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Time-Dependent Analysis for a Two-Unit System with Connecting and Disconnecting Effect

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ABSTRACT

By studying the spectrum of the underlying operator corresponding to the two-unit system with connecting and disconnecting effect, we prove that the time-dependent solution of the system converges strongly to its steady-state solution as time tends to infinity.

Keywords: Two-unit system, Connecting and Disconnecting Effect, Eigenvalue, Resolvent Set.

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1. INTRODUCTION

Earlier researchers [1,4,5] considered the system working in a degraded state instantaneously after the failure of one of the units. In most cases, the repair is also done instantaneously after the failure of one unit, but in practice it can be seen that the criterion is not always possible. In certain systems, the failed unit should be disconnected for the smooth working of the other unit. In 1997 Goel et al. [3] considered such cases and established the corresponding model by supplementary variable technique [2], then they obtained various expression including the system reliability, mean time of the failure by using the Laplace transforms based on the following hypotheses:

Hypothesis 1. The system has a unique and nonnegative time-dependent solution.

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Hypothesis 2. The time-dependent solution of the system converges to its steady-state solution.

In 2013, by using strong continuous semigroup theory of linear operators Mijit et al. [7] proved the model has a unique nonnegative time-dependent solution, that is to say, they proved the Hypothesis 1. In this paper, when repair rates are constants, we study the asymptotic behavior of the time-dependent solution of the system model, i.e., we study the Hypothesis 2. First of all, we prove that 0 is an eigenvalue of the underlying operator with geometric multiplicity one. Next, we determine the expression of the adjoint operator of the underlying operator and obtain the resolvent set of the underlying operator by studying the resolvent set of its adjoint operator. Last, we prove that 0 is an eigenvalue of the adjoint operator with geometric multiplicity one. Therefore, by combining these results with our previous result we obtain the asymptotic behavior of the time-dependent solution of the system.

According to Goel et al. [3], the model for a two-unit system with connecting and disconnecting effect can be expressed by a group of integro-differential equations:

$$\frac{dp_0(t)}{dt} = -(2\lambda + \lambda_{h_0})p_0(t) + \theta p_3(t) + \int_0^\infty \beta p_H(x,t)dx \,, \quad (1.1)$$

$$\frac{dp_1(t)}{dt} = 2\lambda p_0(t) - \theta p_1(t), \quad (1.2)$$

$$\frac{dp_2(t)}{dt} = \theta p_1(t) - (\lambda + \lambda_{h_2} + \mu)p_2(t)\,, \quad (1.3)$$

$$\frac{dp_3(t)}{dt} = \mu p_2(t) - \theta p_3(t) + \int_0^\infty \alpha p_F(x,t)dx \,, \quad (1.4)$$

$$\frac{\partial p_F(x,t)}{\partial t} + \frac{\partial p_F(x,t)}{\partial x} = -\alpha p_F(x,t)\,, \quad (1.5)$$

$$\frac{\partial p_H(x,t)}{\partial t} + \frac{\partial p_H(x,t)}{\partial x} = -\beta p_H(x,t)\,, \quad (1.6)$$

$$p_F(0,t) = \lambda p_2(t)\,, \quad (1.7)$$

$$p_H(0,t) = \lambda_{h_0} p_0(t) + \lambda_{h_2} p_2(t)\,, \quad (1.8)$$

$$\begin{aligned} &p_0(0) = 1,\ p_i(0) = 0,\ i = 1,2,3; \\ &p_i(x,0) = 0,\ i = F,H. \end{aligned} \quad (1.9)$$

Where, $(x,t) \in [0,\infty) \times [0,\infty)$; $p_i(t)$ represents the probability that the system is in state i at time t, $i = 0,1,2,3$; $p_i(x,t)$ represents the probability that at time t the failed system is in the state i and has an elapsed service time x, $i = F,H$; λ represents constant failure rate of one unit in the system; λ_{h_i} represents constant failure rates from the states i to the state H, $i = 0,2$; θ represents constant connecting/ disconnecting rate of the repaired/failed unit to/from the system; μ represents constant repair rate of a unit in the system from degraded state, α / β represents repair rates from states F / H .

2. PROBLEM FORMULATION

We first formulate the system (1.1)-(1.9) as an abstract Cauchy problem on a suitable state space. For convenience we introduce a notation as follows:

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ \lambda_{h_0} & 0 & \lambda_{h_2} & 0 & 0 & 0 \end{pmatrix}.$$

Choose the state space as follows:

$$X = \left\{ p \left| \begin{array}{l} p \in R^4 \times (L^1[0,\infty))^2, \\ \| p \| = \sum_{i=0}^{3} | p_i | + \| p_F \|_{L^1[0,\infty)} \\ \quad + \| p_H \|_{L^1[0,\infty)} \end{array} \right. \right\}.$$

It is obvious that X is a Banach space. In the following we define operators and their domains:

$$D(A) = \left\{ p \in X \left| \begin{array}{l} \frac{dp_i(x)}{dx} \in L^1[0,\infty),\ p_i(x) \\ (i = F, H) \text{ is absolutely} \\ \text{continuous function and} \\ p(0) = \Gamma p(x) \end{array} \right. \right\},$$

$$\begin{aligned} &A(p_{0,}\ p_1, p_2, p_3, p_F(x), p_H(x))^T \\ &= \mathrm{diag}(-(2\lambda + \lambda_{h_0}), -\theta, -(\lambda + \lambda_{h_2} + \mu), \\ &-\theta, -\tfrac{d}{dx} - \alpha, -\tfrac{d}{dx} - \beta)(p_{0,}\ p_1, p_2, p_3, \\ &p_F(x), p_H(x))^T, \\ &U(p_{0,}\ p_1, p_2, p_3, p_F(x), p_H(x))^T \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \theta & 0 & 0 \\ 2\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_F(x) \\ p_H(x) \end{pmatrix},$$

$$E \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_F(x) \\ p_H(x) \end{pmatrix} = \begin{pmatrix} \int_0^\infty \beta p_H(x)dx \\ 0 \\ 0 \\ \int_0^\infty \alpha p_F(x)dx \\ 0 \\ 0 \end{pmatrix},$$

$D(U) = D(E) = X$.

Then the above equations (1.1)-(1.9) can be rewritten as an abstract Cauchy problem in the Banach space X:

$$\begin{cases} \frac{dp(t)}{dt} = (A + U + E)p(t),\ t > 0, \\ p(0) = (1,0,0,0,0,0). \end{cases} \qquad (2.1)$$

In [7], the authors obtained the following result.

Theorem 2.1 $A + U + E$ generates a positive contraction C_0 −semigroup $T(t)$. $T(t)$ is isometric for the initial value. So, the system (2.1) has a unique nonnegative time-dependent solution which satisfies the probability condition.

3. ASYMPTOTIC PROPERTY OF THE SYSTEM (2.1)

Lemma 3.1 0 is an eigenvalue of $A + U + E$ with geometric multiplicity one.

Proof We consider the equation

$$(A + U + E)p = 0.$$

It is equivalent to

$$-(2\lambda + \lambda_{h_0})p_0 + \theta p_3 + \int_0^\infty \beta p_H(x)dx = 0, \qquad (3.1)$$

$$2\lambda p_0 - \theta p_1 = 0, \qquad (3.2)$$

$$\theta p_1 - (\lambda + \lambda_{h_2} + \mu) p_2 = 0, \quad (3.3)$$

$$\mu p_2 - \theta p_3 + \int_0^\infty \alpha p_F(x) dx = 0, \quad (3.4)$$

$$\frac{dp_F(x)}{dx} = -\alpha p_F(x), \quad (3.5)$$

$$\frac{dp_H(x)}{dx} = -\beta p_H(x), \quad (3.6)$$

$$p_F(0) = \lambda p_2, \quad (3.7)$$

$$p_H(0) = \lambda_{h_0} p_0 + \lambda_{h_2} p_2. \quad (3.8)$$

By solving (3.2), (3.3), (3.5), (3.6), we have

$$p_1 = \frac{2\lambda}{\theta} p_0, \quad (3.9)$$

$$p_2 = \frac{2\lambda}{\lambda + \lambda_{h_2} + \mu} p_0, \quad (3.10)$$

$$p_F(x) = a_F e^{-\alpha x}, \quad (3.11)$$

$$p_H(x) = a_H e^{-\beta x}. \quad (3.12)$$

Combining (3.11) and (3.12) with (3.7) and (3.8), we obtain

$$a_F = p_F(0) = \lambda p_2, \quad (3.13)$$

$$a_H = p_H(0) = \lambda_{h_0} p_0 + \lambda_{h_2} p_2. \quad (3.14)$$

By inserting (3.11) into (3.4), and use (3.10), (3.13), we calculate

$$\mu p_2 - \theta p_3 + \alpha \int_0^\infty a_F e^{-\alpha x} dx$$

$$= \mu p_2 - \theta p_3 + a_F = \mu p_2 - \theta p_3 + \lambda p_2 = 0$$

$$\Rightarrow$$

$$p_3 = \frac{\mu + \lambda}{\theta} p_2 = \frac{2\lambda(\mu + \lambda)}{\theta(\lambda + \lambda_{h_2} + \mu)} p_0. \quad (3.15)$$

From (3.9) – (3.15), we deduce

$$\| p \| = \sum_{i=0}^{3} | p_i | + \| p_F \|_{L^1[0,\infty)} + \| p_H \|_{L^1[0,\infty)}$$

$$\le [1 + \frac{2\lambda}{\theta} + \frac{2\lambda}{\lambda + \lambda_{h_2} + \mu} + \frac{2\lambda(\lambda + \mu)}{\theta(\lambda + \lambda_{h_2} + \mu)}$$

$$+ \frac{2\lambda^2}{\alpha(\lambda + \lambda_{h_2} + \mu)} + \frac{\lambda_{h_0}}{\beta}$$

$$+ \frac{2\lambda\lambda_{h_2}}{\beta(\lambda + \lambda_{h_2} + \mu)}] | p_0 | < \infty.$$

Which shows that 0 is an eigenvalue of $A + U + E$. Moreover, by (3.9) – (3.15) we see that the geometric multiplicity of 0 is one.

It is easy to see that X^*, the dual space of X, is equal to

$$X^* = \left\{ q^* \middle| \begin{array}{l} q^* \in R^4 \times (L^\infty[0,\infty))^2, \\ ||| q^* ||| = \sup\{\sup_{0 \le i \le 3} | q_i^* |, \\ \quad \| q_F^* \|_{L^\infty[0,\infty)}, \| q_H^* \|_{L^\infty[0,\infty)}\} \end{array} \right\}$$

It is obvious that X^* is a Banach space, then we have

Lemma 3.2 $(A+U+E)^*$, the adjoint operator of $A+U+E$, is as follows:

$(A+U+E)^* q^* = (H+J+S) q^*$,

$q^* \in D((A+U+E)^*)$,

where

$$Hq^*(x) = \mathrm{diag}(-(2\lambda + \lambda_{h_0}), -\theta,$$
$$-(\lambda + \lambda_{h_2} + \mu), -\theta, \tfrac{d}{dx} - \alpha, \tfrac{d}{dx} - \beta)$$
$$(q_0^* \, q_1^*, q_2^*, q_3^*, q_F^*(x), q_H^*(x))^T,$$

$Jq^*(x)$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_{h_0} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & \lambda_{h_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ q_3^* \\ q_F^*(0) \\ q_H^*(0) \end{pmatrix},$$

$Sq^*(x)$

$$= \begin{pmatrix} 0 & 2\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ q_3^* \\ q_F^*(x) \\ q_H^*(x) \end{pmatrix},$$

$D((A+U+E)^*)$

$$= \left\{ q^* \in X^* \left| \begin{array}{l} \frac{dq_i^*(x)}{dx} \text{ exists and} \\ q_i^*(\infty) = M, i = F, H \end{array} \right. \right\}.$$

Here M is a constant which is irrelevant to i.

Proof: For any $p \in D(A+U+E)$ and $q^* \in D((A+U+E)^*)$, by using the boundary conditions on $p \in D(A+U+E)$ and integration by parts, we have

$$\begin{aligned}
&\langle (A+U+E)p, q^* \rangle \\
&= [-(2\lambda+\lambda_{h_0})p_0 + \theta p_3 + \int_0^\infty \beta p_H(x)dx]q_0^* \\
&+ (2\lambda p_0 - \theta p_1)q_1^* \\
&+ [\theta p_1 - (\lambda+\lambda_{h_2}+\mu)p_2]q_2^* \\
&+ [\mu p_2 - \theta p_3 + \int_0^\infty \alpha p_F(x)dx]q_3^* \\
&+ \int_0^\infty [-\frac{dp_F(x)}{dx} - \alpha p_F(x)]q_F^*(x)dx \\
&+ \int_0^\infty [-\frac{dp_H(x)}{dx} - \beta p_H(x)]q_H^*(x)dx \\
&= p_0[-(2\lambda+\lambda_{h_0})q_0^* + 2\lambda q_1^*] + p_1\theta[-q_1^* + q_2^*] \\
&+ p_2[-(\lambda+\lambda_{h_2}+\mu)q_2^* + \mu q_3^*] + p_3\theta[q_0^* - q_3^*] \\
&+ \int_0^\infty p_F(x)\alpha[q_3^* - q_F^*(x)]dx \\
&+ \int_0^\infty p_H(x)\beta[q_0^* - q_H^*(x)]dx \\
&- \int_0^\infty \frac{dp_F(x)}{dx}q_F^*(x)dx - \int_0^\infty \frac{dp_H(x)}{dx}q_H^*(x)dx \\
&= p_0[-(2\lambda+\lambda_{h_0})q_0^* + 2\lambda q_1^*] + p_1\theta[-q_1^* + q_2^*] \\
&+ p_2[-(\lambda+\lambda_{h_2}+\mu)q_2^* + \mu q_3^*] + p_3\theta[q_0^* - q_3^*] \\
&+ \int_0^\infty p_F(x)\alpha[q_3^* - q_F^*(x)]dx \\
&+ \int_0^\infty p_H(x)\beta[q_0^* - q_H^*(x)]dx \\
&+ p_F(0)q_F^*(0) + \int_0^\infty p_F(x)\frac{dq_F^*(x)}{dx}dx \\
&+ p_H(0)q_H^*(0) + \int_0^\infty p_H(x)\frac{dq_H^*(x)}{dx}dx \\
&= p_0[-(2\lambda+\lambda_{h_0})q_0^* + 2\lambda q_1^* + \lambda_{h_0}q_H^*(0)] \\
&+ p_1\theta[-q_1^* + q_2^*] + p_2[-(\lambda+\lambda_{h_2}+\mu)q_2^* \\
&+ \mu q_3^* + \lambda q_F^*(0) + \lambda_{h_2}q_H^*(0)] + p_3\theta[q_0^* - q_3^*] \\
&+ \int_0^\infty p_F(x)[\alpha q_3^* - \alpha q_F^*(x) + \frac{dq_F^*(x)}{dx}]dx \\
&+ \int_0^\infty p_H(x)[\beta q_0^* - \beta q_H^*(x) + \frac{dq_H^*(x)}{dx}]dx \\
&= \langle p, (H+J+S)q^* \rangle.
\end{aligned}$$

From which together with the definition of adjoint operator we know that the result of the lemma is right.

Lemma 3.3

$$\left\{ \gamma \in C \left| \begin{array}{l} \sup\{\frac{2\lambda+\lambda_{h_0}}{|\gamma+2\lambda+\lambda_{h_0}|}, \frac{\theta}{|\gamma+\theta|}, \\ \frac{\lambda+\lambda_{h_2}+\mu}{|\gamma+\lambda+\lambda_{h_2}+\mu|}, \frac{\alpha}{|\gamma+\alpha|}, \\ \frac{\beta}{|\gamma+\beta|}\} < 1 \end{array} \right. \right\}$$

belongs to the resolvent set of $(A+U+E)^*$. Particularly, all points on the imaginary axis except zero belong to the resolvent set of $(A+U+E)^*$. In other words, all points on the imaginary axis except zero belong to the resolvent set of $A+U+E$.

Proof First we prove that $(\gamma I-H)^{-1}(J+S)$ and $(\gamma I-H)^{-1}$ exist and are bounded for some γ, then by using the perturbation theory we obtain the desired result. For any given $y^*\in X^*$, consider the equation

$$(\gamma I-H)q^*=(J+S)y^*,$$

that is

$$(\gamma+2\lambda+\lambda_{h_0})q_0^*=2\lambda y_1^*+\lambda_{h_0}y_H^*(0), \quad (3.16)$$

$$(\gamma+\theta)q_1^*=\theta y_2^*, \quad (3.17)$$

$$(\gamma+\lambda+\lambda_{h_2}+\mu)q_2^*=\mu y_3^*+\lambda y_F^*(0) + \lambda_{h_2}y_H^*(0), \quad (3.18)$$

$$(\gamma+\theta)q_3^*=\theta y_0^*, \quad (3.19)$$

$$\frac{dq_F^*(x)}{dx}=(\gamma+\alpha)q_F^*(x)-\alpha y_3^*, \quad (3.20)$$

$$\frac{dq_H^*(x)}{dx}=(\gamma+\beta)q_H^*(x)-\beta y_0^*, \quad (3.21)$$

$$q_F^*(\infty)=q_H^*(\infty)=M. \quad (3.22)$$

By solving (3.20) and (3.21), we know

$$q_F^*(x)=a_F e^{(\gamma+\alpha)x} - e^{(\gamma+\alpha)x}\int_0^x \alpha y_3^* e^{-(\gamma+\alpha)\tau}d\tau, \quad (3.23)$$

$$q_H^*(x)=a_H e^{(\gamma+\beta)x} - e^{(\gamma+\beta)x}\int_0^x \beta y_0^* e^{-(\gamma+\beta)\tau}d\tau. \quad (3.24)$$

By multiplying $e^{-(\gamma+\alpha)x}$ and $e^{-(\gamma+\beta)x}$ to two side of (3.23) and (3.24), taking the limit $x\to\infty$ and using (3.22), it gives

$$a_F=\int_0^\infty \alpha y_3^* e^{-(\gamma+\alpha)\tau}d\tau, \quad (3.25)$$

$$a_H=\int_0^\infty \beta y_0^* e^{-(\gamma+\beta)\tau}d\tau. \quad (3.26)$$

Inserting (3.25) and (3.26) into (3.23) and (3.24), we calculate

$$q_F^*(x)=e^{(\gamma+\alpha)x}\int_x^\infty \alpha y_3^* e^{-(\gamma+\alpha)\tau}d\tau = \frac{\alpha}{\gamma+\alpha}y_3^*, \quad (3.27)$$

$$q_H^*(x)=\frac{\beta}{\gamma+\beta}y_0^*. \quad (3.28)$$

If we assume

$$|\gamma+2\lambda+\lambda_{h_2}|\neq 0, |\gamma+\lambda+\lambda_{h_2}+\mu|\neq 0,$$
$$|\gamma+\theta|\neq 0, Re\gamma+\alpha>0, Re\gamma+\beta>0,$$

then by (3.16) – (3.19), (3.27), (3.28), we estimate

$$|q_0^*|\leq \frac{1}{|\gamma+2\lambda+\lambda_{h_0}|}\{2\lambda|y_1^*| + \lambda_{h_0}\|y_H^*\|_{L^\infty[0,\infty)}\} \leq \frac{2\lambda+\lambda_{h_0}}{|\gamma+2\lambda+\lambda_{h_0}|}|||y^*|||, \quad (3.29)$$

$$|q_1^*|\leq \frac{\theta}{|\gamma+\theta|}|||y^*|||, \quad (3.30)$$

$$|q_2^*|\leq \frac{\lambda+\lambda_{h_2}+\mu}{|\gamma+\lambda+\lambda_{h_2}+\mu|}|||y^*|||, \quad (3.31)$$

$$|q_3^*|\leq \frac{\theta}{|\gamma+\theta|}|||y^*|||, \quad (3.32)$$

$$\|q_F^*\|_{L^\infty[0,\infty)}\leq \frac{\alpha}{|\gamma+\alpha|}|||y^*|||, \quad (3.33)$$

$$\|q_H^*\|_{L^\infty[0,\infty)}\leq \frac{\beta}{|\gamma+\beta|}|||y^*|||. \quad (3.34)$$

(3.29) – (3.34) mean

$$||| q^* |||= \sup\{\sup_{0\le i\le 3} | q_i^* |, \| q_F^* \|_{L^\infty[0,\infty)}, \| q_H^* \|_{L^\infty[0,\infty)}\}$$

$$\le \sup\{\frac{2\lambda+\lambda_{h_0}}{|\gamma+2\lambda+\lambda_{h_0}|}, \frac{\theta}{|\gamma+\theta|}, \frac{\alpha}{|\gamma+\alpha|}, \frac{\lambda+\lambda_{h_2}+\mu}{|\gamma+\lambda+\lambda_{h_2}+\mu|}, \frac{\beta}{|\gamma+\beta|}\} ||| y^* |||.$$

That is to say

$$\| (\gamma I - H)^{-1}(J+S) \| \le \sup\{\frac{2\lambda+\lambda_{h_0}}{|\gamma+2\lambda+\lambda_{h_0}|}, \frac{\theta}{|\gamma+\theta|}, \frac{\alpha}{|\gamma+\alpha|}, \frac{\lambda+\lambda_{h_2}+\mu}{|\gamma+\lambda+\lambda_{h_2}+\mu|}, \frac{\beta}{|\gamma+\beta|}\}. \quad (3.35)$$

In the same way, for any given $y^* \in X^*$, by considering the equation $(\gamma I - H)q^* = y^*$, we can prove the existence and boundness of $(\gamma I - H)^{-1}$, and get that

$$\| (\gamma I - H)^{-1} \| \le \sup\{\frac{1}{|\gamma+2\lambda+\lambda_{h_0}|}, \frac{1}{|\gamma+\theta|}, \frac{1}{\operatorname{Re}\gamma+\alpha}, \frac{1}{\operatorname{Re}\gamma+\beta}\}. \quad (3.36)$$

By combining

$$[\gamma I - (A+U+E)^*]^{-1} = [\gamma I - (H+J+S)]^{-1} = [I-(\gamma I - H)^{-1}(J+S)]^{-1}(\gamma I - H)^{-1}$$

with (3.35) and (3.36) we know that the $[\gamma I - (A+U+E)^*]^{-1}$ exists and is bounded when

$$\| (\gamma I - H)^{-1}(J+S) \| \le \sup\{\frac{2\lambda+\lambda_{h_0}}{|\gamma+2\lambda+\lambda_{h_0}|}, \frac{\theta}{|\gamma+\theta|}, \frac{\alpha}{|\gamma+\alpha|}, \frac{\lambda+\lambda_{h_2}+\mu}{|\gamma+\lambda+\lambda_{h_2}+\mu|}, \frac{\beta}{|\gamma+\beta|}\} < 1. \quad (3.37)$$

In other words, all γ satisfy (3.37) belong to the resolvent set of $(A+U+E)^*$.

In particular, if $\gamma = ia,\ a \in R \setminus \{0\}$, then all γ automatically satisfy (3.37). In fact,

$$\frac{2\lambda+\lambda_{h_0}}{\sqrt{a^2+(2\lambda+\lambda_{h_0})^2}} < 1, \quad \frac{\theta}{\sqrt{a^2+\theta^2}} < 1,$$

$$\frac{\lambda+\lambda_{h_2}+\mu}{\sqrt{a^2+(\lambda+\lambda_{h_2}+\mu)^2}} < 1,$$

$$\frac{\alpha}{\sqrt{a^2+\alpha^2}} < 1, \quad \frac{\beta}{\sqrt{a^2+\beta^2}} < 1.$$

The above inequalities mean that all points on the imaginary axis except zero belong to the resolvent set of $(A+U+E)^*$. From the relation between the spectrum $A+U+E$ and the spectrum $(A+U+E)^*$ we know that all points on the imaginary axis except zero belong to the resolvent set of $A+U+E$.

Lemma 3.4 0 is an eigenvalue of $(A+U+E)^*$ with geometric multiplicity one.

Proof Consider the equation

$$(A+U+E)^* q^* = 0.$$

That is

$$-(2\lambda+\lambda_{h_0})q_0^* + \lambda_{h_0} q_H^*(0) + 2\lambda q_1^* = 0, \quad (3.38)$$

$$-\theta q_1^* + \theta q_2^* = 0, \quad (3.39)$$

$$-(\lambda+\lambda_{h_2}+\mu)q_2^* + \lambda q_F^*(0)$$

$$+\lambda_{h_2} q_H^*(0) = 0, \qquad (3.40)$$

$$-\theta q_3^* + \theta q_0^* = 0, \qquad (3.41)$$

$$(\frac{d}{dx} - \alpha) q_F^*(x) + \alpha q_3^* = 0, \qquad (3.42)$$

$$(\frac{d}{dx} - \beta) q_H^*(x) + \beta q_0^* = 0, \qquad (3.43)$$

$$q_F^*(\infty) = q_H^*(\infty) = M. \qquad (3.44)$$

Through solving (3.42) and (3.43), and combining with (3.41), we have

$$q_F^*(x) = b_F e^{\alpha x} - e^{\alpha x} \int_0^x \alpha q_3^* e^{-\alpha \tau} d\tau$$

$$= (b_F - q_0^*) e^{\alpha x} + q_0^*, \qquad (3.45)$$

$$q_H^*(x) = (b_H - q_0^*) e^{\beta x} + q_0^*. \qquad (3.46)$$

Combining (3.45), (3.46) with (3.44), we get

$$b_F = b_H = q_0^*.$$

Which imply

$$q_F^*(x) = q_H^*(x) = q_0^*. \qquad (3.47)$$

From (3.38) – (3.41) and (3.47), we have

$$q_1^* = q_2^* = q_3^* = q_F^*(x) = q_H^*(x) = q_0^*.$$

Which shows that the eigenvector space corresponding to 0 is one dimensional. Therefore the geometric multiplicity of 0 is one.

From Lemma 3.1, Lemma 3.4 and Lemma 27 in [6] we know that the algebraic multiplicity of 0 is one. Which together with Theorem 2.1, Lemma 3.3 and the Theorem 14 in Gupur et al. [6], we conclude the desired result in this paper:

Theorem 3.1 The time-dependent solution of the system (2.1) converges strongly to its steady-state solution as time tends infinity, that is,

$$\lim_{t \to \infty} p(x,t) = p(x),$$

where $p(x)$ is the eigenvector corresponding to 0 in Lemma 3.1.

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interests regarding the publication of this paper.

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