)$.*

Let $\lim_{t\to\infty} M(x,y,t) = 1, \forall x,y \in X$ ....(A).

Lemma 1.6 ([20]). *Let $\{y_n\}$ be a sequence in fuzzy metric space $(X,M,*)$ satisfying (A). If there exists a positive number $k < 1$ such that*

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t),\ t > 0,\ n = 1,2,...,$$

then $\{y_n\}$ is a Cauchy sequence in X.

Now, we prove a lemma slight different from Lemma 1.6.

Lemma 1.7 *Let $\{z_n\}$ and $\{p_n\}$ be sequences in fuzzy metric space $(X,M,*)$* satisfying (A). If there exists a positive number $k < 1$ such that

$$\min\{M(z_n,z_{n+1},kt), M(p_n,p_{n+1},kt)\} \geq \min\{M(z_{n-1},z_n,t), M(p_{n-1},p_n,t)\}$$

for all $t > 0,\ n = 1,2,...,$ then $\{z_n\}$ and $\{p_n\}$ are Cauchy sequences in X.

Proof. We have

$$\min\{M(z_n,z_{n+1},t), M(p_n,p_{n+1},t)\}$$
$$\geq \min\left\{M(z_{n-1},z_n,\frac{t}{k}), M(p_{n-1},p_n,\frac{t}{k})\right\}$$
$$\geq \min\left\{M(z_{n-2},z_{n-1},\frac{t}{k^2}), M(p_{n-2},p_{n-1},\frac{t}{k^2})\right\}$$

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$$\geq \min\left\{M(z_0,z_1,\frac{t}{k^n}), M(p_0,p_1,\frac{t}{k^n})\right\}.$$

Hence

$$M(z_n,z_{n+1},t) \geq min\{M(z_0,z_1,\frac{t}{k^n}), M(p_0,p_1,\frac{t}{k^n})\}.$$

Now, for any positive integer p,

$$M(z_n,z_{n+p},t)$$
$$\geq M(z_n,z_{n+1},\frac{t}{p}) * M(z_{n+1},z_{n+2},\frac{t}{p}) * ..... * M(z_{n+p-1},z_{n+p},\frac{t}{p})$$
$$\geq \min\left\{M(z_0,z_1,\frac{t}{pk^n}), M(p_0,p_1,\frac{t}{pk^n})\right\} *$$
$$\min\left\{M(z_0,z_1,\frac{t}{pk^{n+1}}), M(p_0,p_1,\frac{t}{pk^{n+1}})\right\}$$
$$* .... *$$
$$\min\left\{M(z_0,z_1,\frac{t}{pk^{n+p-1}}), M(p_0,p_1,\frac{t}{pk^{n+p-1}})\right\}.$$

Letting $n \to \infty$ and using (A), we have

$$\lim_{n\to\infty} M(z_n,z_{n+p},t) \geq 1*1*......*1 = 1.$$

Hence $\lim_{n\to\infty} M(z_n,z_{n+p},t) = 1$.

Thus $\{z_n\}$ is a Cauchy sequence in X. Similalrly, we can show that $\{p_n\}$ is also a Cauchy sequence in X.

Lemma 1.8 ([20]). *Let $(X,M,*)$ be a fuzzy metric space satisfying (A). If there exists $k \in (0,1)$ such that $M(x,y,kt) \geq M(x,y,t)$ for all $x,y \in X$ and $t > 0$, then $x = y$.*

Now, we give the following lemma.

Lemma 1.9 *Let $(X,M,*)$ be a fuzzy metric space satisfying (A). Let $f: X \to X$ be a mapping such that $min\{M(fx,x,kt), M(fy,y,kt)\} \geq min\{M(fx,x,t), M(fy,y,t)\}$* for all $x,y \in X, t > 0$ and $k \in (0,1)$. Then $fx = x$ and $fy = y$.

***Proof*.** We have

$$min\{M(fx,x,t),M(fy,y,t)\} \geq min\left\{M(fx,x,\frac{t}{k}),M(fy,y,\frac{t}{k})\right\}$$
$$\geq min\left\{M(fx,x,\frac{t}{k^2}),M(fy,y,\frac{t}{k^2})\right\}$$
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$$\geq min\left\{M(fx,x,\frac{t}{k^n}),M(fy,y,\frac{t}{k^n})\right\}$$
$\to 1$ *as* $n \to \infty$ from the condition (A).

Hence $M(fx,x,t) = M(fy,y,t) = 1$ for all $t > 0$. Thus $fx = x$ and $fy = y$.

In 2010, Sedghi, Altun and Shobe [22] introduced n-property in fuzzy metric spaces as follows:

Definition 1.10 ([22]) *. Let $(X,M,*)$ be a fuzzy metric space. M is said to satisfy the n-property on $X^2 \times (0,\infty)$ if $\lim_{n\to\infty}\left[M(x,y,k^n t)\right]^{n^p} = 1$ whenever $x,y \in X, k > 1$ and $p > 0$.*

Based on this they [22] obtained the Lemma 1.6 without the condition (A).

Recently Xin-Qi Hu [24] observed that if M satisfies the n-property then the condition (A) is satisfied. He also given an example (Ex.2, [24]) to show that the condition (A) need not imply the n-property .

In 2006, Bhaskar and Lakshmikantham [23] introduced the notion of a coupled fixed point in partially ordered metric spaces, also discussed some problems of the uniqueness of a coupled fixed point and applied their results to the problems of the existence and uniqueness of a solution for the periodic boundary value problems.

In this paper, we prove coupled fixed point theorems for two and four mappings in fuzzy metric spaces.

Definition 1.11 ([23]) *. Let X be a nonempty set. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x,y)$ and $y = F(y,x)$.*

Definition 1.12 ([14]) *. Let X be a nonempty set. An element $(x,y) \in X \times X$ is called*

(i) a coupled coincidence point of $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x,y)$ and $gy = F(y,x)$.

(ii) a common coupled fixed point of $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x,y)$ and $y = gy = F(y,x)$.

(iii) a point $x \in X$ is called a common fixed point of $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x,x)$.

Definition 1.13 ([9]) *. Let X be a nonempty set. The mappings $F : X \times X \to X$ and $g : X \to X$ are called W-compatible if $g(F(x,y)) = F(gx,gy)$ and $g(F(y,x)) = F(gy,gx)$ whenever $gx = F(x,y)$ and $gy = F(y,x)$ for some $(x,y) \in X \times X$.*

2. MAIN RESULTS

Theorem 2.1. *Let $(X,M,*)$ be a fuzzy metric space satisfying (A) and $f,g : X \to X$* and *$F,G : X \times X \to X$* be mappings satisfying

(2.1.1) $$M\big(F(x,y),G(u,v),kt\big) \geq \min\left\{\begin{matrix}M(fx,gu,t),M(fy,gv,t),\\ M(fx,F(x,y),t),M(gu,G(u,v),t)\end{matrix}\right\}$$
for all $x,y,u,v \in X$, $\forall\ t > 0$ and $k \in (0,1)$,

(2.1.2) $F(X \times X) \subseteq g(X)$ and $G(X \times X) \subseteq f(X)$,

(2.1.3) one of $f(X)$ and $g(X)$ is complete and

(2.1.4) the pairs (f,F) and (g,G) are W-compatible.

Then f,g,F and G have a unique common coupled fixed point in $X \times X$ and also they have a unique common fixed point in X .

Proof. Let x_0 and y_0 be in X .

Since $F(X \times X) \subseteq g(X)$, we can choose x_1 , $y_1 \in X$ such that $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$.

Since $G(X \times X) \subseteq f(X)$, we can choose x_2 , $y_2 \in X$ such that $fx_2 = G(x_1,y_1)$ and $fy_2 = G(y_1,x_1)$.

Continuing this process we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$gx_{2n+1} = F(x_{2n},y_{2n}) = z_{2n}$, say;
$gy_{2n+1} = F(y_{2n},x_{2n}) = p_{2n}$, say;

$fx_{2n+2} = G(x_{2n+1},y_{2n+1}) = z_{2n+1}$, say and
$fy_{2n+2} = G(x_{2n+1},x_{2n+1}) = p_{2n+1}$, say for $n = 0,1,2,......$

$$M(z_{2n},z_{2n+1},kt) = M(F(x_{2n},y_{2n}),G(x_{2n+1},y_{2n+1}),kt)$$
$$\geq \min\{M(z_{2n-1},z_{2n},t),M(p_{2n-1},p_{2n},t)),M(z_{2n-1},z_{2n},t),M(z_{2n},z_{2n+1},t)\}$$
$$= \min\{M(z_{2n-1},z_{2n},t),M(p_{2n-1},p_{2n},t)\}$$

$$M(p_{2n},p_{2n+1},kt) = M(F(y_{2n},x_{2n}),G(y_{2n+1},x_{2n+1}),kt)$$
$$\geq \min\left\{\begin{matrix}M(p_{2n-1},p_{2n},t),M(z_{2n-1},z_{2n},t),M(p_{2n-1},p_{2n},t),\\ M(p_{2n},p_{2n+1},t)\end{matrix}\right\}$$
$$= \min\{M(p_{2n-1},p_{2n},t),M(z_{2n-1},z_{2n},t)\}$$

Thus

$$\min\left\{\begin{matrix}M(z_{2n},z_{2n+1},kt),\\ M(p_{2n},p_{2n+1},kt)\end{matrix}\right\} \geq \min\left\{\begin{matrix}M(z_{2n-1},z_{2n},t),\\ M(p_{2n-1},p_{2n},t)\end{matrix}\right\} \text{-------(I)}$$

Similarly we can show that

$$\min\left\{\begin{matrix} M(z_{2n+1}, z_{2n+2}, kt), \\ M(p_{2n+1}, p_{2n+2}, kt) \end{matrix}\right\} \geq \min\left\{\begin{matrix} M(z_{2n}, z_{2n+1}, t), \\ M(p_{2n}, p_{2n+1}, t) \end{matrix}\right\} \text{ --- (II)}$$

Thus from (I) and (II) we have

$$\min\{M(z_n, z_{n+1}, kt), M(p_n, p_{n+1}, kt))\} \geq \min\{M(z_{n-1}, z_n, t), M(p_{n-1}, p_n, t)\}.$$

From Lemma 1.7 , it follows that $\{z_n\}$ and $\{p_n\}$ are Cauchy sequences in X .

Suppose $f(X)$ is complete .

Then $\{z_{2n+1}\} \to fx = \alpha$, say and $\{p_{2n+1}\} \to fy = \beta$, say for some $x, y \in X$.

Since $\{z_n\}$ and $\{p_n\}$ are Cauchy, we have $\{z_{2n+2}\} \to \alpha$ and $\{p_{2n+2}\} \to \beta$.

$$M(F(x,y), z_{2n+1}, kt) = M(F(x,y), G(x_{2n+1}, y_{2n+1}), kt)$$

$$\geq \min\{M(fx, z_{2n}, t), M(fy, p_{2n}, t), M(fx, F(x,y), t), M(z_{2n}, z_{2n+1}, t)\}$$

Letting $n \to \infty$, we get

$$M(F(x,y), fx, kt) \geq min\{1, 1, M(fx, F(x,y), t), 1\} = M(fx, F(x,y), t).$$

From Lemma 1.8, we have $F(x,y) = fx = \alpha$.

$$M(F(y,x), p_{2n+1}, kt) = M(F(y,x), G(y_{2n+1}, x_{2n+1}), kt)$$

$$\geq \min\left\{\begin{matrix} M(fy, p_{2n}, t), M(fx, z_{2n}, t), M(fy, F(y,x), t), \\ M(p_{2n}, p_{2n+1}, t) \end{matrix}\right\}$$

Letting $n \to \infty$, we get

$$M(F(y,x), fy, kt) \geq min\{1, 1, M(fy, F(y,x), t), 1\} = M(fy, F(y,x), t).$$

From Lemma 1.8, we have $F(y,x) = fy = \beta$.

Since the pair (F, f) is W-compatible, we have

$$f\alpha = f(fx) = f(F(x,y)) = F(fx, fy) = F(\alpha, \beta) \text{ and}$$

$$f\beta = f(fy) = f(F(y,x)) = F(fy, fx) = F(\beta, \alpha) \text{ .... (III)}$$

$$M(f\alpha, z_{2n+1}, kt) = M(F(\alpha,\beta), G(x_{2n+1}, y_{2n+1}), kt)$$

$$\geq \min\{M(f\alpha, z_{2n}, t), M(f\beta, p_{2n}, t), 1, M(z_{2n}, z_{2n+1}, t)\}.$$

Letting $n \to \infty$, we get

$$M(f\alpha, \alpha, kt) \geq \min\{M(f\alpha, \alpha, t), M(f\beta, \beta, t), 1, 1\} = \min\{M(f\alpha, \alpha, t), M(f\beta, \beta, t)\}.$$

Also,

$$M(f\beta, p_{2n+1}, kt) = M(F(\beta,\alpha), G(y_{2n+1}, x_{2n+1}), kt)$$

$$\geq \min\{M(f\beta, p_{2n}, t), M(f\alpha, z_{2n}, t), 1, M(p_{2n}, p_{2n+1}, t)\}.$$

Letting $n \to \infty$, we get

$$M(f\beta, \beta, kt) \geq \min\{M(f\beta, \beta, t), M(f\alpha, \alpha, t)\} .$$

Thus

$$\min\{M(f\alpha, \alpha, kt), M(f\beta, \beta, kt)\} \geq \min\{M(f\alpha, \alpha, t), M(f\beta, \beta, t)\}.$$

From Lemma 1.9, we have $f\alpha = \alpha$ and $f\beta = \beta$.

Thus $\alpha = f\alpha = F(\alpha, \beta)$ ....(IV) and

$\beta = f\beta = F(\beta, \alpha)$ ....(V)

Since $F(X \times X) \subseteq g(X)$, there exist γ and δ in X such that

$g\gamma = F(\alpha,\beta) = f\alpha = \alpha$ and $g\delta = F(\beta,\alpha) = f\beta = \beta$.

$$M(g\gamma, G(\gamma,\delta), kt) = M(F(\alpha,\beta), G(\gamma,\delta), kt) = \min\{1, 1, 1, M(g\gamma, G(\gamma,\delta), t)\} = M(g\gamma, G(\gamma,\delta), t).$$

From Lemma 1.8, we have $G(\gamma, \delta) = g\gamma$.

Similarly, we can show that $G(\delta, \gamma) = g\delta$.

Since the pair (G, g) is weakly compatible, we have

$$g\alpha = g(g\gamma) = g(G(\gamma,\delta)) = G(g\gamma, g\delta) = G(\alpha,\beta) \text{ and}$$

$$g\beta = g(g\delta) = g(G(\delta,\gamma)) = G(g\delta, g\gamma) = g(\beta,\alpha) .$$

$$M(z_{2n}, G(\alpha,\beta), kt) = M(F(x_{2n}, y_{2n}), G(\alpha,\beta), kt) \geq \min\{M(z_{2n-1}, g\alpha, t), M(z_{2n-1}, g\beta, t), M(z_{2n-1}, z_{2n}, t), 1\}.$$

Letting $n \to \infty$, we get

$$M(\alpha, g\alpha, kt) \geq \min\{M(\alpha, g\alpha, t), M(\beta, g\beta, t), 1, 1\} = \min\{M(\alpha, g\alpha, t), M(\beta, g\beta, t)\}.$$

Similarly, we have

$$M(\beta, g\beta, kt) \geq \min\{M(\alpha, g\alpha, t), M(\beta, g\beta, t)\}.$$

Thus

$$\min\{M(\alpha, g\alpha, kt), M(\beta, g\beta, kt)\} \geq \min\{M(\alpha, g\alpha, t), M(\beta, g\beta, t)\}.$$

From Lemma 1.9, we have $g\alpha = \alpha$ and $g\beta = \beta$.

Hence $\alpha = g\alpha = G(\alpha, \beta)$ ....(VI) and

$\beta = g\beta = G(\beta, \alpha)$ ....(VII)

From (IV), (V), (VI), and (VII), we have

$f\alpha = g\alpha = \alpha = F(\alpha,\beta) = G(\alpha,\beta)$ and

$f\beta = g\beta = \beta = F(\beta,\alpha) = G(\beta,\alpha)$. Thus (α, β) is a coupled common fixed point of f, g, F and G .

Suppose (α_1, β_1) is another coupled common fixed point of f, g, F and G .

$$M(\alpha_1, \alpha, kt) = M(F\alpha_1, \beta_1), G(\alpha,\beta), kt) \geq \min\{M(\alpha_1, \alpha, t), M(\beta_1, \beta, t), 1, 1\} = \min\{M(\alpha_1, \alpha, t), M(\beta_1, \beta, t)\}.$$

Similarly we can show that

$$M(\beta_1, \beta, kt) \geq \min\{M(\alpha_1, \alpha, t), M(\beta_1, \beta, t)\} .$$

Thus

$\min\{M(\alpha_1,\alpha,kt),M(\beta_1,\beta,kt)\} \ge \min\{M(\alpha_1,\alpha,t),M(\beta_1,\beta,t)\}$.
From Lemma 1.9, we have $\alpha_1 = \alpha$ and $\beta_1 = \beta$.

Thus (α,β) is the unique common coupled fixed point of f,g,F and G.

Now, we prove that $\alpha = \beta$. Consider

$$\begin{aligned}M(\alpha,\beta,kt) &= M(F(\alpha,\beta),G(\beta,\alpha),kt)\\ &\ge \min\{M(\alpha,\beta,t),M(\beta,\alpha,t),1,1\} = M(\alpha,\beta,t).\end{aligned}$$

Hence $\alpha = \beta$.

Thus $\alpha = f\alpha = F(\alpha,\alpha) = G(\alpha,\alpha)$.

That is α is a common fixed point of f,g,F and G.

Suppose α' is another common fixed point of f,g,F and G. Then

$$\begin{aligned}M(\alpha,\alpha',kt) &= M(F(\alpha,\alpha),G(\alpha',\alpha'),kt)\\ &\ge \min\{M(\alpha,\alpha',t),M(\alpha,\alpha',t),1,1\} = M(\alpha,\alpha',t).\end{aligned}$$

Hence $\alpha = \alpha'$. Thus α' is the unique common fixed point of f,g,F and G.

Corollary 2.2. *Let $(X,M,*)$ be a fuzzy metric space and $f : X \to X$ and $F : X \times X \to X$* be mappings satisfying

(2.2.1) $$M\big(F(x,y),F(u,v),kt\big) \ge \min\left\{\begin{matrix} M(fx,fu,t),M(fy,fv,t),\\ M(fx,F(x,y),t),M(fu,F(u,v),t)\end{matrix}\right\}$$
for all $x,y,u,v \in X$, $\forall\ t > 0$ and $k \in (0,1)$,

(2.2.2) $F(X \times X) \subseteq f(X)$,

(2.2.3) $f(X)$ is complete and

(2.2.4) the pair (f,F) is W-compatible.

Then f,F have a unique common coupled fixed point in $X \times X$ and also they have a unique common fixed point in X.

Example 2.3. *Let $(X,M,*)$ be a fuzzy metric space, where $X = [0,1]$ and $M(x,y,t) = \dfrac{t}{t+|x-y|}$ for all $x,y \in X$ and $t > 0$. Define $f : X \to X$ by $fx = \dfrac{2x+1}{3}$ and $F : X \times X \to X$ by $F(x,y) = 1$ for all $x,y \in X$.*

It is easy to see that all conditions of Corollary 2.2 are satisfied. Consequently, 1 is the unique common fixed point of f and F.

Finally using the boundedness of a fuzzy metric space, we prove a common fixed point theorem for two maps satisfying a general contractive condition.

Theorem 2.4. *Let $(X,M,*)$ be a bounded fuzzy metric space and $f : X \to X$, $F : X \times X \to X$ be mappings satisfying*

(2.4.1) $$M\big(F(x,y),F(u,v),t\big) \ge \phi\left(\min\left\{\begin{matrix} M(fx,fu,t),M(fy,fv,t),\\ M(fx,F(x,y),t),M(fu,F(u,v),t),\\ M(fx,F(u,v),t),M(fu,F(x,y),t)\end{matrix}\right\}\right)$$
for all $x,y,u,v \in X$, $\forall\ t > 0$, where $\phi : [0,1] \to [0,1]$ is continuous monotonically increasing such that $\phi(s) > s$ for all $s \in [0,1)$,

(2.4.2) $F(X \times X) \subseteq f(X)$ and $f(X)$ is complete,

(2.4.3) the pair (f,F) is W-compatible.

Then f and F have a unique common coupled fixed point in $X \times X$ and also they have a unique common fixed point in X.

Proof. Let x_0 and y_0 be in X.

From (2.4.2), we can find $\{x_n\}$ and $\{y_n\}$ in X such that

$fx_{n+1} = F(x_n,y_n) = z_n$, say;

$fy_{n+1} = F(y_n,x_n) = p_n$, say for $n = 0,1,2,3,\ldots\ldots$.

For $n \in N$, let $\alpha_n(t) = inf\,\{M(z_i,z_j,t)/\ i \ge n,\ j \ge n\}$ and

$\beta_n(t) = inf\,\{M(p_i,p_j,t)/\ i \ge n,\ j \ge n\}$ for all $t > 0$.

Then $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ monotonically increasing sequences of real numbers between 0 and 1 for all $t > 0$.

Hence $\lim_{n\to\infty}\alpha_n(t) = \alpha(t)$ for some $0 \le \alpha(t) \le 1$ and

$\lim_{n\to\infty}\beta_n(t) = \beta(t)$ for some $0 \le \beta(t) \le 1$.

For any $n \in N$ and integers $i \ge n,\ j \ge n$ we have

$$M(z_i,z_j,t) = M(F(x_i,y_i),F(x_j,y_j),t)$$

$$\ge \phi\left(\min\left\{\begin{matrix} M(z_{i-1},z_{j-1},t),M(p_{i-1},p_{j-1},t),M(z_{i-1},z_i,t),\\ M(z_{j-1},z_j,t),M(z_{i-1},z_j,t),M(z_{j-1},z_i,t)\end{matrix}\right\}\right)$$

$$\ge \phi\big(\min\{\alpha_{n-1}(t),\ \beta_{n-1}(t)\}\big).$$

Taking supremum over all $i \ge n,\ j \ge n$ we get

$\alpha_n(t) \ge \phi(\min\{\alpha_{n-1}(t),\ \beta_{n-1}(t)\})$.

Similarly we can show that
$\beta_n(t) \ge \phi(\min\{\alpha_{n-1}(t),\ \beta_{n-1}(t)\})$.

Thus

$\min\ \{\alpha_n(t),\ \beta_n(t)\} \ge \phi(\min\{\alpha_{n-1}(t),\ \beta_{n-1}(t)\})$.

Letting $n \to \infty$, we get

$\min\ \{\alpha(t),\ \beta(t)\} \ge \phi(\min\{\alpha(t),\ \beta(t)\})$.

It is contradiction if $\min\ \{\alpha(t),\ \beta(t)\} < 1$.

Hence $\alpha(t) = 1$ and $\beta(t) = 1$.

Thus $\lim_{n\to\infty} \alpha_n(t) = 1 = \lim_{n\to\infty} \beta_n(t)$.

Hence $\{z_n\}$ and $\{p_n\}$ are Cauchy sequences in X.

Since $f(X)$ is complete, it follows that $\{z_n\}$ and $\{p_n\}$ converge to some p and q respectively in $f(X)$. Hence there exist x and y in X such that $p = fx$ and $q = fy$.

$$M(z_n, F(x,y),t) = M(F(x_n, y_n), F(x,y),t)$$

$$\geq \phi\left(\min\left\{\begin{matrix} M(z_{n-1}, fx,t), M(p_{n-1}, fy,t), M(z_{n-1}, z_n,t), \\ M(fx, F(x,y),t), M(z_{n-1}, F(x,y),t), M(fx, z_{n+1},t)\end{matrix}\right\}\right)$$

Letting $n \to \infty$, we get

$$M(fx, F(x,y),t) \geq \phi\left(\min\left\{\begin{matrix} 1,1,1, M(fx, F(x,y),t), \\ M(fx, F(x,y),t), 1\end{matrix}\right\}\right)$$

$$= \phi(M(fx, F(x,y),t))$$

$$> M(fx, F(x,y),t) \text{ if } M(fx, F(x,y),t) < 1.$$

Hence $M(fx, F(x,y),t) = 1$ so that $fx = F(x,y)$.

Similarly we can show that $fy = F(y,x)$.

Since (F, f) is a W-compatible pair, we have

$fp = f(fx) = f(F(x,y)) = F(fx, fy) = F(p,q)$ and

$fq = f(fy) = f(F(y,x)) = F(fy, fx) = F(q,p).$

$$M(z_n, F(p,q),t) = M(F(x_n, y_n), F(p,q),t)$$

$$\geq \phi\left(\min\left\{\begin{matrix} M(z_n, fp,t), M(p_n, fq,t), M(z_{n-1}, z_n,t), \\ 1, M(z_{n-1}, F(p,q),t), M(fp, z_n,t)\end{matrix}\right\}\right)$$

Letting $n \to \infty$, we get

$$M(p, fp,t) \geq \phi\left(\min\left\{\begin{matrix} M(p, fp,t), M(q, fq,t), 1, 1, \\ M(p, fp,t), M(fp, p,t)\end{matrix}\right\}\right)$$

$$= \phi(\min\{M(p, fp,t), M(q, fq,t)\})$$

Similarly we can show that
$M(q, fq,t) \geq \phi(\min\{M(p, fp,t), M(q, fq,t)\})$.

Thus
$\min\{M(p, fp,t), M(q, fq,t)\} \geq \phi(\min\{M(p, fp,t), M(q, fq,t)\})$.
It is contradiction if $\min\{M(p, fp,t), M(q, fq,t)\} < 1$.
Hence from this we can conclude that $fp = p$ and $fq = q$.
Thus $p = fp = F(p,q)$ and $q = fq = F(q,p)$.
Using (2.4.1) two times , one can show that (p,q) is the unique common coupled fixed point of F and f.
Now we will show that $p = q$. Consider

$$M(p,q,t) = M(F(p,q), F(q,p),t)$$

$$\geq \phi(\min\{M(p,q,t), M(q,p,t), 1, 1, M(p,q,t), M(q,p,t)\})$$

$$= \phi(M(p,q,t)).$$

Hence $p = q$.
Thus $p = fp = F(p,p)$. i.e p is a common fixed point of F and f.
Using (2.4.1), we can show that p is the unique common fixed point of F and f.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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