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A New Estimator for Stationary Distribution of the Inventory Model of Type (*s*, *S*)

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ABSTRACT

We consider inventory model of type (*s*, *S*) which is used mostly in stock control policy. It is very important to know characteristics of an inventory model of type (*s*, *S*), such as stationary distribution. Using the straight line approach of Frees [1], we establish estimator for ergodic distribution of inventory model of type (*s*, *S*) and investigate asymptotic properties of this estimator such as consistency, asymptotic unbiasedness and asymptotic normality.

***Key Words*:** Inventory model of type (*s*, *S*), Ergodic distribution, Estimation, Consistency, Asymptotic unbiasedness, Asymptotic normality.

1. INTRODUCTION

Renewal process, renewal-reward process, random walk and various modifications of these processes have a wide range of applications in reliability theory, queuing theory, insurance applications and etc. There are many valuable studies on these processes in the literature [2-7]. Many interested problems of stock control are also expressed by means of these processes. For example, in the analysis of most inventory processes, it is customary to assume that the pattern of demands forms a renewal process. Most of the standard inventory policies induce renewal sequences, e.g., the times of replenishment of stock. The stock control policy is one of the important topics related to production policy of managements, since running a right stock policy for managements will increase company's profit. One of the most used inventory models is the inventory model of type (*s*, *S*). The inventory model of type (*s*, *S*) has been extensively considered in recent years [8-18; etc.]. It is very important to know characteristics of an inventory model of type (*s*, *S*) in real world application, such as stationary distribution, moments, etc. For this reason, the ergodic distribution of an inventory model of type (*s*, *S*) is considered in this study. The stationary ergodic distribution is obtained by using renewal function of demand quantity which has distribution function F. However, the functional form of the distribution function F or the parameters of the distribution or both of them are unknown in many cases. Thus, it is desirable to have an estimator of the renewal function. Frees (1986a) proposed estimation of a straight line approximation of the renewal function instead of direct estimation of the renewal function. This approximation is based on a limit expression for large values of time parameter *t*. It is easy to apply this estimator in practice, especially for large value of *t*. To estimate of renewal function, especially in the cases of fixed values of *t*, there are many studies on this topic in the literature [19-27]. To calculate these estimators, it is needed to the

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considerable amount of computation, especially in the cases of large values of t. Contrary to Frees's estimator, it is a disadvantage for these estimators. Because of simple form of Frees's estimator, it can be used as estimator for applying in stochastic models including the complicated function associated with renewal function for large value of t, for example, estimation of ergodic distribution of inventory model of type (s, S).

1.1. The model

In this section, we define the inventory model of type (s, S) and process expressed this model. Before giving the mathematical definition, let us give the essential notations as follows:

s : Stock control level,
S : The maximum stock level,
$Y(t)$: Stock level in a depot at time t,
X_n : Demand quantity,
ξ_n : Interarrival time,
τ_1 : The first crossing time of the control level s by the process $Y(t)$,
N_1 : The necessary number of demands up to the first crossing time τ_1

It is assumed that stock level ($Y(t)$) in a depot at the initial time (t=0) is $Y(0)=Y_0=S$. Furthermore, it is assumed that in a depot at random times T_1, T_2,...,T_n,... the stock level ($Y(t)$) in the depot decreases according to X_1, X_2,..., X_n,... until the stock level falls below the predetermined control level s as follows:

$Y(T_1)=Y_1=S-X_1$; $Y(T_2)=Y_2=S-(X_1+X_2)$; ... ;
$Y(T_n)=Y_n=S-(\sum_{i=1}^{n} X_i)$, ...

where X_n represents the quantity of the n^{th} demand. This change of the system continues until the certain random time τ_1 which is the first crossing time of the control level s by the process $Y(t)$. At time τ_1, stock level in a depot is immediately replenished to the maximum stock level S. Thus, the first period has been completed. Afterwards, the system continues its variation by similar way to the first period.
The estimator for ergodic distribution makes it possible to estimate many characteristics arising in inventory model of type (s, S). Although the literature of inventory model of type (s, S) include many articles, none of them investigate the estimation problem for this model. Because of that, in this study, we deal with estimator for the ergodic distribution of inventory model of type (s, S). Thus the this article is organized as follows. In Section 2 it is definded the process which expressed the inventory model of type (s, S) and in Section 3 the ergodic distribution of this process is defined. In section 4, we give estimator for ergodic distribution of inventory model of type (s, S) based on Frees's estimator. Besides, we investigate some important statistical properties of this estimator, such as consistency, asymptotic unbiasedness and asymptotic normality. In section 5, the estimator is applied to a simulated dataset. Concluding remarks are summarized in section 6.

2. MATHEMATICAL CONSTRUCTION OF THE PROCESS *Y(t)*

Let $\left\{\left(\xi_i, X_i\right)\right\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed pairs of positive random variables defined on some probability space $\{\Omega, F, P\}$ We denote
$\Phi(t)=P\left(\xi_1 \le t\right)$,
$F\left(x\right)=P\left(X_1 \le x\right),\ x\in\left[s,S\right], t\ge 0.$ Define the renewal sequences $\{T_n\}$ and $\{Z_n\}$ as follows:

$$T_n=\sum_{i=1}^{n}\xi_i, \quad Z_n=\sum_{i=1}^{n}X_i, \quad n\ge 1, \quad T_0=Z_0=0,$$

and a sequence of integer valued random variables $\{N_n\}$, $n\ge 0$ as

$$N_0=0; \quad N_1=\min\left\{k\ge 1: S-Z_k<s\right\};$$
$$N_{n+1}=\min\left\{k\ge N_n+1: S-\left(Z_k-Z_{N_n}\right)<s\right\}.$$

n^{th} crossing time of the control level s by the process $Y(t)$ can be written as

$$\tau_n=T_{N_n}=\sum_{i=1}^{N_n}\xi_i,\ n\ge 1,\ \tau_0=0.$$

Then the desired stochastic process ($Y(t)$) expressed by inventory level at time t can be written as

$$Y(t)=S-Z_{\nu(t)}+Z_{N_n}, \quad \tau_n\le t<\tau_{n+1}, \quad n=0,1,2,...,$$

where $\nu(t)=\max\left\{n\ge 0: T_n\le t\right\},\ t>0.$

This model is known as an inventory model of type (s, S). A sample path of the process $Y(t)$ is given in Figure 1.

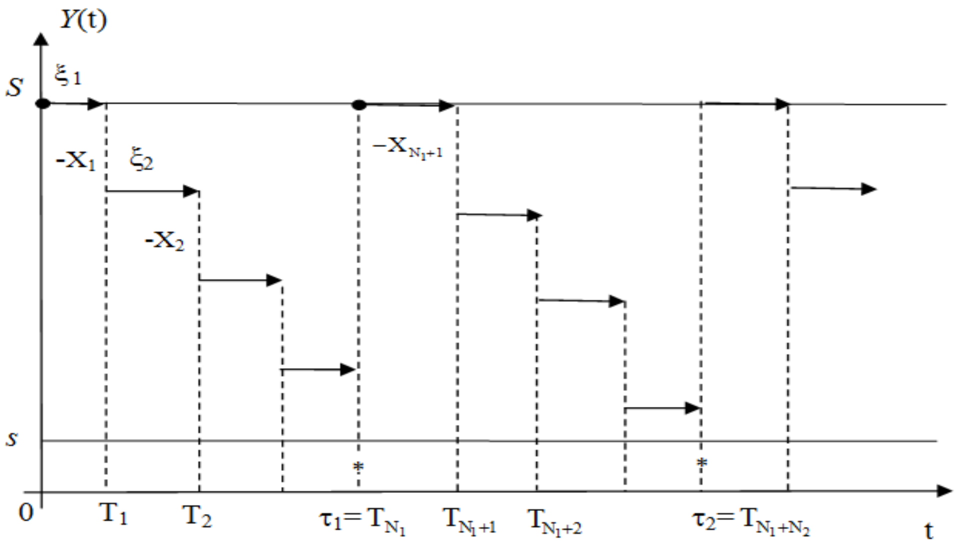

Figure 1. A sample path of the process $Y(t)$

3. ERGODIC DISTRIBUTION FOR THE INVENTORY MODEL OF TYPE (*s, S*)

The ergodic distribution of the process Y(t) is given as $Q(x)=\lim_{t\to\infty} P(\mathrm{Y}(t)\le x),\ \ x\in[s,S)$. Nasirova et al. [28] obtained the ergodic distribution as following Lemma 3.1.

Lemma 3.1. Suppose that E(ξ_n)<∞, then, the ergodic distribution of the process Y(t) is given as:

$$Q(x)=1-\frac{U_X(S-x)}{U_X(S-s)},\quad s\le x< S \qquad (3.1)$$

Here $U_X(z)$ is the renewal function generated by the sequence of {X_n, n=1,2,…}. Note that $U_X(S\text{-}x)$ and $U_X(S\text{-}s)$ can be written as follows for sufficiently large values of (S-s) and (S-x) [29]:

$$U_X(S-s)=\frac{(S-s)}{\mu_1}+\frac{\mu_2}{2\mu_1^2}+o(1)$$

and

$$U_X(S-x)=\frac{(S-x)}{\mu_1}+\frac{\mu_2}{2\mu_1^2}+o(1),$$

where $\mu_k=E\left(X_n^k\right),\ k=1,2$ is k^{th} moment of the demands. Then the ergodic distribution $Q(x)$ can be represent as follows when $\beta=S\text{-}s\to\infty$:

$$Q(x)=Q_1(x;\mu_1,\mu_2)+o(1). \qquad (3.2)$$

Here $Q_1(x;\mu_1,\mu_2)=\dfrac{2(x-s)\mu_1}{2\beta\mu_1+\mu_2}$.

Note that if ξ_1 and X_1 are mutually independent and $0<E(\xi_1)<\infty$, then the ergodic distribution of the process $Y(t)$ is independent from the moments of the random variable ξ_1.

4. ESTIMATOR FOR ERGODIC DISTRIBUTION OF INVENTORY MODEL OF TYPE (*s, S*)

In this section, we give estimator for ergodic distribution of inventory model of type (s, S) based on Frees's estimator [1] of renewal function.

$$\hat{Q}_{1n}(x)=1-\frac{\hat{U}_X(S-x)}{\hat{U}_X(S-s)},\qquad s\le x<S \qquad (4.1)$$

where

$$\hat{U}_X(S-s)=\frac{(S-s)}{\hat{\mu}_1}+\frac{\hat{\mu}_2}{2\hat{\mu}_1^2}$$

and

$$\hat{U}_X(S-x)=\frac{(S-x)}{\hat{\mu}_1}+\frac{\hat{\mu}_2}{2\hat{\mu}_1^2}.$$

Here $\hat{\mu}_1=\overline{X}=\sum_{i=1}^{n}X_i\big/n$ and $\hat{\mu}_2=\overline{X^2}=\sum_{i=1}^{n}X_i^2\big/n$ are estimators of μ_1 and μ_2 based on the random sample $X_1,X_2,\dots,X_n$. Then this estimator can be rewritten as follows:

$$\hat{Q}_{1n}(x)=\frac{2(x-s)\overline{X}}{2\beta\overline{X}+\overline{X^2}}. \qquad (4.2)$$

We first investigate some important statistical properties of the proposed estimator $\hat{Q}_{1n}(x)$ for

$Q_1(x;\mu_1,\mu_2)$. After that we will show that the proposed estimator $\hat{Q}_{1n}(x)$ converges in probability to $Q(x)$ when the value of parameter β =S-s is sufficiently large and $n\to\infty$.

Let us investigate the asymptotic unbiasedness of this estimator. To show unbiasedness properties of this estimator, we first need to prove following Proposition 4.1 about the covariance between $\bar{X}$ and $\overline{X^2}$.

Proposition 4.1. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with distribution function F and μ_i (i=1,2,...) are ith moment of the distribution F. $\overline{X^k} = \sum_{i=1}^n X_i^k / n$ (k=1,2,...) and $\overline{X^m} = \sum_{i=1}^n X_i^m / n$ (m=1,2,…) are kth and mth sample moment, respectively. The covariance between $\overline{X^k}$ and $\overline{X^m}$ can be given as: $Cov\left(\overline{X^k}, \overline{X^m}\right) = \left(\mu_{k+m} - \mu_k\mu_m\right)/n$.

Proof: See the Appendix A.

Theorem 4.1. Suppose that $\mu_4 = E\left(X_1^4\right) < \infty$, then

$$\lim_{n\to\infty} E\left(\hat{Q}_{1n}(x)\right) = \frac{2(x-s)\mu_1}{2\beta\mu_1 + \mu_2} = Q_1(x;\mu_1,\mu_2)$$

holds, that is, $\hat{Q}_{1n}(x)$ is asymptotic unbiased estimator of $Q_1(x;\mu_1,\mu_2)$.

Hereafter, for simplicity, we take $Q_1(\mu_1,\mu_2)$ instead of $Q_1(x;\mu_1,\mu_2)$.

Proof. Let us first find the second order Taylor expansion of $\hat{Q}_{1n}(x)$ at μ_1 and μ_2

$$\begin{aligned}\hat{Q}_{1n}(x) &= Q_1(\mu_1,\mu_2) + \left(\bar{X}-\mu_1\right)\left(\frac{\partial \hat{Q}_{1n}(x)}{\partial\left(\bar{X}\right)}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} + \frac{\left(\bar{X}-\mu_1\right)^2}{2}\left(\frac{\partial^2 \hat{Q}_{1n}(x)}{\partial\left(\bar{X}\right)^2}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} \\ &+ \left(\overline{X^2}-\mu_2\right)\left(\frac{\partial \hat{Q}_{1n}(x)}{\partial\left(\overline{X^2}\right)}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} + \frac{\left(\overline{X^2}-\mu_2\right)^2}{2}\left(\frac{\partial^2 \hat{Q}_{1n}(x)}{\partial\left(\overline{X^2}\right)^2}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} \\ &+ \left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)\left(\frac{\partial \hat{Q}_{1n}(x)}{\partial\left(\bar{X}\right)\partial\left(\overline{X^2}\right)}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} + \hat{R}_{21}, \qquad (4.3)\end{aligned}$$

where $\hat{R}_{21}$ is the remainder term in Lagrange form. The expected value of $\hat{Q}_{1n}(x)$ could be given as follows.

$$\begin{aligned}E\left(\hat{Q}_{1n}(x)\right) &= E\left(Q_1(\mu_1,\mu_2)\right) + E\left(\bar{X}-\mu_1\right)\left(\frac{\partial \hat{Q}_{1n}(x)}{\partial\left(\bar{X}\right)}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} + \frac{E\left(\bar{X}-\mu_1\right)^2}{2}\left(\frac{\partial^2 \hat{Q}_{1n}(x)}{\partial\left(\bar{X}\right)^2}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} + E\left(\overline{X^2}-\mu_2\right)\left(\frac{\partial \hat{Q}_{1n}(x)}{\partial\left(\overline{X^2}\right)}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} \\ &+ \frac{E\left(\overline{X^2}-\mu_2\right)^2}{2}\left(\frac{\partial^2 \hat{Q}_{1n}(x)}{\partial\left(\overline{X^2}\right)^2}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} + E\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)\left(\frac{\partial \hat{Q}_{1n}(x)}{\partial\left(\bar{X}\right)\partial\left(\overline{X^2}\right)}\right)\Bigg|_{\substack{\bar{X}=\mu_1\\ \overline{X^2}=\mu_2}} + E\left(\hat{R}_{21}\right) \qquad (4.4)\end{aligned}$$

It is seen that $E\left(\bar{X}-\mu_1\right)=0,\ E\left(\overline{X^2}-\mu_2\right)=0,\ E\left(\left(\bar{X}-\mu_1\right)^2\right)=\left(\mu_2-\mu_1^2\right)/n,\ E\left(\overline{X^2}-\mu_2\right)^2=\left(\mu_4-\mu_2^2\right)/n$ and $E\left(\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)\right)=Cov\left(\bar{X},\overline{X^2}\right)=\left(\mu_3-\mu_1\mu_2\right)/n$ from Proposition 4.1. It can be proved that $E\left(\hat{R}_{21}\right)\le D/n^3$ (see, Appendix B). So $E\left(\hat{Q}_{1n}(x)\right)$ is obtained as below:

$$E\left(\hat{Q}_{1n}(x)\right)=\frac{2(x-s)\mu_1}{2\beta\mu_1+\mu_2}+\left(\frac{-4(x-s)\beta\mu_2}{\left(2\beta\mu_1+\mu_2\right)^3}\right)\left(\frac{\mu_2-\mu_1^2}{n}\right)$$

$$+\left(\frac{2(x-s)\mu_1}{\left(2\beta\mu_1+\mu_2\right)^3}\right)\left(\frac{\mu_4-\mu_2^2}{n}\right)+\left(\frac{2(x-s)(2\beta\mu_1-\mu_2)}{\left(2\beta\mu_1+\mu_2\right)^3}\right)\left(\frac{\mu_3-\mu_2\mu_1}{n}\right)+o\left(\frac{1}{n}\right)$$

By simplifying, we get:

$$E\left(\hat{Q}_{1n}(x)\right)=Q_1(x)+\frac{2(x-s)A_1}{n}+o\left(\frac{1}{n}\right), \tag{4.5}$$

where $A_1=\dfrac{\left(2\beta(\mu_3\mu_1-\mu_2^2)+\mu_4\mu_1-\mu_3\mu_2\right)}{\left(2\beta\mu_1+\mu_2\right)^3}$. If $\mu_4<\infty$, $A_1<\infty$ for all $0<\beta<\infty$. We have the bias of $\hat{Q}_{1n}(x)$ as follows:

$$Bias\left(\hat{Q}_{1n}(x)\right)=\frac{2(x-s)A_1}{n}+o\left(\frac{1}{n}\right),\quad x\in(s,S]. \tag{4.6}$$

Thus, $Bias\left(\hat{Q}_{1n}(x)\right)\xrightarrow{n\to\infty}0$ holds. That is, it is seen that $\hat{Q}_{1n}(x)$ is asymptotic unbiased estimator for $Q_1(x)$.

We need to obtain the variance of $\hat{Q}_{1n}(x)$ before show that $\hat{Q}_{1n}(x)$ is a consistent for $Q_1(x)$. Thus, Lemma 4.1 gives the variance of $\hat{Q}_{1n}(x)$.

Lemma 4.1. Suppose that $\mu_4=E\left(X_1^4\right)<\infty$. Then, the variance of $\hat{Q}_{1n}(x)$ can be represented as follows as $n\to\infty$

$$Var\left(\hat{Q}_{1n}(x)\right)=\frac{4A_2}{n}(x-s)^2+o\left(\frac{1}{n}\right),$$

where $A_2=\dfrac{\left[2\beta\left(\mu_3\mu_1^2+2\mu_2\mu_1^3-3\mu_2^2\mu_1\right)+5\mu_4\mu_1^2-3\mu_3\mu_2\mu_1-4\mu_2^2\mu_1^2+2\mu_2^3\right]}{\left(2\beta\mu_1+\mu_2\right)^4}$.

Proof. It is known that

$$Var\left(\hat{Q}_{1n}(x)\right)=E\left(\hat{Q}_{1n}^2(x)\right)-\left[E\left(\hat{Q}_{1n}(x)\right)\right]^2$$

$$=E\left(\frac{2(x-s)\bar{X}}{2\beta\bar{X}+\overline{X^2}}\right)^2-\left[E\left(\frac{2(x-s)\bar{X}}{2\beta\bar{X}+\overline{X^2}}\right)\right]^2. \tag{4.7}$$

For calculating the first term in Eq. (4.7), we need to obtain the second order Taylor expansion of $\hat{Q}_{1n}^2(x)=\left(2(x-s)\overline{X}\Big/\left(2\beta\overline{X}+\overline{X^2}\right)\right)^2$ at μ_1 and μ_2 as given Eq. (4.8). We have the following result:

$$\hat{Q}_{1n}^2(x)=Q_1^2(\mu_1,\mu_2)+(\overline{X}-\mu_1)\left(\frac{\partial\hat{Q}_{1n}^2(x)}{\partial(\overline{X})}\right)\Bigg|_{\substack{\overline{X}=\mu_1\\ \overline{X^2}=\mu_2}}+\frac{(\overline{X}-\mu_1)^2}{2}\left(\frac{\partial^2\hat{Q}_{1n}^2(x)}{\partial(\overline{X})^2}\right)\Bigg|_{\substack{\overline{X}=\mu_1\\ \overline{X^2}=\mu_2}}+\left(\overline{X^2}-\mu_2\right)\left(\frac{\partial\hat{Q}_{1n}^2(x)}{\partial\left(\overline{X^2}\right)}\right)\Bigg|_{\substack{\overline{X}=\mu_1\\ \overline{X^2}=\mu_2}}$$

$$+\frac{\left(\overline{X^2}-\mu_2\right)^2}{2}\left(\frac{\partial^2\hat{Q}_{1n}^2(x)}{\partial\left(\overline{X^2}\right)^2}\right)\Bigg|_{\substack{\overline{X}=\mu_1\\ \overline{X^2}=\mu_2}}+(\overline{X}-\mu_1)\left(\overline{X^2}-\mu_2\right)\left(\frac{\partial\hat{Q}_{1n}^2(x)}{\partial(\overline{X})\partial\left(\overline{X^2}\right)}\right)\Bigg|_{\substack{\overline{X}=\mu_1\\ \overline{X^2}=\mu_2}}+\hat{R}_{22}. \quad (4.8)$$

The expected value of $\hat{Q}_{1n}^2(x)$ could be given as shown below:

$$E\left(\frac{2(x-s)\overline{X}}{2\beta\overline{X}+\overline{X^2}}\right)^2=\frac{4(x-s)^2\mu_1^2}{(2\beta\mu_1+\mu_2)^2}+\frac{8(x-s)^2\left(3\mu_4\mu_1^2-2\mu_3\mu_2\mu_1-2\mu_2^2\mu_1^2+\mu_2^3\right)}{n(2\beta\mu_1+\mu_2)^4}$$

$$+\frac{16(x-s)^2\beta\left(\mu_2\mu_1^3-2\mu_2^2\mu_1+\mu_3\mu_1^2\right)}{n(2\beta\mu_1+\mu_2)^4}+o\left(\frac{1}{n}\right). \quad (4.9)$$

The second term given in Eq. (4.7) is the square of the expression given in Eq. (4.5), i.e.,

$$\left[E\left(\frac{2(x-s)\overline{X}}{2\beta\overline{X}+\overline{X^2}}\right)\right]^2=\frac{4(x-s)^2\mu_1^2}{(2\beta\mu_1+\mu_2)^2}+4(x-s)^2\left(\frac{B_1}{n}+\frac{B_2}{n^2}\right)+o\left(\frac{1}{n^2}\right), \quad (4.10)$$

where $B_1=\dfrac{\left(2\beta(\mu_3\mu_1^2-\mu_2^2\mu_1)+\mu_4\mu_1^2-\mu_3\mu_2\mu_1\right)}{(2\beta\mu_1+\mu_2)^4}$ and

$$B_2=\frac{4\beta^2\left(\mu_3^2\mu_1^2-2\mu_3^2\mu_2^2\mu_1+\mu_2^4\right)+4\beta\left(\mu_4\mu_3\mu_1^2-\mu_4\mu_2^2\mu_1-\mu_3^2\mu_2\mu_1+\mu_3\mu_2^3\right)+\mu_4^2\mu_1^2-2\mu_4\mu_3\mu_2\mu_1+\mu_3^2\mu_2^2}{(2\beta\mu_1+\mu_2)^6}.$$

Consequently, the variance of $\hat{Q}_{1n}(x)$, by using the results shown in Eq. (4.9) and Eq. (4.10), is given by

$$Var\left(\hat{Q}_{1n}(x)\right)=4(x-s)^2\left(\frac{A_2}{n}+o\left(\frac{1}{n}\right)\right),$$

where $A_2=\dfrac{2\beta\left(\mu_3\mu_1^2+2\mu_2\mu_1^3-3\mu_2^2\mu_1\right)+5\mu_4\mu_1^2-3\mu_3\mu_2\mu_1-4\mu_2^2\mu_1^2+2\mu_2^3}{(2\beta\mu_1+\mu_2)^4}.$

This completes the proof. □

Corollary 4.2. It can be shown that

$$nVar\left(\hat{Q}_{1n}(x)\right)\rightarrow 4(x-s)^2A_2, \text{ as } n\rightarrow\infty. \quad (4.11)$$

Now we can show the consistency property of estimator $\hat{Q}_{1n}(x)$ as given in Lemma 4.2.

Lemma 4.2. Suppose that $E\left(X_i^4\right)=\mu_4<\infty$, then, $\hat{Q}_{1n}(x)$ is a consistent estimator of $Q_1(x)$.

Proof. Recall that $\lim_{n\to\infty} E\left(\hat{Q}_{1n}(x)\right)=Q_1(x)$ from Theorem 4.1. To show that $\hat{Q}_{1n}(x)$ is a consistent estimator of $Q_1(x)$, we need to prove that $\lim_{n\to\infty} P\left(\left|\hat{Q}_{1n}(x)-Q_1(x)\right|\geq\varepsilon\right)=0$. It is well known that $P\left(\left|\hat{Q}_{1n}(x)-Q_1(x)\right|\geq\varepsilon\right)\leq E\left(\hat{Q}_{1n}(x)-Q_1(x)\right)^2/\varepsilon^2$ from Markov inequality. Then, we need to show that $\lim_{n\to\infty} E\left(\hat{Q}_{1n}(x)-Q_1(x)\right)^2=0$. It is seen that

$$\lim_{n\to\infty} E\left(\hat{Q}_{1n}(x)-Q(x)\right)^2=\lim_{n\to\infty}\left(Var\left(\hat{Q}_{1n}(x)\right)+\left(Bias(\hat{Q}_{1n}(x))\right)^2\right)=\lim_{n\to\infty}\left(4(x-s)^2A_2/n+\left(2(x-s)A_1\right)^2/n^2\right)=0,$$

that is, $\hat{Q}_{1n}(x)$ is consistent estimator for $Q_1(x)$.

Now we can show that $\hat{Q}_{1n}(x)$ converges in probability to $Q(x)$ when the value of parameter β is sufficiently large.

Theorem 4.2. Suppose that $E\left(X_i^4\right)=\mu_4<\infty$, then, $\hat{Q}_{1n}(x)\xrightarrow[n\to\infty]{P}Q(x)$ when the value of parameter β is sufficiently large.

Proof. It is known that

$$\begin{aligned}P\left(\left|\hat{Q}_{1n}(x)-Q(x)\right|<\varepsilon\right)&=P\left(\left|\left(\hat{Q}_{1n}(x)-Q_1(x)\right)+\left(Q_1(x)-Q(x)\right)\right|<\varepsilon\right)\\&\geq P\left(\left|\left(\hat{Q}_{1n}(x)-Q_1(x)\right)\right|<\varepsilon/2 \text{ and } \left|\left(Q_1(x)-Q(x)\right)\right|<\varepsilon/2\right).\end{aligned}$$

We have $P\left(\left|\left(\hat{Q}_{1n}(x)-Q_1(x)\right)\right|<\varepsilon\right)\to 1$ from Lemma 4.2. It is possible to find a number of $\beta_0>0$ for each $\beta>\beta_0$, so that $\left|\left(Q_1(x)-Q(x)\right)\right|<\varepsilon$ and $P\left(\left|\left(Q_1(x)-Q(x)\right)\right|<\varepsilon\right)=1$. If E_n and F_n are two sequences of events, then $P\left(E_n\right)\xrightarrow[n\to\infty]{}1$, $P\left(F_n\right)\xrightarrow[n\to\infty]{}1$ implies $P\left(E_n\cap F_n\right)\xrightarrow[n\to\infty]{}1$ (Lemma 2.1.2 in [30]).

Thus, $P\left(\left|\left(\hat{Q}_{1n}(x)-Q_1(x)\right)\right|<\varepsilon/2 \text{ and } \left|\left(Q_1(x)-Q(x)\right)\right|<\varepsilon/2\right)\to 1$, implies $P\left(\left|\left(\hat{Q}_{1n}(x)-Q(x)\right)\right|<\varepsilon\right)\to 1$, that is, $\hat{Q}_{1n}(x)\xrightarrow[n\to\infty]{P}Q(x)$ when the value of parameter β is sufficiently large. □

Remark. Since $\hat{Q}_{1n}(x)\xrightarrow[n\to\infty]{P}Q(x)$ then $\hat{Q}_{1n}(x)$ is a consistent and asymptotic unbiased estimator for $Q(x)$ when the value of parameter β is sufficiently large.

To show asymptotic normality of the estimator $\hat{Q}_{1n}(x)$, we first need to following Lemma 4.3.

Lemma 4.3. (Multivariate Delta Method, [31]) Define the random vector $\mathbf{Z}=(Z_1,\dots,Z_r)$ with mean $\boldsymbol{\mu}=(\mu_1,\dots,\mu_r)$ and covariances $Cov\left(Z_i, Z_j\right)=\sigma_{ij}$. Let $\mathbf{Z}^{1)},\dots,\mathbf{Z}^{(n)}$ be a random samples of the population $\mathbf{Z}$ and $\hat{Z}_i=\sum_{k=1}^{n} Z_i^{(k)}$, i=1,...,r be the sum of observation for each variables. For a given function g(**Z**) with continues first partial derivatives and a specific value of $\boldsymbol{\mu}=(\mu_1,\dots,\mu_r)$ for which $\tau^2=\sum_i\sum_j\sigma_{ij}g_i'(\boldsymbol{\mu})g_j'(\boldsymbol{\mu})>0$. Then $\sqrt{n}\left(g(\hat{Z}^{(1)},\dots,\hat{Z}^{(n)})-g(\mu_1,\dots,\mu_p)\right)\xrightarrow[n\to\infty]{d}N\left(0,\tau^2\right)$. Here $g_i'(\boldsymbol{\mu})=\partial g\left(\mu_1,\dots,\mu_r\right)/\partial\mu_i$ (i=1,…r). Now we can give asymptotic normality of the estimator $\hat{Q}_{1n}(x)$.

Theorem 4.3. Suppose that $E\left(X_i^4\right)=\mu_4<\infty$, then $\hat{Q}_{1n}(x)$ is asymptotically normal, that is,

$$\sqrt{n}\left(\hat{Q}_{1n}(x)-Q_1(x)\right)\xrightarrow[n\to\infty]{d}N\left(0,\,\sigma^2\right),$$

where $\sigma^2=\dfrac{4(x-s)^2}{\left(2\beta\mu_1+\mu_2\right)^4}\left(\mu_4\mu_1^2-2\mu_3\mu_2\mu_1+\mu_2^3\right).$

Proof. By the Multivariate Delta Method [31] we obtain that

$$\sigma^2=Var(X_1)\left(\frac{\partial\hat{Q}_{1n}(\mu_1,\mu_2)}{\partial(\mu_1)}\right)^2+Var(X_1^2)\left(\frac{\partial\hat{Q}_{1n}(\mu_1,\mu_2)}{\partial(\mu_2)}\right)^2$$

$$+2Cov\left(X_1,X_1^2\right)\frac{\partial\hat{Q}_{1n}(\mu_1,\mu_2)}{\partial(\mu_1)}\frac{\partial\hat{Q}_{1n}(\mu_1,\mu_2)}{\partial(\mu_2)},$$

where $Var(X_1)=\mu_2-\mu_1^2,\ Var(X_1^2)=\mu_4-\mu_2^2$ and $Cov\left(X_1,X_1^2\right)=\mu_3-\mu_2\mu_1$.
Then,

$$\sigma^2=\frac{4(x-s)^2}{\left(2\beta\mu_1+\mu_2\right)^4}\left(\mu_4\mu_1^2-2\mu_3\mu_2\mu_1+\mu_2^3\right).$$

Thus, we have that $\sqrt{n}\left(\hat{Q}_{1n}(x)-Q_1(x)\right)\xrightarrow[n\to\infty]{d}N\left(0,\,\sigma^2\right)$.

This completes the proof. □

As a result, it is shown that $\hat{Q}_{1n}(x)$ is consistent, asymptotic unbiased and asymptotically normal estimator for $Q(x)$ when the value of parameter β is sufficiently large.

5. NUMERICAL EXAMPLE

A company operating in the energy sector produces, stores, fills, and distributes liquefied petroleum gas (LPG). Domestic LPG distribution is carried out through pipelines and land transport. Where there is no pipeline installation, gas is distributed through land transport. LPG is carried from the LPG production center (a city in Turkey) to the 30 dealers by tankers. After delivering the needed amount of gas to the dealer, the tanker waits in its position until the nest order of any dealer. Each dealer has a storage capacity of S=30m^3. Random amounts of LPG (X_n) are sold from these storage tanks at random times (ξ_n) . The level of LPG in the tank of the dealer falls below the control level s=S/10, a demand signal is automatically sent online to the production center. As a response to this demand, the nearest tanker to the dealer is directed to the demanding dealer. If there is no tanker near to the dealer, a full tanker is sent from the production center. The dealers fill the full capacity S of their tanks. Therefore, the process that expresses the working principle of the storage tank can be considered as the renewal process with inventory model of type (s, S). A dealer need to know ergodic distribution of the process $Y(t)$ to obtain some important characteristics of the process. To do this we should know the distribution of demands and its parameters at least their first two moments. Knowing the distribution or its parameters are almost impossible in most of the times. Nevertheless these basic characteristics, e.g. first two moments of the demands can be estimated from the calculated data. To obtain these moments dealer collected data until full storage tank emptied twice. For generating this kind of data we use pseudo random numbers from Gamma (2,1). The generated data is given at Table 5.

Table 5.1.The demand quantity collected from dealers

1,14	1,50	0,28	1,99	2,89	1,62	3,05	1,87	3,32
1,91	1,90	0,40	1,59	2,62	1,88	1,56	10,42	0,61
2,56	2,75	3,01	0,29	1,36	0,80	0,77	2,08	1,58

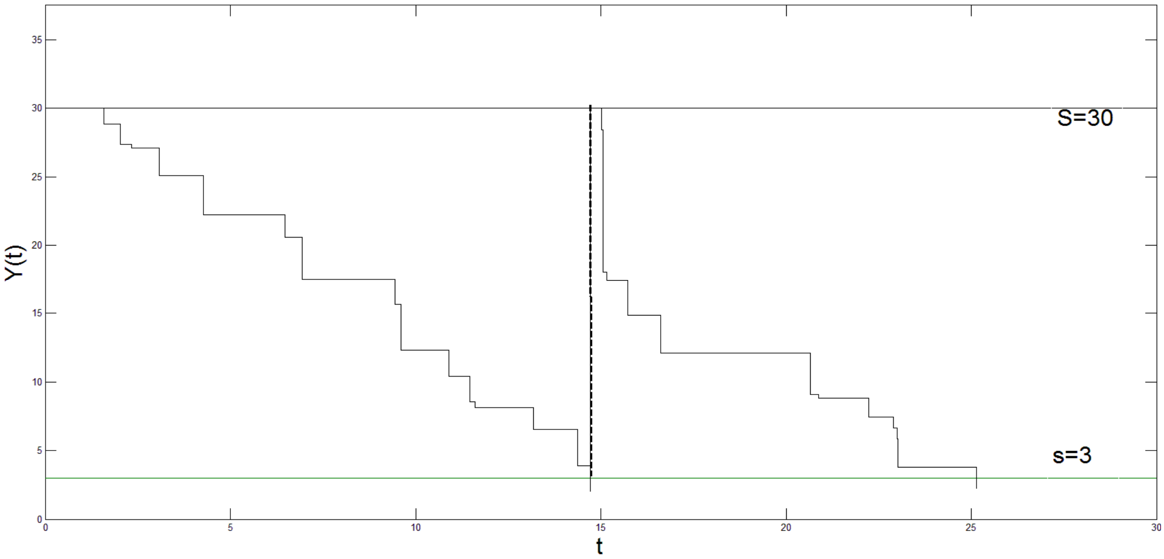

Figure 2. A figure of the process according to example

The first and second sample moments are $\overline{X} = 1.923$ and $\overline{X^2} = 7.147.$ By using the moments, estimator for $Q(x)$ given in Eq. (4.2) is obtained as

$$\hat{Q}_{1n}(x) = \frac{2(x-s)\overline{X}}{\left(2\beta\overline{X} + \overline{X^2}\right)} = 0.035x - 0.104.$$

To compare $Q(x)$ and $\hat{Q}_{1n}(x)$ we take different values of *x* from *s* to *S*. Then values of $Q(x)$ and $\hat{Q}_{1n}(x)$ are obtained for 13 value of *x*. Also we compare these values according to their absolute difference, $\Delta = \left|\hat{Q}_{1n}(x) - Q(x)\right|$, and accurate percentage (AP) values, AP(%)=100-$\left|\hat{Q}_{1n}(x) - Q(x)\right| / Q(x).$ These values are given in Table 5.2.

Table 5.2. A comparison of the values of $Q(x)$ and $\hat{Q}_{1n}(x)$

x	$Q(x)$	$\hat{Q}_{1n}(x)$	Δ	AP
4	0,0351	0,0350	0.0001	99,71
6	0,1053	0,1051	0.0002	99,81
8	0,1754	0,1751	0.0003	99,82
10	0,2456	0,2452	0.0004	99,83
12	0,3158	0,3153	0.0005	99,84
14	0,3860	0,3853	0.0007	99,81
16	0,4561	0,4554	0.0007	99,84
18	0,5263	0,5254	0.0009	99,82
20	0,5965	0,5955	0.0010	99,83
22	0,6667	0,6656	0.0011	99,83
24	0,7368	0,7356	0.0012	99,83
26	0,8070	0,8057	0.0013	99,83
28	0,8769	0,8755	0.0014	99,84

Here $U(z) = z/2 + 3/4 + (1/4)\exp(-2z)$ for Gamma (2,1).

As seen from Table 5.2 the all AP values are greater than 99%. These results show that the proposed estimator $\hat{Q}_{1n}(x)$ for $Q(x)$ gives very accurate results.

In the rest of this section, we will show the goodness of fit of $\hat{Q}_{1n}(x)$ to $Q(x)$ for Gamma distribution with parameter (2,1) by using Kolmogrov-Simirnov One Sample Test. The test statistics of Kolmogorov-Simirnov is $D_{\max} = \max\left\{\hat{Q}_{1n}(x) - Q(x);\ \forall x\right\}$. We use different β=S-s and n values to obtain the mean and the maximum of D_{max} . To obtain this values, we generate 10.000 pseudo random number set for each combination of β and n. The obtained results are given at the Table 5.3.

Table 5.3. The mean and maximum values of D_{max} for Gamma (2,1)

β	n	$\overline{D_{max}}$	Max(D_{max})
5	10	0.0346	0.2001
5	20	0.0265	0.1195
5	30	0.0229	0.1000
5	50	0.0188	0.0765
5	100	0.0151	0.0644
10	10	0.0241	0.1395
10	20	0.0186	0.0977
10	30	0.0157	0.0840
10	50	0.0131	0.0528
10	100	0.0103	0.0418
20	10	0.0147	0.0970
20	20	0.0111	0.0740
20	30	0.0096	0.0481
20	50	0.0079	0.0336
20	100	0.0061	0.0267
50	10	0.0060	0.0473
50	20	0.0044	0.0290
50	30	0.0037	0.0232
50	50	0.0030	0.0178
50	100	0.0022	0.0112
100	10	0.0029	0.0364
100	20	0.0022	0.0156
100	30	0.0018	0.0145
100	50	0.0014	0.0094
100	100	0.0010	0.0055
200	10	0.0015	0.0138
200	20	0.0011	0.0100
200	30	0.0009	0.0056
200	50	0.0007	0.0045
200	100	0.0005	0.0030

As seen from Table 5.3, both maximum and mean of the D_{max} statistics are decreased when n and β are increased. Especially for great values of *n* and β, these values are very small. This indicates that the estimator is much better at the great values of *n* and β.

6. CONCLUSION AND REMARKS

In real world application, it is very important to know characteristics of an inventory model of type (*s, S*), such as ergodic distribution, moments, etc. The ergodic distribution is obtained by using renewal function of demand quantity which has distribution function F. However, the functional form of the distribution function F or the parameters of the distribution or both of them are unknown in many cases. Thus, it is desirable to have a estimator of the renewal function. For this reason, in this article, we give an estimator for ergodic distribution of inventory model of type (s, S) by using the straight line approach for Frees [1]. We show that the proposed estimator is consistent, asymptotically unbiased and asymptotically normal for large values of β. The proposed estimator is also applied to a dataset. The results indicate that the accurate percentage (AP) values are all above 99%. It is meant that the estimator gives the results very close to $Q(x)$. However, it is observed that the AP values get smaller as the *x* values get closer to *S*. The estimator given in this study can be applied to other complex stochastic models including renewal function. Therefore, in future studies, it can be considered to apply this estimator in different inventory models of type (*s*, *S*).

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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Appendix A

Proof. The covariance between $\overline{X^k}$ and $\overline{X^m}$ is obtained as follows:

$$Cov\left(\overline{X^k}, \overline{X^m}\right) = E\left[\left(\overline{X^k} - \mu_k\right)\left(\overline{X^m} - \mu_m\right)\right] = E\left(\overline{X^k}\,\overline{X^m}\right) - \mu_k \mu_m .$$

Here $E\left(\overline{X^k}\,\overline{X^m}\right)$ term is obtained as

$$E\left(\overline{X^k}\,\overline{X^m}\right)=E\left(\frac{1}{n}\sum_{i=1}^{n}X_i^k\frac{1}{n}\sum_{i=1}^{n}X_i^m\right)$$
$$=\frac{1}{n^2}E\left(\sum_{i=1}^{n}X_i^{k+m}+2\sum_{1\le i<j\le n}X_i^kX_j^m\right)$$
$$=\frac{1}{n^2}\left[\sum_{i=1}^{n}E\left(X_i^{k+m}\right)+2\sum_{1\le i<j\le n}E\left(X_i^kX_j^m\right)\right]$$

$$\sum_{i=1}^{n}E\left(X_i^{k+m}\right)=\left(n\mu_{k+m}\right)$$
$$\sum_{1\le i<j\le n}E\left(X_i^kX_j^m\right)=E\left(X_1^kX_2^m+X_1^kX_3^m+\ldots+X_1^kX_n^m+X_2^kX_3^m+\ldots+X_2^kX_n^m+\ldots+X_{n-1}^kX_n^m\right)$$
$$=E\left(X_1^k\right)E\left(X_2^m\right)+E\left(X_1^k\right)E\left(X_3^m\right)+\ldots+E\left(X_1^k\right)E\left(X_n^m\right)+E\left(X_2^k\right)E\left(X_3^m\right)$$
$$+\ldots+E\left(X_2^k\right)E\left(X_n^m\right)+\ldots+E\left(X_{n-1}^k\right)E\left(X_n^m\right)$$
$$=\frac{n(n-1)}{2}\mu_k\mu_m$$

$$E\left(\overline{X^k}\,\overline{X^m}\right)=\frac{1}{n^2}\left[\sum_{i=1}^{n}E\left(X_i^{k+m}\right)+2\sum_{1\le i<j\le n}E\left(X_i^kX_j^m\right)\right]$$
$$=\frac{1}{n^2}\left[n\mu_{k+m}+2\frac{n(n-1)}{2}\mu_k\mu_m\right]$$
$$=\frac{\mu_{k+m}}{n}+\mu_k\mu_m-\frac{\mu_k\mu_m}{n}$$

Then, $Cov\left(\overline{X^k},\overline{X^m}\right)=\left(\mu_{k+m}-\mu_k\mu_m\right)/n.$

Corollary A.1. The covariance between $\overline{X}$ and $\overline{X^2}$ can be given as $Cov\left(\overline{X},\overline{X^2}\right)=\left(\mu_3-\mu_1\mu_2\right)/n.$

Remark. As it is seen from Corollary 4.1, $\lim_{n\to\infty}Cov\left(\overline{X},\overline{X^2}\right)=0$ when $\mu_3=E\left(X_1^3\right)<\infty.$ This result shows that $\overline{X}$ and $\overline{X^2}$ are asymptotically uncorrelated.

By using Corollary 4.1, we can prove that the estimator $\hat{Q}_{1n}(x)$ is an asymptotic unbiased estimator of $Q_1(x)$.

Appendix B

The expected value of remainder term $\hat{R}_2$ goes to zero as $n\to\infty$. To show this, initially, we need to obtain Lagrange form of the remainder term $\hat{R}_2$ as follows:

$$\hat{R}_{21}=\left(\bar{X}-\mu_1\right)^3\left\{\frac{8(x-s)\beta^2}{C_1^3}-\frac{16(x-s)\beta^3\left(\mu_1+\theta\left(\bar{X}-\mu_1\right)\right)}{C_1^4}\right\}+$$

$$\left(\bar{X}-\mu_1\right)^2\left(\overline{X^2}-\mu_2\right)\left\{\frac{8(x-s)\beta}{C_1^3}-\frac{24(x-s)\beta^2\left(\mu_1+\theta\left(\bar{X}-\mu_1\right)\right)}{C_1^4}\right\}+$$

$$\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)^2\left\{\frac{2(x-s)}{C_1^3}-\frac{12(x-s)\beta\left(\mu_1+\theta\left(\bar{X}-\mu_1\right)\right)}{C_1^4}\right\}+$$

$$\left(\overline{X^2}-\mu_2\right)^3\left\{\frac{-2(x-s)\left(\mu_1+\theta\left(\bar{X}-\mu_1\right)\right)}{C_1^4}\right\}$$

where $0<\theta<1$ and $C_1=\left[2\beta\left(\mu_1+\theta\left(\bar{X}-\mu_1\right)\right)+\mu_2+\theta\left(\bar{X}-\mu_2\right)\right]$. According to law of large numbers, $\bar{X}\xrightarrow{P}\mu_1$ and $\overline{X^2}\xrightarrow{P}\mu_2$, then n can be chosen so large that $\left|\bar{X}-\mu_1\right|<\frac{\delta_1}{2n}$ and $\left|\overline{X^2}-\mu_2\right|<\frac{\delta_2}{2n}$ where $0<\delta_1;\delta_2<1$. Hence we have inequality as shown below:

$\mu_1+\theta\left(\bar{X}-\mu_1\right)\geq\mu_1-\frac{\theta\delta_1}{2n}$ and $\mu_2+\theta\left(\overline{X^2}-\mu_2\right)\leq\mu_2+\frac{\theta\delta_2}{2n}$.

Then, we obtain the upper bound of $\left|\hat{R}_{21}\right|$ as follows:

$$\left|\hat{R}_{21}\left(\bar{X},\overline{X^2}\right)\right|\leq\left(\frac{\delta_1}{2n}\right)^3\left\{\frac{8(x-s)\beta^2}{C_2^3}-\frac{16(x-s)\beta^3\left(\mu_1+\left(\theta\delta_1/2n\right)\right)}{C_2^4}\right\}$$

$$+\left(\frac{\delta_1}{2n}\right)^2\left(\frac{\delta_2}{2n}\right)\left\{\frac{8(x-s)\beta}{C_2^3}-\frac{24(x-s)\beta^2\left(\mu_1+\left(\theta\delta_1/2n\right)\right)}{C_2^4}\right\}$$

$$+\left(\frac{\delta_1}{2n}\right)\left(\frac{\delta_2}{2n}\right)^2\left\{\frac{2(x-s)}{C_2^3}-\frac{12(x-s)\beta\left(\mu_1+\left(\theta\delta_1/2n\right)\right)}{C_2^4}\right\}$$

$$+\left(\frac{\delta_2}{2n}\right)^3\left\{\frac{-2(x-s)\left(\mu_1+\left(\theta\delta_1/2n\right)\right)}{C_2^4}\right\} \qquad \text{(A.1)}$$

where $C_2=\left[2\beta\left|\mu_1-\left(\theta\delta_1/2n\right)\right|+\left|\mu_2-\left(\theta\delta_2/2n\right)\right|\right]$. δ_1 $(0<\delta_1<1)$ and δ_2 $(0<\delta_2<1)$ are chosen so small that let $\left|\mu_1-\left(\theta\delta_1/2\right)\right|\geq\mu_1/2$ and $\left|\mu_2-\left(\theta\delta_2/2\right)\right|\geq\mu_2/2$. In this case, Eq. (A.1) similarly holds by using $C_3=\left[2\beta\,\mu_1/2+\mu_2/2\right]$ instead of C_2.

δ_1 $(0<\delta_1<1)$ and δ_2 $(0<\delta_2<1)$ are chosen so small that let $\theta\delta_1/2\leq\mu_1$ and $\theta\delta_2/2\leq\mu_2$. Then,

$$\left|\hat{R}_{21}\left(\bar{X},\overline{X^2}\right)\right| \le \left(\frac{\delta_1}{2n}\right)^3\left[\frac{8\beta^2(x-s)}{C_3^3}-\frac{32\beta^3(x-s)\mu_1}{C_3^4}\right]+\left(\frac{\delta_1}{2n}\right)^2\left(\frac{\delta_2}{2n}\right)\left\{\frac{8(x-s)\beta}{C_3^3}-\frac{48(x-s)\beta^2\mu_1}{C_3^4}\right\}$$

$$+\left(\frac{\delta_1}{2n}\right)\left(\frac{\delta_2}{2n}\right)^2\left\{\frac{2(x-s)}{C_3^3}-\frac{24\beta(x-s)\mu_1}{C_3^4}\right\}+\left(\frac{\delta_2}{2n}\right)^3\left\{\frac{-4(x-s)\mu_1}{C_3^4}\right\}.$$

Let max(δ_1; δ_2)= δ, then

$$\left|\hat{R}_{21}\left(\bar{X},\overline{X^2}\right)\right| \le \frac{\delta^3}{n^3}\left\{\frac{(x-s)\left(\beta^2+\beta+1/4\right)}{C_3^3}-\frac{(x-s)\mu_1\left(4\beta^3+6\beta^2+3\beta+1/2\right)}{C_3^4}\right\}$$

Then,

$$\left|\hat{R}_{21}\left(\bar{X},\overline{X^2}\right)\right| \le \frac{D}{n^3},$$

where $D=\delta^3\left\{\frac{(x-s)\left(\beta^2+\beta+1/4\right)}{C_3^3}-\frac{(x-s)\mu_1\left(4\beta^3+6\beta^2+3\beta+1/2\right)}{C_3^4}\right\}$.

It is known that if $X_1 \le X_2$ in probability 1, then $E\left(X_1\right)\le E\left(X_2\right)$. So, $\left|E\left(\hat{R}_{21}\left(\bar{X},\overline{X^2}\right)\right)\right| \le E\left(\left|\hat{R}_{21}\left(\bar{X},\overline{X^2}\right)\right|\right)\le E\left(D/n^3\right)=D/n^3$. Therefore, $E\left(\hat{R}_{21}\left(\bar{X},\overline{X^2}\right)\right)\le D/n^3$.