Gazi University Journal of Science
GU J Sci
27(3):917-921 (2014)

A Study of D −Operator on Probabilistic Metric Space

Arvind BHATT[1,♠]

[1]Department of Applied Science (Mathematics), Bipin Tripathi Kumaun Institute of Technology Dwarahat(Almora), Uttarakhand Technical University,Dehradun, India, 263653

Received:14/12/2013 Accepted: 27/02/2014

ABSTRACT

In this paper, we obtain some common fixed point theorems for recently introduced notion of D-operator on a set X equipped with the function $F: X \times X \to \Delta^+$ without using the triangle inequality besides relaxing symmetric condition. Our results extend the results of Pathak and Rai.

2010 Mathematics Subject Classification, Primary: 47H09, 47H10, 54E35, 54E70.

Key words and phrases: fixed point theorem, Common fixed point, Probabilistic metric space, PD-operator, D −operator.

1. INTRODUCTION

The theory of probabilistic metric spaces was introduced by Menger [2] in connection with some measurements in Physics. The first effort in this direction was made by Sehgal [4], who in his doctoral dissertation initiated the study of contraction mapping theorems in probabilistic metric spaces. Since then, Sehgal and Bharucha - Reid [7] obtained a generalization of Banach Contraction Principle on a complete Menger space which is an important step in the development of fixed point theorems in Mengar space. Over the years, the theory has found several important applications in the investigation of physical quantities in quantum particle physics and string theory as studied by El.Naschie [17, 18]. The area of probabilistic metric spaces is also of fundamental importance in probabilistic functional analysis.

In 1976, Jungck [9] initiated a study of common fixed points of commuting maps. On the other hand in 1982 Sessa [10] initiated the tradition of improving commutativity in fixed- point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [12] soon enlarged this concept to compatible maps. The notion of compatible mappings in a Mengar space has been introduced by Mishra [13]. After this, Jungck [15] gave the concept of weakly compatible maps. Aamri and El Moutawakil [21] introduced the $(E.A.)$ property and thus generalized the concept of non-compatible maps. The results obtained in the metric fixed point theory by using the notion of non-compatible maps or the $(E.A.)$ property is very interesting. Al-Thagafi and Shahzad [25] (see also, Jungck and Rhoades [22]) defined the concept of occasionally weakly compatible mappings which is more general than the concept of weakly compatible maps. Bhatt et al. [29] have given application of occasionally weakly compatible mappings in dynamical programming. Pathak and Hussain [30] defined the concept of P-operators. Hussain et.al [31] gave the concepts of JH-operators and occasionally weakly g-biased. Recently Pathak and Rai [33] proved some common fixed point theorems for more generalized non commuting notion, namely, PD-operators and gave some applications in variational inequalities and dynamical

♠Corresponding author, e-mail: arvindbhu_6june@rediffmail.com

programming. After this Pathak and Rai [34] gave the notion of D-operator.

In this paper, we extend some common fixed point theorems for D-operator under relaxed condition on probabilistic metric space. Our results extend the results of Pathak and Rai [33], Hussain et al. [31], Bhatt et al. [29] and others [21, 24, 26, 27, 32, 35].

We begin with the following basic definitions of concepts relating to probabilistic metric spaces for ready reference and also for the sake of completeness.

Definition 1.1. [20]. A distribution function (on $[-\infty, +\infty]$) is a function

$F: [-\infty, +\infty] \to [0, 1]$ which is left-continuous on R, non-decreasing and $F(-\infty) = 0$, $F(+\infty) = 1$. The Heaviside function H is a distribution function defined by

$$H(t) = \begin{cases} 0, & if\ t \leq 0 \\ 1, & if\ t > 0. \end{cases}$$

Definition 1.2. [20]. A distance distribution function F: $[-\infty, +\infty] \to [0, 1]$ is a distribution function with support contained in $[0, \infty]$. The family of all distance distribution functions will be denoted by Δ^+. We denote

$$D^+ = \left\{F: F \in \Delta^+, \lim_{x\to\infty} F(x) = 1\right\}.$$

Definition 1.3. [11]. A probabilistic metric space in the sense of Schweizer and Sklar is an ordered pair (X, F), where X is a nonempty set and $F : X \times X \to \Delta^+$, if and only if the following conditions are satisfied ($F(x, y) = F_{x,y}$ for every $x, y \in X \times X$):

(i) for every $(x, y) \in X \times X$, $F_{x,y}(0) = 0$;
(ii) for every $(x, y) \in X \times X$, $F_{x,y} = F_{y,x}$;
(iii) $F_{x,y} = 1$, for every $t > 0 \Leftrightarrow x = y$;
(iv) for every $(x, y, z) \in X \times X \times X$ and for every $t_1, t_2 > 0$,
$F_{x,y}(t_1) = 1$, $F_{y,z}(t_2) = 1, \Longrightarrow F_{x,z}(t_1 + t_2) = 1$.

For each x and y in X and for each real number $t \geq 0$, $F_{x,y}(t)$ is to be thought of as the probability that the distance between x and y is less than t. Indeed, if (X, d) is a metric space, then the distribution function $F_{x,y}(t)$ defined by the relation $F_{x,y}(t) = H(t - d(x, y))$ induces a probabilistic metric space.

Definition 1.4. Let an ordered pair (X, F), where X is a nonempty set and F is a mapping from $X \times X$ into Δ^+ satisfying the following condition:

$$(1) \qquad F_{x,y}(t) = 1 \forall t > 0 \Leftrightarrow x = y.$$

Where $F: X \times X \to \Delta^+$, defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$ and d be a function $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ iff $x = y, \forall x, y \in X$ (symmetric and triangle conditions are not required). A topology $\tau(d)$ on X is given by $U \in \tau(d)$ if and only if for each $x \in U, B(x, \in) \subset U$ for some $\epsilon > 0$, where $B(x, \in) = \{y \in X : d(x, y) < \epsilon\}$.

Remark 1.1. We note that every symmetric (semi-metric) space (X, d) [1] can be realized as a probabilistic semi - metric space by taking $F : X \times X \to \Delta^+$, defined by $F_{x,y}(t) = H(t - d(x, y))$ for all x, y in X. So probabilistic semi metric spaces provide a wider framework than that of the symmetric spaces and are better suited in many situations. In this paper we have relaxed the symmetric condition from probabilistic semi metric space.

Definition 1.5. [6, 28]. Let (X, F) be a probabilistic metric space and A be a nonempty subset of X. The probabilistic diameter $\delta_A: [0, +\infty) \to [0, 1]$ is defined by,

$$\delta_A(x) = \sup_{t<x} \inf_{p,q\in A} \{F_{p,q}(t)\}.$$

If (X, F) satisfies condition (1), the probabilistic diameter is defined by,

$$\delta_A(x) = \sup_{t<x} \inf_{p,q\in A} \{F_{p,q}(t), F_{q,p}(t)\}.$$

Let X be a non-empty set together with the function $F: X \times X \to \Delta^+$ satisfying the condition (1). A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case $w = fx = gx$ is called a point of coincidence of f and g. Let $C(f, g)$ and $PC(f, g)$ denote the sets of coincidence points and points of coincidence, respectively, of the pair (f, g).

Definition 1.6. Let X be a non-empty set together with the function $F: X \times X \to \Delta^+$ satisfying the condition (1), two self maps f and g of a space (X, F) are called D-operators iff for some t, there is a point $x \in C(f, g)$ and their exists $R > 0$ such that $F_{fgx,gfx}(t) \geq \delta_{PC(f,g)}\left(\frac{t}{R}\right)$ and $F_{gfx,fgx}(t) \geq \delta_{PC(f,g)}\left(\frac{t}{R}\right)$.

Example 1.1. Let $X = [0, \infty)$ and $F_{x,y}(t) = H(t - d(x, y))$, where,

$$d(x, y) = \begin{cases} e^{x-y} - 1, & if\ x \geq y \\ e^{y-x}, & otherwise. \end{cases}$$

Define $f, g : X \to X$ by,

$f(x) = 2x$ and $g(x) = 2x^2$, for all $x \neq 0$ and $f(0) = g(0) = 1$.
Here $C(f, g) = \{0,1\}$ and $PC(f, g) = \{1, 2\}$. Since $F_{fg(0),gf(0)}(t) = F_{2,2} = H(t) = F_{gf(0),fg(0)}(t)$ and for $R = 1$, $\delta_{PC(f,g)}\left(\frac{t}{R}\right) = H(t)$ for some $t > 0$. Hence (f, g) is D-operator.

Remark 1.2. Every PD-operator pair is D- operator pair (for $R = 1$), whereas, the converse need not be true. It is obvious that the (f, g) is PD-operator, but not commuting, not weakly compatible and not occasionally weakly compatible. It is also clear that the PD-operator is different from P-operator pair and JH −operator pair. We also observe that if (f, g) is $PD -$ operator then it is not necessary that (f, g) be P-operator and JH-operator pair.

SECTION- II

In this section, we prove some fixed point theorems for a pair of PD- operators on space (X, F) without imposing the restriction of the triangle inequality or symmetry on F.

Theorem 2.1. Let X be a non-empty set together with the function $F: X \times X \to \Delta^+$ satisfying the condition **(1).** Suppose f and g are PD-operators on X satisfying the following condition:

$$\textbf{(2)} \qquad F_{fx,fy}(\mathrm{t}) \geq F_{gx,gy}\left(\frac{t}{a}\right) + \min\left\{F_{fx,gx}\left(\frac{t}{b}\right), F_{fy,gy}\left(\frac{t}{b}\right)\right\} + \min\left\{F_{gx,gy}\left(\frac{t}{c}\right), F_{gx,fx}\left(\frac{t}{c}\right), F_{gy,fy}\left(\frac{t}{c}\right)\right\},$$

for all $x, y \in X$ with $f(x) \neq fy$ and $t > 0$ where $0 < a < 1, 0 < b < 1$ and $0 < c < 1$. Then f and g have a unique common fixed point.

Proof. Since (f, g) is PD -operator pair there exist a point u in X such that $fu = gu$ and

$$\textbf{(3)} \qquad F_{fgu,gfu}(t) \geq \delta_{PC(f,g)}\left(\frac{t}{R}\right).$$

First, we claim that $PC(f, g)$ is singleton. If possible, suppose w and w_1 be two distinct points in X such that $fu = gu = w$ and $fv = gv = w_1$ for some

$u, v \in C(f, g)$. Then from (2), we get,

$$F_{w,w_1}(t) = F_{fu,fv}(t) \geq F_{gu,gv}\left(\frac{t}{a}\right) + 1 + \min\left\{F_{gu,gv}\left(\frac{t}{c}\right), F_{gu,fu}\left(\frac{t}{c}\right), F_{gv,fv}\left(\frac{t}{c}\right)\right\}$$
$$= F_{gu,fu}\left(\frac{t}{a}\right) + 1 + F_{gu,fu}\left(\frac{t}{c}\right) > 1,$$

a contradiction. Hence, $w = w_1$. Thus $PC(f, g)$ is singleton and w is the unique point of coincidence. This further implies $\delta(PC(f, g)) = 1$. Using **(3)**, $fgu = gfu$ for some $u \in C(f, g)$. Now by (**2**). We have,

$$F_{fu,ffu}(t) \geq F_{gu,gfu}\left(\frac{t}{a}\right) + 1 + \min\left\{F_{gu,gfu}\left(\frac{t}{c}\right), F_{gu,fu}\left(\frac{t}{c}\right), F_{gfu,ffu}\left(\frac{t}{c}\right)\right\}$$
$$= F_{gu,ffu}\left(\frac{t}{a}\right) + 1 + F_{gu,ffu}\left(\frac{t}{c}\right) > 1,$$

a contradiction. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g. Uniqueness follows from (2).

Let a function $\emptyset$ be defined by $\emptyset : [0,1] \to [0,1]$ satisfying the condition $\emptyset(q) > q$, for all $0 \leq q < 1$.

Theorem 2.2. Let X be a non-empty set together with the function $F: X \times X \to \Delta^+$ satisfying the condition **(1).** If (f, g) is PD-operator pair. Suppose

$$(4) \qquad F_{fx,fy}(t) \geq \emptyset\left[\min\left\{F_{gx,gy}(t), F_{gx,fy}(t), F_{fx,gy}(t), F_{gy,fy}(t)\right\}\right],$$

for all $x, y \in X$ and $t > 0$. Then f and g have a unique common fixed point.

Proof. Since (f, g) is PD -operator pair there exist a point u in X such that $fu = gu$ and

$$(5) \qquad F_{fgu,gfu}(t) \geq \delta_{PC(f,g)}\left(\frac{t}{R}\right).$$

First, we claim that $PC\ (f,g)$ is singleton. If possible, suppose w and w_1 be two distinct points in X such that $fu = gu = w$ and $fv = gv = w_1$ for some $u, v \in C(f,g)$. Then from **(4)**, we can easily get, w = w_1, $i.e. w = fu = gu = fv = gv = w_1$. Therefore $PC\ (f,g)$ is singleton i.e., w = fu = gu is the unique point of coincidence. $\delta\ (PC\ (f,g)) = 1$. From (**5**), $fgu = gfu$, for some u, v ∈C (f, g). Now, by(**4**)**,** we have,

$$F_{ffu,fu}(t) \geq \emptyset\left[\min\left\{F_{gfu,gu}(t), F_{gfu,fu}(t), F_{ffu,gu}(t), F_{gu,fu}(t)\right\}\right],$$

$$= \emptyset\left[\min\left\{F_{ffu,fu}(t), F_{ffu,fu}(t), F_{ffu,fu}(t), 1\right\}\right],$$

$$= \emptyset\left[F_{ffu,fu}(t)\right].$$

Since $\emptyset : [0,1] \rightarrow [0,1]$ satisfying the condition $\emptyset(q) > q$, for all $0 \leq q < 1$. Therefore, $F_{ffu,fu}(t) > F_{ffu,fu}(t)$ which is a contradiction. Therefore $ffu = fu = gfu$, f and g have a common fixed point. Uniqueness is obvious. Therefore, f and g have a unique common fixed point. This completes the proof of the theorem.

Corollary 2.1. Let X be a non-empty set together with the function $F: X \times X \rightarrow \Delta^+$ satisfying the condition **(1).** If f and g are PD −operator on X. Suppose

$$(6) \qquad F_{fx,fy}(t) \geq \emptyset\left[F_{gx,gy}(t),\right],$$

for some $x, y \in X$ and $t > 0$. Then f and g have a unique fixed point.

The proof of the following theorem can be easily obtained by replacing condition **(4)** by condition (**7**) the proof of Theorem **2**. **2**.

Theorem 2.3. Let X be a non-empty set together with the function $F: X \times X \rightarrow \Delta^+$ satisfying the condition (1). If f and g are PD-operator on X. Suppose

$$(7) \qquad F_{fx,fy}(t) \geq \min\left\{F_{gx,gy}(t), F_{gx,fy}(t), F_{fx,gy}(t), F_{gy,fy}(t)\right\}$$

for some $x, y \in X$ and $t > 0$. Then f and g have a unique common fixed point.

Remark 2.1. Similarly we can proof fixed point theorems for four self-mappings on (X, F), where $F: X \times X \rightarrow \Delta^+$ satisfying the condition (1) (without imposing the restriction of the triangle inequality and symmetry only on point of coincidence).

Remark 2.2. As an application of Corollary 2.1, the existence and uniqueness of a common solution of the functional equations arising in dynamic programming can be established which extends Theorem 4.1 [29].

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES:

[1] Wilson, W.A., On semi metric spaces, ***Amer.J.Math***.53 (1931), 361-373.

[2] Menger, K. Statistical metrices, ***Nat. Acad.Sci***., USA. 28(1942) 535-537.

[3] Schweizer, B. and Skalar, A. Probabilistic metric spaces, ***Pacific J. of Math***.10 (1960) 313 - 324.

[4] Sehgal, V.M. Some fixed point theorems in functional analysis and probability Ph.D.Dissertation, ***Wayne State Univ. Michigan*** (1966).

[5] Kannan, R. Some results on fixed points, ***Bull. Calcutta Math. Soc.*** 60 (1968) 71-76.

[6] Egbert, R.J. Products and quotients of probabilistic metric spaces, ***Pacific J.Math***., 24(1968) 437-455.

[7] Sehgal, V.M. and Bharucha-Reid, A.T. Fixed points of contraction mappings on Probabilistic metric spaces, ***Math. Systems Theo***. 06(1972) 97-102.

[8] Bharucha-Reid, A.T. Fixed point theorems in Probabilistic analysis, ***Bull.Amer. Math. Soc***. 82(1976) 641-657.

[9] Jungck, G. Commuting mappings and fixed points, ***Amer. Math. Month.*** 73(1976) 261-263.

[10] Sessa, S. On a weak commutativity condition of mappings in fixed point considerations, ***Publ. Inst. Math.*** 32(1982) 149-153.

[11] Schweizer, B. and Sklar, A. Statistical metric spaces, North Holland Amsterdam, (1983).

[12] Jungck, G. Compatible mappings and common fixed points, ***Int.J.Math.and Math.Sci***. 09 (1986) 771-779.

[13] Mishra, S.N. Common fixed points of compatible mappings in PM-spaces, ***Math. Japon***. 36(1991) 283-289.

[14] Jungck, G. and Pathak, H.K., Fixed points via biased maps, ***Proc. Amer. Math. Soc***. 123(1995) 2049-2060.

[15] Jungck, G., Common fixed points for noncontinuous nonself maps on nonmetric spaces, ***Far East J. Math. Sci.*** 04(1996) 199-215.

[16] Pant, R.P., Common fixed points of four maps, ***Bull. Calcutta Math.Soc.*** 90(1998) 281-286.

[17] Naschie, MS.EL., On the uncertainty of Cantorian geometry and two-slit experiment, ***Chaos Solitons and Frac.*** 09(03)(1998) 517-529.

[18] Naschie, MS.EL., On the verifications of heterotic string theory and ϵ^{∞}, ***Chaos, Solitons and Frac***. 232(2000) 397-407.

[19] Pant, R.P.and Pant, V., Common fixed points under strict contractive conditions, ***J.Math.Anal.Appl.*** 248(2000) 327-332.

[20] Hadzic, O. and Pap, E.,Fixed Point Theory in Probabilistic Metric Spaces, ***Kluwer Academic Publishers,*** 2001.

[21] Aamri, M. and Moutawakil, D. El., Some new common fixed point theorems under strict contractive conditions, ***J. Math. Anal. Appl***. 270(2002) 181-188.

[22] Jungck, G. and Rhoades, B.E., Fixed point Theorems for occasionally weakly Compatible Mappings, ***Fixed point theory,*** 07(2006) 286-296.

[23] Jungck,G. and Hussain, N., Compatible maps and invariant approximations, ***J. Math. Anal. Appl***., 325(2007) 1003-1012.

[24] Jungck, G. and Rhoades, B.E., Fixed point Theorems for occasionally weakly compatible Mappings, Erratum, ***Fixed point theory***, 9(2008) 383-384.

[25] Al-Thagafi, M.A. and Shahzad, N., Generalized I-nonexpansive selfmaps and invariant approximations, ***Acta Math. Sinica,*** 24(5)(2008) 867-876.

[26] Chandra, H. and Bhatt, A., Fixed point theorems for pairs of Multivalued mappings, ***Kochi Journal of Math.*** 04(2009) 101-108.

[27] Chandra, H. and Bhatt, A., Fixed point theorems for occasionally weakly compatible maps in probabilistic semi-metric space, ***International Journal of Mathematical anal.*** 03(2009) 563-570.

[28] Dorel Mihet, Altering distances in probabilistic Menger spaces, ***Nonlinear Analysis*** 71(2009) 2734 - 2738.

[29] Bhatt, A. et al., Common Fixed point theorems for occasionally weakly compatible mappings under relaxed conditions, ***Non-linear analysis Theory, Methods and appl***.73(2010) 176-182.

[30] Pathak, H.K. and Hussain, N., Common fixed points for P-operator pair with applications, ***Appl. Math. Comput***. 217(2010) 3137-3143.

[31] Hussain, N. et al., Common fixed points for JH-operators and occasionally weakly biased pairs under relaxed conditions, ***Nonlinear Anal***. 74(2011) 2133-2140.

[32] Pant, R.P. and Bisht, R.K., Occasionally weakly compatible mappings and fixed points, ***Bull. Belg. Math. Soc.*** 19(4) (2012) 655-661.

[33] Pathak, H.K. and Rai, D., Common fixed points for PD-operator pairs under relaxed conditions with applications, ***Journal of Computational and Applied Mathematics*** 239(1)(2013) 103-113.

[34] Pathak, H.K. and Rai, D., Some Common fixed point theorems for D-operator pair with applications to nonlinear integral equations, ***Nonlinear Functional Analysis and Applications*** 18(2)(2013) 205-218.

[35] Pathak, H.K. and Shahzad, N., Some results on Best Proximity Points for Cyclic Mappings, ***Bull. Belg. Math. Soc.*** Simon Stevin 20(3) (2013), 559-572.