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Meir-Keeler Type n-Tuplet Fixed Point Theorems in Partially Ordered Metric Spaces

Manoj UGHADE[1,♠], R. D. DAHERIYA[2]

[1]*Department of Mathematics, RKDF Institute of Science & Technology, Bhopal, India-462026*
[2]*Department of Mathematics, J.H .Govt. P.G. College Betul , India-460001*

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ABSTRACT

An n-tuplet fixed point is a generalization of the well-known concept of "coupled fixed point and tripled fixed point". The intent of this paper is to introduce the concept of mixed strict monotone property and generalize Meir-Keeler contraction for mapping $F: X^n \to X$, where n is an arbitrary positive integer. Also establish an n-tuplet fixed point theorem for mappings $F: X^n \to X$ under a generalized Meir-Keeler contraction in the setting of partially ordered metric spaces. Related examples are also given to support our main results.Our results are the generalizations of the results of B. Samet[8] and Hassen et al. [15]. Also as application, some results of integral type are given.

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Key words: n-tuplet fixed point; Meir-Keeler type contraction; mixed strict monotone property; partially ordered metric space.

1. INTRODUCTION

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and is a very active field of research at present. Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tool for proving theexistence and eventually the uniqueness of the solutions to various mathematical models(integral and partial differential equations, variational inequalities).

Basic topological properties of an ordered set like convergence were introduced by Wolk [1]. In 1981, Monjardet [2] considered metrics on partially ordered sets. Ran and Reurings [3] proved and analog of Banach Contraction mapping principle in partially ordered metric spaces. Bhaskar and Lakshmikantham in [4] introduced the concept of coupled fixed point of a mapping $F: X \times X \to X$ and investigated the existence and uniqueness of a coupled fixed point theorem in partially ordered complete metric spaces. Lakshmikantham and Ciric in [5] defined mixed g-monotone property and coupled coincidence point in partially ordered metric spaces. Samet [8] defined generalized Meir-Keeler type functions and proved some coupled fixed point theorems under a generalized Meir-Keeler contractive condition.

Following this trend, Berinde and Borcut [6] introduced the concept of triple fixed point and established some triple fixed point theorems in partially ordered metric spaces. The notion of fixed point of order $N \geq 3$ was first introduced by Samet and Vetro [7]. Hassen Aydi et al [15]

♠Correspondingauthor, e-mail: manojhelpyou@gmail.com

defined generalized Meir-Keeler type functions and established some tripled fixed point theorems under a generalized Meir-Keeler contractive condition.

Karapinar [9] used the notion of quadruple fixed point and obtained some quadruple fixed point theorems in partially ordered metric spaces. Later, various results on quadruple fixed point have been obtained; see, for example, [9-14].Very recently, Muzeyyen Erturk and VatanKarakaya [17] introduced the concept of n-tuplet fixed point and established some n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces.

The purpose of this paper is three fold which can be described as follows.

1. We introduce the concept of mixed strict monotone property and generalize Meir-Keeler type contraction and establish an n-tuplet fixed point theorem for continuous mapping $F: X^n \to X$ under a generalization of Meir-Keeler type contraction in the setting of partially ordered metric spaces. We introduce some examples to illustrate the effectiveness of our results. Also these theorems are still valid for F, not necessarily continuous, assuming $(X, d, \leq)$ is regular.
2. We prove the uniqueness of an n-tuplet fixed point for such mappings in this setup. Related examples are also given to support our main results.
3. Also applications, some results of integral type are given.

2. PRELIMINARIES

Here we recall some basic definitions and results.

The triple $(X, d, \leq)$ is called a partially ordered metric space if $(X, \leq)$ is a partially ordered set and (X, d) is a metric space. Further if (X, d) is complete metric space, then the triple $(X, d, \leq)$ is called a partially ordered complete metric space. Throughout the manuscript, we assume that $X \neq \emptyset$ and

$$X^n = \underbrace{X \times X \times \ldots\ldots\ldots . \times X}_{n-\text{times}} \tag{1}$$

Then the mapping $\rho_n: X^n \times X^n \to [0, \infty)$ such that

$$\rho_n(x, y) := \max\{d(x_1, y_1), d(x_2, y_2), \ldots . d(x_n, y_n)\} \tag{2}$$

forms a metric on X^n, where $x = (x_1, x_2, \ldots . . x_n)$ and $y = (y_1, y_2, \ldots . . y_n) \in X^n$.

Definition 1(see [15]):Let $(X, d, \leq)$ be a partially ordered metric space. A mapping $F: X^3 \to X$ is said to be generalized Meir-Keeler type contraction if for any $\varepsilon > 0$, there exist a $\delta(\varepsilon) > 0$ such that for all $x, y, z, u, v, r \in X$ with $x \leq u, y \geq v$ and $z \leq r$,

$$\varepsilon \leq \max\{d(x, u), d(y, v), d(z, r)\} < \varepsilon + \delta(\varepsilon)$$

$$\Rightarrow d(F(x, y, z), F(u, v, h)) < \varepsilon. \tag{3}$$

Remark 2(see [15]):It is immediate to show that if $F: X^3 \to X$ is a generalized Meir-Keeler type contraction, then it is immediate to show that for all $x, y, z, u, v, r \in X$ with $x \leq u, y \geq v, z \leq r$ or $x < u, y \geq v, z \leq r$,

$$d(F(x, y, z), F(u, v, r)) < max\{d(x, u), d(y, v), d(z, r)\} \tag{4}$$

Definition 3(see [17]):Let $(X, \leq)$ be a partially ordered set and $F: X^n \to X$ be a mapping. We say that F has the mixed monotone property if $F(x_1, x_2, x_3, x_4, \ldots\ldots, x_n)$ is monotone non-decreasing in its odd argument and it is monotone non-increasing in its even argument. That is, for any $x_1, x_2, x_3, x_4, \ldots\ldots, x_n \in X$;

$$y_1, z_1 \in X, \ y_1 \leq z_1$$
$$\Rightarrow F(y_1, x_2, x_3, x_4, \ldots\ldots, x_n) \leq F(z_1, x_2, x_3, x_4, \ldots\ldots, x_n)$$
$$y_2, z_2 \in X, \ y_2 \leq z_2$$
$$\Rightarrow F(x_1, y_2, x_3, x_4, \ldots\ldots, x_n) \geq F(x_1, z_2, x_3, x_4, \ldots\ldots, x_n)$$
$$y_3, z_3 \in X, \ y_3 \leq z_3$$
$$\Rightarrow F(x_1, x_2, y_3, x_4, \ldots\ldots, x_n) \geq F(x_1, x_2, z_3, x_4, \ldots\ldots, x_n)$$

$$
\begin{gathered}
\vdots \\
y_n, z_n \in X,\ \ y_n \le z_n \\
\Rightarrow\ F(x_1, x_2, x_3, x_4, \dots\dots, y_n) \le F(x_1, x_2, x_3, x_4, \dots\dots, z_n)\ \text{(if n is odd)} \\
y_n, z_n \in X,\ \ y_n \le z_n \\
\Rightarrow\ F(x_1, x_2, x_3, x_4, \dots\dots, y_n) \ge F(x_1, x_2, x_3, x_4, \dots\dots, z_n)\ \text{(if n is even)}
\end{gathered} \tag{5}
$$

Definition 4(see [17]): Let X be a nonempty set and $F: X^n \to X$ a given mapping. An element $(x_1, x_2, x_3, x_4, \dots\dots, x_n) \in X^n$ is called a n-tuplet fixed point of F if

$$
\begin{aligned}
&F(x_1, x_2, x_3, x_4, \dots\dots, x_n) = x_1, \\
&F(x_2, x_3, x_4, \dots\dots, x_n, x_1) = x_2, \\
&F(x_3, x_4, \dots\dots, x_n, x_1, x_2) = x_3, \\
&\vdots \\
&F(x_n, x_1, x_2, x_3, \dots\dots, x_{n-1}) = x_n.
\end{aligned} \tag{6}
$$

3. NEW DEFINITIONS

We start this section with the following definitions.

Definition 5 Let$(X, \le)$ be a partially ordered set and $F: X^n \to X$ be a mapping. We say that F has the mixed strict monotone property if $F(x_1, x_2, x_3, \dots\dots, x_n)$ is strict monotone non-decreasing in its odd argument and it is strict monotone non-increasing in its even argument. That is, for any$x_1, x_2, x_3, \dots\dots, x_n \in X$;

$$
\begin{aligned}
&y_1, z_1 \in X,\ \ y_1 < z_1 \ \Rightarrow\ F(y_1, x_2, x_3, x_4, \dots\dots, x_n) < F(z_1, x_2, x_3, x_4, \dots\dots, x_n) \\
&y_2, z_2 \in X,\ \ y_2 < z_2 \ \Rightarrow\ F(x_1, y_2, x_3, x_4, \dots\dots, x_n) > F(x_1, z_2, x_3, x_4, \dots\dots, x_n) \\
&y_3, z_3 \in X,\ \ y_3 < z_3 \ \Rightarrow\ F(x_1, x_2, y_3, x_4, \dots\dots, x_n) < F(x_1, x_2, z_3, x_4, \dots\dots, x_n) \\
&\vdots \\
&y_n, z_n \in X,\ \ y_n < z_n \Rightarrow\ F(x_1, x_2, x_3, x_4, \dots\dots, y_n) < F(x_1, x_2, x_3, x_4, \dots\dots, z_n)\ \text{(if n is odd)} \\
&y_n, z_n \in X,\ \ y_n < z_n \Rightarrow\ F(x_1, x_2, x_3, x_4, \dots\dots, y_n) > F(x_1, x_2, x_3, x_4, \dots\dots, z_n)\ \text{(if n is even)}
\end{aligned} \tag{7}
$$

Definition 6 Let $(X, d, \le)$ be a partially ordered metric space. A mapping $F: X^n \to X$ is said to be generalized Meir-Keeler type contraction if for any $\varepsilon > 0$, there exist a $\delta(\varepsilon) > 0$ such that for all $x_1, x_2, x_3, x_4, \dots\dots, x_n, y_1, y_2, y_3, y_4, \dots\dots, y_n \in X$ with $x_1 \le y_1, x_2 \ge y_2, \dots, x_n \le y_n$(if n is odd),$x_n \ge y_n$ (if n is even),

$$
\begin{aligned}
\varepsilon \le \max\{d(x_1, y_1), d(x_2, y_2), \dots\dots d(x_n, y_n)\} < \varepsilon + \delta \\
\Rightarrow d(F(x_1, x_2, x_3, x_4, \dots\dots, x_n), F(y_1, y_2, y_3, y_4, \dots\dots, y_n)) < \varepsilon.
\end{aligned} \tag{8}
$$

Remark 7 It is immediate to show that if $F: X^n \to X$ is a generalized Meir-Keeler type contraction, then it is immediate to show that for all $x_1, x_2, x_3, x_4 \dots\dots, x_n, y_1, y_2, y_3, y_4, \dots\dots, y_n \in X$ with $x_1 \le y_1, x_2 \ge y_2, \dots, x_n \le y_n$(if n is odd),$x_n \ge y_n$ (if n is even) or $x_1 < y_1, x_2 \ge y_2, \dots, x_n < y_n$(if n is odd),$x_n \ge y_n$ (if n is even)

$$
\begin{aligned}
d\big(F(x_1, x_2, x_3, \dots\dots, x_n), F(y_1, y_2, y_3, \dots\dots, y_n)\big) \\
< max\{d(x_1, y_1), d(x_2, y_2), \dots\dots\dots\dots d(x_n, y_n)\}
\end{aligned} \tag{9}
$$

We introduce some notions used in the sequel. Let $\widetilde{F} : X^n \to X^n$ such that for $a_1, a_2, a_3, \dots\dots, a_n \in X$.

$$
\widetilde{F}(a_1, a_2, a_3, \dots, a_n) = \begin{pmatrix} F(a_1, a_2, a_3, \dots\dots, a_n), F(a_2, a_3, \dots\dots, a_n, a_1), \\ F(a_3, \dots\dots, a_n, a_1, a_2) \dots\dots, F(a_n, a_1, a_2, \dots\dots, a_{n-1}) \end{pmatrix} \tag{10}
$$

Let $x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n \in X$ be such that,

$$
\begin{aligned}
&x_0^1 < F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\
&x_0^2 \ge F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\
&\vdots \\
&x_0^n < F(x_0^n, x_0^1, x_0^2, , \dots\dots, x_0^{n-1})\ \text{(if n is odd)} \\
&x_0^n \ge F(x_0^n, x_0^1, x_0^2, , \dots\dots, x_0^{n-1})\ \text{(if n is even)}
\end{aligned} \tag{11}
$$

We'll consider the n sequences $(x_k^1), (x_k^2), (x_k^3), \dots\dots, (x_k^n)$ as follows:

$$\underbrace{\begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \\ \vdots \\ x_k^n \end{bmatrix}}_{A_k} = \underbrace{\begin{bmatrix} F(x_{k-1}^1, x_{k-1}^2, x_{k-1}^3, \dots\dots, x_{k-1}^n) \\ F(x_{k-1}^2, x_{k-1}^3, \dots\dots, x_{k-1}^n, x_{k-1}^1) \\ F(x_{k-1}^3, \dots\dots, x_{k-1}^n, x_{k-1}^1, x_{k-1}^2) \\ \vdots \\ F(x_{k-1}^n, x_{k-1}^1, x_{k-1}^2, \dots\dots, x_{k-1}^{n-1}) \end{bmatrix}}_{\widetilde{F}(A_{k-1})} = \underbrace{\begin{bmatrix} F^k(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) \\ F^k(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1) \\ F^k(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ F^k(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) \end{bmatrix}}_{\widetilde{F^k}(A_0)} \forall\, k = 1,2,3,.. \qquad (12)$$

4. KEY PROPOSITION

Now, we state and prove the first result of this paper which will be useful in proving our main result.

Proposition 8 Let $(X, d, \leq)$ be a partially ordered metric space and $F: X^n \to X$ be a mapping such that the following hypotheses hold:

I. F has the mixed strict monotone property,
II. F is a generalized Meir-Keeler type contraction,
III. There exist $x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n \in X$ be as in (11),
IV. $\exists\, (x^1, x^2, x^3, \dots\dots, x^n), (y^1, y^2, y^3, \dots\dots, y^n) \in X^n$ such that
$$x^1 < y^1, x^2 \geq y^2, \dots., x^n < y^n (\text{if n is odd}), x^n \geq y^n \ (\text{if n is even})$$

Then $$\rho_n\left(\widetilde{F^k}(x^1, x^2, x^3, \dots\dots, x^n), \widetilde{F^k}(y^1, y^2, y^3, \dots\dots, y^n)\right) \to 0 \text{as } k \to +\infty. \qquad (13)$$

Proof: Set

$$(x^1, x^2, x^3, \dots\dots, x^n) = (x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) \text{ and}$$
$$(y^1, y^2, y^3, \dots\dots, y^n) = (y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n).$$

We assert that$\forall\, k \in \mathbb{N}$

$$\begin{aligned} &x_k^1 = F^k(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) < F^k(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n) = y_k^1, \\ &x_k^2 = F^k(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1) > F^k(y_0^2, y_0^3, \dots\dots, y_0^n, y_0^1) = y_k^2, \\ &x_k^3 = F^k(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2) < F^k(y_0^3, \dots\dots, y_0^n, y_0^1, y_0^2) = y_k^3, \\ &\vdots \\ &x_k^n = F^k(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) < F^k(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) = y_k^n (\text{if n is odd}), \\ &x_k^n = F^k(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) > F^k(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) = y_k^n (\text{if n is even}), \ \text{with} F = F^1. \end{aligned} \qquad (14)$$

For this, we will use the mathematical induction. Because F has the mixed strict monotone property, together with the assumption that

$$x^1 < y^1,\ x^2 \geq y^2,\ x^3 < y^3, \dots\dots., x^n < y^n (\text{if n is odd}), x^n \geq y^n\ (\text{if n is even}).$$

We obtain;

$$\begin{aligned} x_1^1 = F(x^1, x^2, x^3, \dots\dots, x^n) = F^1(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) &< F^1(y_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) \\ &\Rightarrow F^1(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) < F^1(y_0^1, y_0^2, x_0^3, \dots\dots, x_0^n) \\ &\Rightarrow F^1(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) < F^1(y_0^1, y_0^2, y_0^3, \dots\dots, x_0^n) \\ &\vdots \\ &\vdots \\ &\Rightarrow F^1(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) < F^1(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n) \\ &\Rightarrow F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) < F\big((y^1, y^2, y^3, \dots\dots, y^n)\big) = y_1^1. \end{aligned} \qquad (15)$$

Analogously, we get;

$$\begin{aligned} &x_1^2 = F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1) > F(y_0^2, y_0^3, \dots\dots, x_0^n, y_0^1) = y_1^2, \\ &x_1^3 = F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2) < F(y_0^3, \dots\dots, x_0^n, y_0^1, y_0^2) = y_1^3, \\ &\vdots \\ &\vdots \\ &x_1^n = F(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) < F(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) = y_1^n, (\text{if n is odd}) \\ &x_1^n = F(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) > F(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) = y_1^n, (\text{if n is even}). \end{aligned} \qquad (16)$$

Thus, the inequalities in (14) hold for $k = 1$.By using the same arguments, we show that the inequalities in (14) hold also for $n = 2$. In fact,

$$\begin{aligned}
x_2^1 &= F^2(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n)\\
&= F^1(x_1^1, x_1^2, x_1^3, \dots\dots, x_1^n) = F(x_1^1, x_1^2, x_1^3, \dots\dots, x_1^n)\\
&= F\begin{pmatrix} F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \dots\dots, F(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) \end{pmatrix}\\
&< F\begin{pmatrix} F(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n), F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \dots\dots, F(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) \end{pmatrix}\\
&< F\begin{pmatrix} F(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n), F(y_0^2, y_0^3, \dots\dots, x_0^n, y_0^1), \\ F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \dots\dots, F(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) \end{pmatrix}\\
&< F\begin{pmatrix} F(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n), F(y_0^2, y_0^3, \dots\dots, x_0^n, y_0^1), \\ F(y_0^3, \dots\dots, x_0^n, y_0^1, y_0^2), \dots\dots, F(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) \end{pmatrix}\\
&\vdots\\
&< F\begin{pmatrix} F(y_0^1, y_0^2, y_0^3, \dots\dots., y_0^n), F(y_0^2, y_0^3, \dots\dots, x_0^n, y_0^1), \\ F(y_0^3, \dots\dots, x_0^n, y_0^1, y_0^2), \dots\dots, F(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) \end{pmatrix}\\
&< F(F(y_1^1, y_1^2, y_1^3, \dots\dots, y_1^n) = F^2(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n) = y_2^1. \qquad (17)
\end{aligned}$$

Analogously, we get;

$$\begin{aligned}
x_2^2 &= F^2(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1) > F^2(y_0^2, y_0^3, \dots\dots, y_0^n, y_0^1) = y_2^2,\\
x_2^3 &= F^2(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2) < F^2(y_0^3, \dots\dots, y_0^n, y_0^1, y_0^2) = y_2^3,\\
&\vdots \qquad (18)\\
x_2^n &= F^2(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) < F^2(y_0^2, y_0^3, \dots\dots, y_0^n, y_0^1) = y_2^n, \text{(if n is odd)}\\
x_2^n &= F^2(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) > F^2(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) = y_2^n, \text{(if n is even)}.
\end{aligned}$$

Thus, our claim is true for $k = 2$. Now, suppose that the inequalities in (14) hold for $k = m$. In this case,

$$\begin{aligned}
x_m^1 &= F^m(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) < F^m(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n) = y_m^1\\
x_m^2 &= F^m(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1) > F^m(y_0^2, y_0^3, \dots\dots, y_0^n, y_0^1) = y_m^2,\\
x_m^3 &= F^m(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2) < F^m(y_0^3, \dots\dots, y_0^n, y_0^1, y_0^2) = y_m^3,\\
&\vdots \qquad (19)\\
x_m^n &= F^m(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) < F^m(y_0^2, y_0^3, \dots\dots, y_0^n, y_0^1) = y_m^n, \text{(if n is odd)}\\
x_m^n &= F^m(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) > F^m(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) = y_m^n, \text{(if n is even)}.
\end{aligned}$$

Now, we must show that the inequalities in (14) hold for $k = m + 1$. If we consider (12) and mixed strict monotone property of F together with (19), we have;

$$\begin{aligned}
x_{m+1}^1 &= F^{m+1}(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n)\\
&= F^1(x_m^1, x_m^2, x_m^3, \dots\dots, x_m^n) = F(x_m^1, x_m^2, x_m^3, \dots\dots, x_m^n)\\
&= F\begin{pmatrix} F^m(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), F^m(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ F^m(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \dots\dots, F^m(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) \end{pmatrix}\\
&= F\begin{pmatrix} F(x_{m-1}^1, x_{m-1}^2, x_{m-1}^3, \dots\dots, x_{m-1}^n), F(x_{m-1}^2, x_{m-1}^3, \dots\dots, x_{m-1}^n, x_{m-1}^1), \\ F(x_{m-1}^3, \dots\dots, x_{m-1}^n, x_{m-1}^1, x_{m-1}^2), \dots\dots., F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1}) \end{pmatrix}\\
&< F\begin{pmatrix} F(y_{m-1}^1, y_{m-1}^2, y_{m-1}^3, \dots\dots, y_{m-1}^n), F(x_{m-1}^2, x_{m-1}^3, \dots\dots, x_{m-1}^n, x_{m-1}^1), \\ F(x_{m-1}^3, \dots\dots, x_{m-1}^n, x_{m-1}^1, x_{m-1}^2), \dots\dots, F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots.\dots, x_{m-1}^{n-1}) \end{pmatrix}\\
&< F\begin{pmatrix} F(y_{m-1}^1, y_{m-1}^2, y_{m-1}^3, \dots\dots, y_{m-1}^n), F(y_{m-1}^2, y_{m-1}^3, \dots\dots, y_{m-1}^n, y_{m-1}^1), \\ F(x_{m-1}^3, \dots\dots, x_{m-1}^n, x_{m-1}^1, x_{m-1}^2), \dots\dots, F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1}) \end{pmatrix}\\
&< F\begin{pmatrix} F(y_{m-1}^1, y_{m-1}^2, y_{m-1}^3, \dots\dots, y_{m-1}^n), F(y_{m-1}^2, y_{m-1}^3, \dots\dots, y_{m-1}^n, y_{m-1}^1), \\ F(y_{m-1}^3, \dots\dots, y_{m-1}^n, y_{m-1}^1, y_{m-1}^2), \dots\dots, F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1}) \end{pmatrix}\\
&\vdots\\
&< F\begin{pmatrix} F(y_{m-1}^1, y_{m-1}^2, y_{m-1}^3, \dots\dots, y_{m-1}^n), F(y_{m-1}^2, y_{m-1}^3, \dots\dots, y_{m-1}^n, y_{m-1}^1), \\ F(y_{m-1}^3, \dots\dots, y_{m-1}^n, y_{m-1}^1, y_{m-1}^2), \dots\dots, F(y_{m-1}^n, y_{m-1}^1, y_{m-1}^2, \dots\dots, y_{m-1}^{n-1}) \end{pmatrix}\\
&= F(y_m^1, y_m^2, y_m^3, \dots\dots, y_m^n) = F^1(y_m^1, y_m^2, y_m^3, \dots\dots, y_m^n)\\
&= F^{m+1}(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n) = y_{m+1}^1 \qquad (20)
\end{aligned}$$

Analogously, we get;

$$\begin{aligned}
x_{m+1}^2 &= F^{m+1}(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1) > F^{m+1}(y_0^2, y_0^3, \dots\dots, y_0^n, y_0^1) = y_{m+1}^2,\\
x_{m+1}^3 &= F^{m+1}(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2) < F^{m+1}(y_0^3, \dots\dots, y_0^n, y_0^1, y_0^2) = y_{m+1}^3,\\
&\vdots\\
x_{m+1}^n &= F^{m+1}(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) < F^{m+1}(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) = y_{m+1}^n, \text{(if n is odd)}\\
x_{m+1}^n &= F^{m+1}(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) > F^{m+1}(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1}) = y_{m+1}^n, \text{(if n is even)}.
\end{aligned} \tag{21}$$

Thus, (14) is satisfied for all $k \geq 1$.By using Remark 7 and (14), we have;

$$\begin{aligned}
d(x_{k+2}^1, y_{k+2}^1) &= d\left(F^{k+2}(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), F^{k+2}(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n)\right)\\
&= d\left(F(x_{k+1}^1, x_{k+1}^2, x_{k+1}^3, \dots\dots, x_{k+1}^n), F(y_{k+1}^1, y_{k+1}^2, y_{k+1}^3, \dots\dots, y_{k+1}^n)\right)\\
&< max\{d(x_{k+1}^1, y_{k+1}^1), d(x_{k+1}^2, y_{k+1}^2), d(x_{k+1}^3, y_{k+1}^3), \dots\dots, d(x_{k+1}^n, y_{k+1}^n)\}\\
d(x_{k+2}^2, y_{k+2}^2) &= d\left(F^{k+2}(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), F^{k+2}(y_0^2, y_0^3, \dots\dots, y_0^n, y_0^1)\right)\\
&= d\left(F(x_{k+1}^2, x_{k+1}^3, \dots\dots, x_{k+1}^n, x_{k+1}^1), F(y_{k+1}^2, y_{k+1}^3, \dots\dots, y_{k+1}^n, y_{k+1}^1)\right)\\
&< max\{d(x_{k+1}^2, y_{k+1}^2), d(x_{k+1}^3, y_{k+1}^3), \dots\dots, d(x_{k+1}^n, y_{k+1}^n), d(x_{k+1}^1, y_{k+1}^1)\}\\
d(x_{k+2}^3, y_{k+2}^3) &= d\left(F^{k+2}(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), F^{k+2}(y_0^3, \dots\dots, y_0^n, y_0^1, y_0^2)\right)\\
&= d\left(F(x_{k+1}^3, \dots\dots, x_{k+1}^n, x_{k+1}^1, x_{k+1}^2), F(y_{k+1}^3, \dots\dots, y_{k+1}^n, y_{k+1}^1, y_{k+1}^2)\right)\\
&< max\{d(x_{k+1}^3, y_{k+1}^3), \dots\dots, d(x_{k+1}^n, y_{k+1}^n), d(x_{k+1}^1, y_{k+1}^1), d(x_{k+1}^2, y_{k+1}^2)\}\\
&\vdots\\
&\vdots\\
d(x_{k+2}^n, y_{k+2}^n) &= d\left(F^{k+2}(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}), F^{k+2}(y_0^n, y_0^1, y_0^2, \dots\dots, y_0^{n-1})\right)\\
&= d\left(F(x_{k+1}^n, x_{k+1}^1, x_{k+1}^2, \dots\dots, x_{k+1}^{n-1}), F(y_{k+1}^n, y_{k+1}^1, y_{k+1}^2, \dots\dots, y_{k+1}^{n-1})\right)\\
&< max\{d(x_{k+1}^n, y_{k+1}^n), d(x_{k+1}^1, y_{k+1}^1), d(x_{k+1}^2, y_{k+1}^2), \dots\dots, d(x_{k+1}^{n-1}, y_{k+1}^{n-1})\}
\end{aligned} \tag{22}$$

By using (22), we get;

$$\begin{aligned}
&\max\{d(x_{k+2}^1, y_{k+2}^1), d(x_{k+2}^2, y_{k+2}^2), d(x_{k+2}^3, y_{k+2}^3), \dots\dots, d(x_{k+2}^n, y_{k+2}^n)\}\\
&\qquad < max\{d(x_{k+1}^1, y_{k+1}^1), d(x_{k+1}^2, y_{k+1}^2), d(x_{k+1}^3, y_{k+1}^3), \dots\dots, d(x_{k+1}^n, y_{k+1}^n)\}
\end{aligned} \tag{23}$$

For the simplicity, we define

$$\Delta_{k+1} := \max\{d(x_{k+1}^1, y_{k+1}^1), d(x_{k+1}^2, y_{k+1}^2), d(x_{k+1}^3, y_{k+1}^3), \dots\dots, d(x_{k+1}^n, y_{k+1}^n)\} \tag{24}$$

From (23) and (24), we get;

$$\Delta_{k+2} < \Delta_{k+1}, \forall\, k \in \mathbb{N} \tag{25}$$

If we denote $B_k = (y_k^1, y_k^2, y_k^3, \dots\dots, y_k^n)$, then by the definition of ρ_n and (25), we have

$$\rho_n(A_{k+2}, B_{k+2}) < \rho_n(A_{k+1}, B_{k+1}) \quad \forall\, k \in \mathbb{N} \tag{26}$$

Consequently, the sequence $\{t_k\} = \{\rho_n(A_k, B_k)\}$ is decreasing. Hence $\{t_k\}$converges to, say $\varepsilon \geq 0$.Clearly, if $\varepsilon = 0$, we have finished. Suppose, on the contrary, $\varepsilon > 0$.Thus, there exists $r \in \mathbb{N}$such that, for any $k \geq r$,

$$\varepsilon \leq t_k = \rho_n(A_k, B_k) < \varepsilon + \delta(\varepsilon) \tag{27}$$

In particular, for $k = r$ we have,

$$\varepsilon \leq t_r = \rho_n(A_r, B_r) < \varepsilon + \delta(\varepsilon) \tag{28}$$

That is,

$$\varepsilon \leq \Delta_r < \varepsilon + \delta(\varepsilon) \tag{29}$$

It follows from (14) and the hypothesis (ii) that

$$d\big(F(x_r^1, x_r^2, x_r^3, \dots\dots, x_r^n), F(y_r^1, y_r^2, y_r^3, \dots\dots, y_r^n)\big) < \varepsilon \tag{30}$$

This is equivalent to

$$d(x_{r+1}^1, y_{r+1}^1) < \varepsilon \tag{31}$$

Analogously, we can get

$$\begin{aligned} &d(x_{r+1}^2, y_{r+1}^2) < \varepsilon, \\ &d(x_{r+1}^3, y_{r+1}^3) < \varepsilon, \\ &\vdots \\ &d(x_{r+1}^n, y_{r+1}^n) < \varepsilon, \end{aligned} \tag{32}$$

That is;

$$\Delta_{r+1} < \varepsilon \tag{33}$$

Thus,

$$t_{r+1} = \rho_n(A_{r+1}, B_{r+1}) < \varepsilon$$

This is a contradiction. Thus, $\varepsilon = 0$, that is,

$$\lim_{k\to+\infty} t_k = \lim_{k\to+\infty} \rho_n(A_k, B_k) = 0. \tag{34}$$

We conclude that,

$$\begin{aligned} &\rho_n\left(F^k(x^1, x^2, x^3, \dots\dots, x^n), F^k(y^1, y^2, y^3, \dots\dots, y^n)\right) \\ &\quad = \rho_n\left(F^k(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), F^k(y_0^1, y_0^2, y_0^3, \dots\dots, y_0^n)\right) \\ &\quad = \rho_n\left(\widetilde{F^k}(A_0), \widetilde{F^k}(B_0)\right) = \rho_n\left(\tilde{F}(A_{k-1}), \tilde{F}(B_{k-1})\right) \\ &\quad = \rho_n(A_k, B_k) \to 0 \ \text{ as } k \to +\infty. \end{aligned} \tag{35}$$

Remark 9 The previous proposition remains true if in (iii), we change the assumption:$\exists\ (x_1, x_2, x_3, x_4, \dots\dots, x_n), (y_1, y_2, y_3, y_4, \dots\dots, y_n) \in X^n$ such that $x_1 < y_1, x_2 \geq y_2, \dots., x_n < y_n$ (if n is odd),$x_n \geq y_n$(if n is even) with the following,

$$\begin{gathered} \exists\ (x_1, x_2, x_3, x_4, \dots\dots, x_n), (y_1, y_2, y_3, y_4, \dots\dots, y_n) \in X^n \text{ such that} \\ x_1 < y_1, x_2 > y_2, \dots., x_n < y_n (\text{if n is odd}), x_n > y_n \ \ (\text{if n is even}). \end{gathered} \tag{36}$$

5. EXISTENCE OF N-TUPLET FIXED POINT

Following theorem is the main result this of paper.

Theorem10 Let $(X, d, \leq)$ be a partially ordered metric space. (X, d) be a complete metric space and $F: X^n \to X$ be a mapping such that the following hypotheses hold:

I. F is continuous,
II. F has the mixed strict monotone property,
III. F is a generalized Meir-Keeler type contraction,
IV. There exist $x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n \in X$ be as in (11).

Then F has a n-tuplet fixed point, that is, there exists $x^1, x^2, x^3, \dots\dots, x^n \in X$ such that

$$\begin{aligned} &F(x^1, x^2, x^3, \dots\dots, x^n) = x^1, \\ &F(x^2, x^3, \dots\dots, x^n, x^1) = x^2, \\ &F(x^3, \dots\dots, x^n, x^1, x^2) = x^3, \\ &\vdots \\ &F(x^n, x^1, x^2, \dots\dots, x^{n-1}) = x^n. \end{aligned}$$

Proof: Let $x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n \in X$be as in (11). We construct the sequences $(x_k^1), (x_k^2), (x_k^3), \dots\dots, (x_k^n)$ according to (12). We claim that;

$$x_{k-1}^1 < x_k^1,\ x_{k-1}^2 > x_k^2, \dots\dots\dots\ x_{k-1}^n < x_k^n, \quad (\text{if n is odd})\ \ x_{k-1}^n > x_k^n \quad (\text{if n is even}) \tag{37}$$

Indeed, we will use the mathematical induction to prove (37). The inequalities in (37) hold for $k = 1$ because of (11), that is, we have

$$\begin{aligned}
&x_0^1 < F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n) = x_1^1,\\
&x_0^2 \geq F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1) = x_1^2,\\
&\vdots\\
&x_0^n < F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}) = x_1^n \text{ (if n is odd)}\\
&x_0^n \geq F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}) = x_1^n \text{ (if n is even)}
\end{aligned}$$

Now, suppose that the inequalities in (37) hold for $k = m$. In that case,

$$x_{m-1}^1 < x_m^1,\ x_{m-1}^2 \geq x_m^2, \dots\dots\dots x_{m-1}^n < x_m^n, \quad \text{(if n is odd)}, x_{m-1}^n \geq x_m^n \quad \text{(if n is even)} \tag{38}$$

If we consider (12) and mixed strict monotone property of F together with (38), we have

$$\begin{aligned}
x_m^1 &= F^m(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n)\\
&= F^1(x_{m-1}^1, x_{m-1}^2, x_{m-1}^3, \dots\dots, x_{m-1}^n)\\
&= F(x_{m-1}^1, x_{m-1}^2, x_{m-1}^3, \dots\dots, x_{m-1}^n)\\
&< F(x_m^1, x_{m-1}^2, x_{m-1}^3, \dots\dots, x_{m-1}^n)\\
&\leq F(x_m^1, x_m^2, x_{m-1}^3, \dots\dots, x_{m-1}^n)\\
&< F(x_m^1, x_m^2, x_m^3, \dots\dots, x_{m-1}^n)\\
&\ \vdots\\
&< F(x_m^1, x_m^2, x_m^3, \dots\dots, x_m^n) = x_{m+1}^1\\
x_m^2 &= F^m(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1)\\
&= F^1(x_{m-1}^2, x_{m-1}^3, \dots\dots, x_{m-1}^n, x_{m-1}^1)\\
&= F(x_{m-1}^2, x_{m-1}^3, \dots\dots, x_{m-1}^n, x_{m-1}^1)\\
&\geq F(x_m^2, x_{m-1}^3, \dots\dots, x_{m-1}^n, x_{m-1}^1)\\
&> F(x_m^2, x_m^3, \dots\dots, x_{m-1}^n, x_{m-1}^1)\\
&\ \vdots\\
&> F(x_m^2, x_m^3, \dots\dots, x_m^n, x_{m-1}^1)\\
&\geq F(x_m^2, x_m^3, \dots\dots, x_m^n, x_m^1) = x_{m+1}^2\\
&\ \vdots\\
&\ \vdots\\
x_m^n &= F^m(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1})\\
&= F^1(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1})\\
&= F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1})\\
&< F(x_m^n, x_{m-1}^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1})\\
&\leq F(x_m^n, x_m^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1})\\
&< F(x_m^n, x_m^1, x_m^2, \dots\dots, x_{m-1}^{n-1})\\
&\ \vdots\\
&< F(x_m^n, x_m^1, x_m^2, \dots\dots, x_m^{n-1}) = x_{m+1}^n \text{(if n is odd)}\\
x_m^n &= F^m(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1})\\
&= F^1(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1})\\
&= F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1})\\
&\geq F(x_m^n, x_{m-1}^1, x_{m-1}^2, \dots\ \dots, x_{m-1}^{n-1})\\
&> F(x_m^n, x_m^1, x_{m-1}^2, \dots\dots, x_{m-1}^{n-1})\\
&\ \vdots\\
&> F(x_m^n, x_m^1, x_m^2, \dots\dots, x_m^{n-1})\\
&\geq F(x_m^n, x_m^1, x_m^2, \dots\dots, x_m^{n-1}) = x_{m+1}^n \text{(if n is even)}
\end{aligned} \tag{39}$$

Thus, (38) is satisfied for all $k \geq 1$. So, we have

$$\begin{aligned}
&\dots\dots\dots\dots\dots.. > x_k^1 > x_{k-1}^1 > \cdots \dots > x_3^1 > x_2^1 > x_1^1 > x_0^1,\\
&\dots\dots\dots\dots\dots.. < x_k^2 < x_{k-1}^2 < \cdots \dots < x_3^2 < x_2^2 < x_1^2 < x_0^2,\\
&\vdots\\
&\vdots\\
&\dots\dots\dots\dots\dots.. > x_k^n > x_{k-1}^n > \cdots \dots > x_3^n > x_2^n > x_1^n > x_0^n, \text{(if n is odd)}
\end{aligned} \tag{40}$$

$$\dots\dots\dots\dots\dots.. < x_k^n < x_{k-1}^n < \cdots \dots < x_3^n < x_2^n < x_1^n < x_0^n, \text{(if n is even)}.$$

Now putting $(x^1, x^2, x^3, \dots\dots, x^n) = A_0, (y^1, y^2, y^3, \dots\dots, y^n) = A_1$ and Proposition 8, we get;

$$\rho_n\left(\widetilde{F^k}(A_0), \widetilde{F^k}(A_1)\right) \to 0 \text{ as } k \to +\infty. \tag{41}$$

This is equivalent to

$$\rho_n(A_k, A_{k+1}) \to 0 \text{ as } k \to +\infty. \tag{42}$$

Take an arbitrary $\varepsilon > 0$. it follows from (42) that there exists $r \in \mathbb{N}$ such that

$$\rho_n(A_r, A_{r+1}) < \delta(\varepsilon) \tag{43}$$

Without loss of generality, assume that$\delta(\varepsilon) \le \varepsilon$ and define the following set:

$$\prod := \left\{\begin{matrix} A = (x^1, x^2, x^3, \dots\dots, x^n) \in X^n | \rho_n\left(\tilde{F}(A_r), \tilde{F}(A)\right) < \varepsilon + \delta(\varepsilon) \\ \text{and } x^1 > x_r^1, x^2 \le x_r^2, x^3 > x_r^3, \dots\dots, x^n > x_r^n \text{(if n is odd)}, \\ x^n \le x_r^n, \text{(if n is even)} \end{matrix}\right\} \tag{44}$$

We claim:

$$\tilde{F}(A) \in \prod, \ \forall\ A \in \prod \tag{45}$$

Take $A \in \prod$, then by (43) and the triangle inequality, we have

$$\begin{aligned}
\rho_n\left((A_r), \tilde{F}(A)\right) &= \max\left\{\begin{matrix} d\big(x_r^1, F(x^1, x^2, x^3, \dots\dots, x^n)\big), d\big(x_r^2, F(x^2, x^3, \dots\dots, x^n, x^1)\big), \\ d\big(x_r^3, F(x^3, \dots, x^n, x^1, x^2)\big), \dots\dots, d\big(x_r^n, F(x^n, x^1, x^2, \dots, x^{n-1})\big) \end{matrix}\right\} \\
&\le \max\left\{\begin{matrix} d(x_r^1, x_{r+1}^1) + d\big(x_{r+1}^1, F(x^1, x^2, x^3, \dots\dots, x^n)\big), \\ d(x_r^2, x_{r+1}^2) + d\big(x_{r+1}^2, F(x^2, x^3, \dots\dots, x^n, x^1)\big), \\ \vdots \\ d(x_r^n, x_{r+1}^n) + d\big(x_{r+1}^n, F(x^n, x^1, x^2, \dots\dots, x^{n-1})\big) \end{matrix}\right\} \\
&\le \max\left\{\begin{matrix} d(x_r^1, x_{r+1}^1) + d\big(F(x_r^1, x_r^2, x_r^3, \dots\dots, x_r^n), F(x^1, x^2, x^3, \dots\dots, x^n)\big), \\ d(x_r^2, x_{r+1}^2) + d\big(F(x_r^2, x_r^3, \dots\dots, x_r^n, x_r^1), F(x^2, x^3, \dots\dots, x^n, x^1)\big), \\ \vdots \\ d(x_r^n, x_{r+1}^n) + d\big(F(x_r^n, x_r^1, x_r^2, \dots\dots, x_r^{n-1}), F(x^n, x^1, x^2, \dots\dots, x^{n-1})\big) \end{matrix}\right\} \\
&\le \max\{d(x_r^1, x_{r+1}^1), d(x_r^2, x_{r+1}^2), \dots\dots, d(x_r^n, x_{r+1}^n)\} \\
&\quad + \max\left\{\begin{matrix} d\big(F(x_r^1, x_r^2, x_r^3, \dots\dots, x_r^n), F(x^1, x^2, x^3, \dots\dots, x^n)\big), \\ d\big(F(x_r^2, x_r^3, \dots\dots, x_r^n, x_r^1), F(x^2, x^3, \dots\dots, x^n, x^1)\big), \\ \dots\dots\dots\dots\dots\dots, \\ d\big(F(x_r^n, x_r^1, x_r^2, \dots\dots, x_r^{n-1}), F(x^n, x^1, x^2, \dots\dots, x^{n-1})\big) \end{matrix}\right\} \\
&= \rho_n(A_r, A_{r+1}) + \rho_n\left(\tilde{F}(A_r), \tilde{F}(A)\right) \\
&< \delta(\varepsilon) + \rho_n\left(\tilde{F}(A_r), \tilde{F}(A)\right) (46)
\end{aligned}$$

We consider the following two cases:

Case. 1$\rho_n(A_r, A) \le \varepsilon$. By Remark 7 and the definition of $\prod$, inequality (46) turns into

$$\begin{aligned}\rho_n\left((A_r), \tilde{F}(A)\right) &< \delta(\varepsilon) + \rho_n\left(\tilde{F}(A_r), \tilde{F}(A)\right) \\ &< \delta(\varepsilon) + \rho_n(A_r, A) < \delta(\varepsilon) + \varepsilon \end{aligned} \tag{47}$$

Case. 2 $\varepsilon < \rho_n(A_r, A) < \delta(\varepsilon) + \varepsilon$, that is

$$\varepsilon < max\{d(x_r^1, x^1), d(x_r^2, x^2), d(x_r^3, x^3), \dots\dots, d(x_r^n, x^n)\} < \delta(\varepsilon) + \varepsilon \tag{48}$$

Since$x^1 > x_r^1, x^2 \le x_r^2, x^3 > x_r^3, \ldots\ldots, x^n > x_r^n$(if n is odd)$, x^n \le x_r^n$, (if n is even), then by hypothesis (iii) we have

$$
\begin{aligned}
&d\big(F(x^1,x^2,x^3,\ldots\ldots,x^n),F(x_r^1,x_r^2,x_r^3,\ldots\ldots,x_r^n)\big) < \varepsilon\\
&d\big(F(x^2,x^3,\ldots\ldots,x^n,x^1),F(x_r^2,x_r^3,\ldots\ldots,x_r^n,x_r^1)\big) < \varepsilon\\
&d\big(F(x^3,\ldots\ldots,x^n,x^1,x^2),F(x_r^3,\ldots\ldots,x_r^n,x_r^1,x_r^2)\big) < \varepsilon\\
&\vdots\\
&d\big(F(x^n,x^1,x^2,\ldots\ldots,x^{n-1}),F(x_r^n,x_r^1,x_r^2,\ldots\ldots,x_r^{n-1})\big) < \varepsilon
\end{aligned}
\tag{49}
$$

Hence, from (46) and (50), we get;

$$\rho_n\left((A_k),\tilde{F}(A)\right) < \delta(\varepsilon) + \varepsilon \tag{50}$$

On other hand, using (II), one can easily check that

$$
\begin{aligned}
&F(x^1,x^2,x^3,\ldots\ldots,x^n) > x_r^1\\
&F(x^2,x^3,\ldots\ldots,x^n,x^1) \le x_r^2\\
&F(x^3,\ldots\ldots,x^n,x^1,x^2) > x_r^3\\
&\vdots\\
&F(x^n,x^1,x^2,\ldots\ldots,x^{n-1}) > x_r^n \text{ (if n is odd)}\\
&F(x^n,x^1,x^2,\ldots\ldots,x^{n-1}) \le x_r^n \text{ (if n is even)}
\end{aligned}
\tag{51}
$$

Hence, we conclude that (45) holds. By (44), we have that $A_{k+1} \in \prod$ and so by (45), we get

$$
\begin{aligned}
A_{k+1} \in \prod &\Rightarrow \tilde{F}(A_{k+1}) = A_{k+2} \in \prod\\
&\Rightarrow \tilde{F}(A_{k+2}) = A_{k+3} \in \prod\\
&\vdots\\
&\vdots\\
&\Rightarrow \tilde{F}(A_r) \in \prod \ \forall\, r \ge k
\end{aligned}
\tag{52}
$$

Then, for all $k, m > r$ we have,

$$\rho_n(A_k, A_m) \le \rho_n(A_k, A_r) + \rho_n(A_r, A_m) < 2\big(\varepsilon + \delta(\varepsilon)\big) \le 4\varepsilon \tag{53}$$

Hence$(x_k^1),(x_k^2),(x_k^3),\ldots\ldots,(x_k^n)$ are Cauchy sequences in the metric space(X^n,ρ_n). Since(X,d) is a complete,(X^n,ρ_n) is also complete. Then, there exists a point $(z^1,z^2,z^3,\ldots\ldots,z^n) \in X^n$.

$$d(x_k^1,z^1), d(x_k^2,z^2), d(x_k^3,z^3),\ldots\ldots, d(x_k^n,z^n) \to 0 \text{ as } k \to +\infty$$

That is,

$$\lim_{k\to+\infty} x_k^1 = z^1, \lim_{k\to+\infty} x_k^2 = z^2, \lim_{k\to+\infty} x_k^3 = z^3, \ldots\ldots\ldots\ldots, \lim_{k\to+\infty} x_k^n = z^n \tag{54}$$

By the continuity of F, we obtain

$$
\begin{aligned}
z^1 &= \lim_{k\to+\infty} x_{k+1}^1 = \lim_{k\to+\infty} F\big(x_k^1,x_k^2,x_k^3,\ldots\ldots,x_k^n\big)\\
&= F\left(\lim_{k\to+\infty} x_k^1, \lim_{k\to+\infty} x_k^2, \lim_{k\to+\infty} x_k^3,\ldots\ldots, \lim_{k\to+\infty} x_k^n\right)\\
&= F(z^1,z^2,z^3,\ldots\ldots,z^n)
\end{aligned}
$$

Analogously

$$
\begin{aligned}
z^2 &= \lim_{k\to+\infty} x_{k+1}^2 = \lim_{k\to+\infty} F\big(x_k^2,x_k^3,\ldots\ldots,x_k^n,x_k^1\big)\\
&= F\left(\lim_{k\to+\infty} x_k^2, \lim_{k\to+\infty} x_k^3,\ldots\ldots, \lim_{k\to+\infty} x_k^n, \lim_{k\to+\infty} x_k^1\right)\\
&= F(z^2,z^3,\ldots\ldots,z^n,z^1)\\
&\vdots\\
&\vdots\\
z^n &= \lim_{k\to+\infty} x_{k+1}^n = \lim_{k\to+\infty} F\big(x_k^n,x_k^1,x_k^2,\ldots\ldots,x_k^{n-1}\big)\\
&= F\left(\lim_{k\to+\infty} x_k^n, \lim_{k\to+\infty} x_k^1, \lim_{k\to+\infty} x_k^2,\ldots\ldots, \lim_{k\to+\infty} x_k^{n-1}\right)\\
&= F(z^n,z^1,z^2,\ldots\ldots,z^{n-1})
\end{aligned}
$$

Thus, we have

$$z^1 = F(z^1,z^2,z^3,\ldots\ldots,z^n),$$

$$\begin{aligned}
z^2 &= F(z^2, z^3, \ldots\ldots, z^n, z^1),\\
z^3 &= F(z^3, \ldots\ldots, z^n, z^1, z^2),\\
&\vdots\\
z^n &= (z^n, z^1, z^2, \ldots\ldots, z^{n-1})
\end{aligned}$$

We have proved that F has an n-tuplet fixed point. This finishes the proof of theorem 10.

Remark 11 Theorems remains true if we replace (IV) with one of the following statements. There exists $x_0, y_0, z_0, w_0 \in X$ such that,

$$1.\begin{cases}
x_0^1 \le F(x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n),\\
x_0^2 > F(x_0^2, x_0^3, \ldots\ldots, x_0^n, x_0^1),\\
\vdots\\
\vdots\\
x_0^n \le F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is odd)}\\
x_0^n > F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is even)}
\end{cases}$$

$$2.\begin{cases}
x_0^1 < F(x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n),\\
x_0^2 > F(x_0^2, x_0^3, \ldots\ldots, x_0^n, x_0^1),\\
\vdots\\
\vdots\\
x_0^n < F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is odd)}\\
x_0^n > F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is even)}
\end{cases}$$

$$3.1\begin{cases}
x_0^1 \le F(x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n),\\
x_0^2 > F(x_0^2, x_0^3, \ldots\ldots, x_0^n, x_0^1),\\
x_0^3 < F(x_0^3, \ldots\ldots, x_0^n, x_0^1, x_0^2),\\
\vdots\\
x_0^n < F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is odd)}\\
x_0^n > F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is even)}
\end{cases}$$

$$3.2\begin{cases}
x_0^1 < F(x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n),\\
x_0^2 \ge F(x_0^2, x_0^3, \ldots\ldots, x_0^n, x_0^1),\\
x_0^3 < F(x_0^3, \ldots\ldots, x_0^n, x_0^1, x_0^2),\\
\vdots\\
x_0^n < F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is odd)}\\
x_0^n > F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is even)}
\end{cases}$$

$$3.3\begin{cases}
x_0^1 < F(x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n),\\
x_0^2 > F(x_0^2, x_0^3, \ldots\ldots, x_0^n, x_0^1),\\
x_0^3 \le F(x_0^3, \ldots\ldots, x_0^n, x_0^1, x_0^2),\\
\vdots\\
x_0^n < F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is odd)}\\
x_0^n > F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is even)}
\end{cases}$$

$\vdots$

$\vdots$

$$3.n\begin{cases}
x_0^1 < F(x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n),\\
x_0^2 > F(x_0^2, x_0^3, \ldots\ldots, x_0^n, x_0^1),\\
x_0^3 < F(x_0^3, \ldots\ldots, x_0^n, x_0^1, x_0^2),\\
\vdots\\
x_0^n \le F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is odd)}\\
x_0^n \ge F(x_0^n, x_0^1, x_0^2,, \ldots\ldots, x_0^{n-1}), \text{(if n is even)}
\end{cases}$$

$$4.1\begin{cases} x_0^1 \le F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\ x_0^2 \ge F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ x_0^3 < F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \\ \vdots \\ x_0^n < F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is odd)} \\ x_0^n > F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is even)} \end{cases}$$

$$4.2\begin{cases} x_0^1 < F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\ x_0^2 > F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ x_0^3 \le F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \\ x_0^4 \ge F(x_0^4, \dots\dots, x_0^n, x_0^1, x_0^2, x_0^3), \\ \vdots \\ \vdots \\ \begin{cases} x_0^{n-1} < F(x_0^{n-1}, x_0^n, x_0^1,, \dots\dots, x_0^{n-2}), \text{(if n is odd)}, \\ x_0^{n-1} > F(x_0^{n-1}, x_0^n, x_0^1,, \dots\dots, x_0^{n-2}), \text{(if n is even)}, \end{cases} \\ \begin{cases} x_0^n < F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is odd)}, \\ x_0^n > F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is even)}, \end{cases} \end{cases}$$

$\vdots$

$\vdots$

$$4.n\begin{cases} x_0^1 < F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\ x_0^2 > F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ x_0^3 < F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \\ x_0^4 > F(x_0^4, \dots\dots, x_0^n, x_0^1, x_0^2, x_0^3), \\ \vdots \\ \vdots \\ \begin{cases} x_0^{n-1} \le F(x_0^{n-1}, x_0^n, x_0^1,, \dots\dots, x_0^{n-2}), \text{(if n is odd)}, \\ x_0^{n-1} \ge F(x_0^{n-1}, x_0^n, x_0^1,, \dots\dots, x_0^{n-2}), \text{(if n is even)}, \end{cases} \\ \begin{cases} x_0^n \le F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is odd)}, \\ x_0^n \ge F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is even)}, \end{cases} \end{cases}$$

$$5.1\begin{cases} x_0^1 < F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\ x_0^2 \ge F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ x_0^3 \le F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \\ \vdots \\ x_0^n \le F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is odd)} \\ x_0^n \ge F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is even)} \end{cases}$$

$$5.2\begin{cases} x_0^1 \le F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\ x_0^2 > F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ x_0^3 \le F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \\ \vdots \\ x_0^n \le F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is odd)} \\ x_0^n \ge F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is even)} \end{cases}$$

$$5.3\begin{cases} x_0^1 \le F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\ x_0^2 \ge F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ x_0^3 < F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \\ \vdots \\ x_0^n \le F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is odd)} \\ x_0^n \ge F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is even)} \end{cases}$$

$\vdots$

$\vdots$

$$5.n\begin{cases} x_0^1 \le F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\ x_0^2 \ge F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ x_0^3 \le F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \\ \vdots \\ x_0^n < F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is odd)} \\ x_0^n > F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is even)} \end{cases}$$

$\vdots$

$$n.\begin{cases} x_0^1 \le F(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), \\ x_0^2 \ge F(x_0^2, x_0^3, \dots\dots, x_0^n, x_0^1), \\ x_0^3 \le F(x_0^3, \dots\dots, x_0^n, x_0^1, x_0^2), \\ x_0^4 \ge F(x_0^4, \dots\dots, x_0^n, x_0^1, x_0^2, x_0^3), \\ \vdots \\ \vdots \\ \begin{cases} x_0^{n-1} \le F(x_0^{n-1}, x_0^n, x_0^1,, \dots\dots, x_0^{n-2}), \text{(if n is odd)}, \\ x_0^{n-1} \ge F(x_0^{n-1}, x_0^n, x_0^1,, \dots\dots, x_0^{n-2}), \text{(if n is even)}, \end{cases} \\ \begin{cases} x_0^n < F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is odd)}, \\ x_0^n > F(x_0^n, x_0^1, x_0^2,, \dots\dots, x_0^{n-1}), \text{(if n is even)}, \end{cases} \end{cases}$$

6. EXAMPLES

We state some examples showing that our result is effective.

Example 12 (k-tuplet fixed point) Let $X = R$ be complete metric space under usual metric and natural ordering $\le$ of real numbers.Define the mapping $F: X^k \to X$ (where k is an arbitrary positive odd integer) be defined by

$$F(x^1, x^2, x^3, \dots\dots, x^k) = \frac{x^1 - x^2 + x^3 - \cdots \dots + x^k}{k+1}$$

It is clear that F is continuous and has the mixed strict monotone property. Moreover taking $x_0^1 = x_0^2 = \cdots \dots . = x_0^k = 0$, the condition (IV) of Theorem 10 holds for k . On the other hand, for $x^1, x^2, \dots, x^k, y^1, y^2, \dots ., y^k \in X$ with $x^1 < y^1, x^2 \ge y^2, \dots\dots, x^k < y^k$, we have

$$\begin{aligned} \varepsilon &\le \max\{d(x^1,y^1), d(x^2,y^2), \dots\dots\dots, d(x^k,y^k)\} \\ &= \max\{x^1 - y^1, x^2 - y^2, \dots\dots\dots.., x^k - y^k\} < \varepsilon + \delta(\varepsilon) . \end{aligned}$$

$$\begin{aligned} d\left(F(x^1, x^2, x^3, \dots\dots, x^k), (y^1, y^2, y^3, \dots\dots, y^k)\right) &= \left|\frac{x^1 - x^2 + x^3 - \cdots \dots + x^k}{k+1} - \frac{y^1 - y^2 + y^3 - \cdots \dots + y^k}{k+1}\right| \\ &\le \left|\frac{x^1 - y^1}{k+1}\right| + \left|\frac{x^2 - y^2}{k+1}\right| + \cdots \dots + \left|\frac{x^k - y^k}{k+1}\right| \end{aligned}$$

Then condition (III) of theorem 10 hold for $\delta(\varepsilon) < \frac{1}{k}\varepsilon$. That is, F is a generalized Meir-Keeler type contraction. It is clear that all conditions of Theorem 10 are satisfied (without order) and in view of Theorem 10, the k-tuplet point $(0,0, \dots .,0) \in X^k$ is the desired k-tuplet fixed point of F.

Example 13 (5-tuplet fixed point) Let $X = R$ with the metric $d(x, y) = |x - y|$ and the usual ordering. Clearly $(X, d, \le)$ is a partially ordered complete metric space. Let$F: X^5 \to X$ be defined by

$$F(z^1, z^2, z^3, z^4, z^5) = \frac{2z^1 - z^2 + 2z^3 - z^4 + 2z^5 + 4}{24}$$

for all $z^1, z^2, z^3, z^4, z^5 \in X$. Obviously, F is continuous and has the mixed strict monotone property. Moreover taking $z^1 = 0, z^2 = \frac{1}{5}, z^3 = 0, z^4 = \frac{1}{5}, z^5 = 0$, the condition (IV) of Theorem 10 holds for $n = 5$ and $(z_0^1, z_0^2, z_0^3, z_0^4, z_0^5) = \left(0, \frac{1}{5}, 0, \frac{1}{5}, 0\right)$. On the other hand, for $z^1, z^2, z^3, z^4, z^5, w^1, w^2, w^3, w^4, w^5 \in X$ with $z^1 < w^1, z^2 \ge w^2, z^3 < w^3, z^4 \ge w^4, z^5 < w^5$, we have

$$\varepsilon \le \max\{d(z^1, w^1), d(z^2, w^2), d(z^3, w^3), d(z^4, w^4), d(z^5, w^5)\}$$

$$= \max\{z^1 - w^1, z^2 - w^2, z^3 - w^3, z^4 - w^4, z^5 - w^5\} < \varepsilon + \delta(\varepsilon) .$$

$$d\Big(F(z^1, z^2, z^3, z^4, z^5), (w^1, w^2, w^3, w^4, w^5)\Big)$$
$$= \left|\frac{2z^1 - z^2 + 2z^3 - z^4 + 2z^5 + 4}{24} - \frac{2w^1 - w^2 + 2w^3 - w^4 + 2w^5 + 4}{24}\right|$$
$$\le \left|\frac{z^1 - w^1}{12}\right| + \left|\frac{w^2 - z^2}{24}\right| + \left|\frac{z^3 - w^3}{12}\right| + \left|\frac{w^4 - z^4}{24}\right| + \left|\frac{z^5 - w^5}{12}\right|$$

Then condition (III) of theorem 10 hold for $\delta(\varepsilon) < 2\varepsilon$. That is, F is a generalized Meir-Keeler type contraction. It is clear that all the hypotheses of Theorem 10 are satisfies and in view of Theorem 10, $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$ is the desired 5-tuplet fixed point of F. In the following theorem, we omit the continuity hypothesis of F. We need the following definition.

Definition 14 Let $(X, d, \le)$ be a partially ordered metric space. We say that $(X, d, \le)$ is regular if the following conditions hold in X:

a) If strict non-decreasing sequence $x_k \to x$in X , then $x_k < x \; \forall \, k$.
b) If strict non-increasing sequence$y_k \to y$in X, then $y_k > y \; \forall \, k$.

Theorem: 15 Let $(X, d, \le)$ be a partially ordered complete metric space. Suppose $(X, d, \le)$ is regular.Assume that $F: X^n \to X$ satisfies the following hypothesis.

1. F has the mixed strict monotone property,
2. F is a generalized Meir-Keeler type contraction,
3. There exist $x_0^1, x_0^2, x_0^3, \dots \dots, x_0^n \in X$ be as in (11).

Then F has an n-tuplet fixed point**.**

Proof: Following the same lines of the proof of the Theorem 10, we remain to prove that F has an n-tuplet fixed point in X, that is, there exist $(z^1, z^2, z^3, \dots \dots, z^n) \in X^n$ such that

$$\begin{aligned} z^1 &= F(z^1, z^2, z^3, \dots \dots, z^n), \\ z^2 &= F(z^2, z^3, \dots \dots, z^n, z^1), \\ z^3 &= F(z^3, \dots \dots, z^n, z^1, z^2), \\ &\vdots \\ z^n &= (z^n, z^1, z^2, \dots \dots, z^{n-1}) \end{aligned}$$

To this aim, suppose $(X, d, \le)$ is regular. If n is odd since $(x_k^1), (x_k^3), \dots \dots, (x_k^n)$ are strict non-decreasing and $(x_k^2), (x_k^4), \dots \dots, (x_k^{n-1})$ are strict non-increasing, if n is even since $(x_k^1), (x_k^3), \dots \dots, (x_k^{n-1})$ are strict non-decreasing and $(x_k^2), (x_k^4), \dots \dots, (x_k^n)$ are strict non-increasing. Since

$$\begin{aligned} x_k^1 &= F(x_{k-1}^1, x_{k-1}^2, x_{k-1}^3, \dots \dots, x_{k-1}^n) = F^k(x_0^1, x_0^2, x_0^3, \dots \dots, x_0^n) \to z^1, \\ x_k^2 &= F(x_{k-1}^2, x_{k-1}^3, \dots \dots, x_{k-1}^n, x_{k-1}^1) = F^k(x_0^2, x_0^3, \dots \dots, x_0^n, x_0^1) \to z^2, \\ x_k^3 &= F(x_{k-1}^3, \dots \dots, x_{k-1}^n, x_{k-1}^1, x_{k-1}^2) = F^k(x_0^3, \dots \dots, x_0^n, x_0^1, x_0^2) \to z^3, \\ &\vdots \\ &\vdots \\ x_k^n &= F(x_{k-1}^n, x_{k-1}^1, x_{k-1}^2, \dots \dots, x_{k-1}^{n-1}) = F^k(x_0^n, x_0^1, x_0^2, \dots \dots, x_0^{n-1}) \to z^n \end{aligned}$$

For any given $\epsilon > 0$, there exists $k_1, k_2, k_3 \dots \dots, k_n \in \mathbb{N}$ such that

$$\begin{aligned} d(x_{r_1}^1, z^1) &= d(F^{r_1}(x_0^1, x_0^2, x_0^3, \dots \dots, x_0^n), z^1) < \epsilon, \\ d(x_{r_2}^2, z^2) &= d(F^{r_2}(x_0^2, x_0^3, \dots \dots, x_0^n, x_0^1), z^2) < \epsilon, \\ d(x_{r_3}^3, z^3) &= d(F^{r_3}(x_0^3, \dots \dots, x_0^n, x_0^1, x_0^2), z^3) < \epsilon, \\ &\vdots \\ &\vdots \\ d(x_{r_n}^n, z^n) &= d(F^{r_n}(x_0^n, x_0^1, x_0^2, \dots \dots, x_0^{n-1}), z^n) < \epsilon. \end{aligned} \tag{54}$$

for all $r_1 \ge k_1, r_2 \ge k_2, r_3 \ge k_3, \dots \dots, r_n \ge k_n$.Now taking $k = \max_{1 \le i \le n}\{k_i\}$ and using Remark 7with the assumption,

$$\begin{aligned} x_k^1 &= F^k(x_0^1, x_0^2, x_0^3, \dots \dots, x_0^n) < z^1, \\ x_k^2 &= F^k(x_0^2, x_0^3, \dots \dots, x_0^n, x_0^1) > z^2, \\ x_k^3 &= F^k(x_0^3, \dots \dots, x_0^n, x_0^1, x_0^2) < z^3, \\ &\vdots \end{aligned} \tag{55}$$

$$\vdots$$
$$x_k^n = F^k(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) < z^n \text{ (if n is odd)},$$
$$x_k^n = F^k(x_0^n, x_0^1, x_0^2, \dots\dots, x_0^{n-1}) > z^n \text{ (if n is even)}.$$

From (9), (12) and (55), we get

$$\begin{aligned} d\big(z^1, F(z^1, z^2, z^3, \dots\dots, z^n)\big) &\le d\big(z^1, x_{k+1}^1\big) + d\left(x_{k+1}^1, F(z^1, z^2, z^3, \dots\dots, z^n)\right) \\ &= \big(z^1, x_{k+1}^1\big) + d\left(F^{k+1}(x_0^1, x_0^2, x_0^3, \dots\dots, x_0^n), F(z^1, z^2, z^3, \dots\dots, z^n)\right) \\ &= \big(z^1, x_{k+1}^1\big) + d\left(F(x_k^1, x_k^2, x_k^3, \dots\dots, x_k^n), F(z^1, z^2, z^3, \dots\dots, z^n)\right) \\ &= \big(z^1, x_{k+1}^1\big) + \max\{d\big(x_k^1, z^1\big), d\big(x_k^2, z^2\big), d\big(x_k^3, z^3\big), \dots\dots, d(x_k^n, z^n)\} \\ &< \varepsilon + max\{\varepsilon, \varepsilon, \varepsilon, \dots., \varepsilon\} = 2\varepsilon \end{aligned} \tag{56}$$

Analogously, we get that

$$\begin{gathered} d\big(z^2, F(z^2, z^3, \dots\dots, z^n, z^1)\big) \le 2\varepsilon \\ d\big(z^3, F(z^3, \dots\dots, z^n, z^1, z^2)\big) \le 2\varepsilon \\ \vdots \\ d\big(z^n, (z^n, z^1, z^2, \dots\dots, z^{n-1})\big) \le 2\varepsilon \end{gathered} \tag{57}$$

This yield that

$$\begin{gathered} z^1 = F(z^1, z^2, z^3, \dots\dots, z^n), \\ z^2 = F(z^2, z^3, \dots\dots, z^n, z^1), \\ z^3 = F(z^3, \dots\dots, z^n, z^1, z^2), \\ \vdots \\ z^n = F(z^n, z^1, z^2, \dots\dots, z^{n-1}). \end{gathered}$$

So $(z^1, z^2, z^3, \dots\dots, z^n)$ is an n-tuplet fixed point of F.

7. UNIQUENESS OF N-TUPLET FIXED POINT

In this section, we shall prove the uniqueness of n-tuplet fixed point. For a product X^n of a partial ordered set $(X, \le)$, we define a partial ordering in the following way: For all$(u^1, u^2, u^3, \dots\dots, u^n), (v^1, v^2, v^3, \dots\dots, v^n) \in X^n$,

$$\begin{gathered} (u^1, u^2, u^3, \dots\dots, u^n) \le (v^1, v^2, v^3, \dots\dots, v^n) \\ \Leftrightarrow u^1 \le v^1, \quad u^2 \le v^2, \quad \dots\dots\dots\dots, \quad u^n \le v^n \text{(if n is odd)}, u^n \ge v^n \text{ (if n is even)}. \end{gathered}$$

We say that $(u^1, u^2, u^3, \dots\dots, u^n)$ and $(v^1, v^2, v^3, \dots\dots, v^n)$ are comparable if $(u^1, u^2, u^3, \dots\dots, u^n) \le (v^1, v^2, v^3, \dots\dots, v^n)$ or $(v^1, v^2, v^3, \dots\dots, v^n) \le (u^1, u^2, u^3, \dots\dots, u^n)$. Also, we say that $(u^1, u^2, u^3, \dots\dots, u^n)$ is equal to $(v^1, v^2, v^3, \dots\dots, v^n)$ if and only if $u^1 = v^1, u^2 = v^2, = \cdots\dots\dots, = u^n = v^n$.

Theorem 16 In addition to the hypotheses of the Theorem 10, assume that for all$(u^1, u^2, u^3, \dots\dots, u^n), (v^1, v^2, v^3, \dots\dots, v^n) \in X^n$, there exists $(w^1, w^2, w^3, \dots\dots, w^n) \in X^4$. such that

$$(F(w^1, w^2, w^3, .., w^n), F(w^2, w^3, \dots, w^n, w^1), \dots\dots, F(w^n, w^1, w^2, \dots, w^{n-1})) \tag{58}$$

is comparable to

$$(F(u^1, u^2, u^3, \dots\dots, u^n), F(u^2, u^3, \dots\dots, u^n, u^1), \dots\dots, F(u^n, u^1, u^2, \dots\dots, u^{n-1})) \tag{59}$$

and

$$(F(v^1, v^2, v^3, \dots\dots, v^n), F(v^2, v^3, \dots\dots, v^n, v^1), \dots\dots, F(v^n, v^1, v^2, \dots\dots, v^{n-1})). \tag{60}$$

Then F has a unique n-tuplet fixed point.

Proof: The set of n-tuplet fixed point of F is nonempty due to Theorem 10. Assume now, $A = (u^1, u^2, u^3, \dots\dots, u^n)$and $A^\star = (v^1, v^2, v^3, \dots\dots, v^n) \in X^n$ are two n-tuplet fixed point of F, that is,

$$\begin{gathered} u^1 = F(u^1, u^2, u^3, \dots\dots, u^n), \\ u^2 = F(u^2, u^3, \dots\dots, u^n, u^1), \\ u^3 = F(u^3, \dots\dots, u^n, u^1, u^2), \\ \vdots \\ u^n = F(u^n, u^1, u^2, \dots\dots, u^{n-1}). \end{gathered}$$

and

$$\begin{gathered} v^1 = F(v^1, v^2, v^3, \dots\dots, v^n), \\ v^2 = F(v^2, v^3, \dots\dots, v^n, v^1), \\ v^3 = F(v^3, \dots\dots, v^n, v^1, v^2), \\ \vdots \\ v^n = F(v^n, v^1, v^2, \dots\dots, v^{n-1}). \end{gathered}$$

We shall show that$A = A^\star$. We distinguish the following two cases.

Case: 1. If$(u^1, u^2, u^3, \dots\dots, u^n)$ is comparable to $(v^1, v^2, v^3, \dots\dots, v^n)$with respect to the ordering in X^n. where,

$$\begin{aligned} &lim_{k\to+\infty} F^k(u_0^1, u_0^2, u_0^3, \dots\dots, u_0^n) = u^1, \\ &lim_{k\to+\infty} F^k(u_0^2, u_0^3, \dots\dots, u_0^n, u_0^1) = u^1, \\ &\vdots \\ &lim_{k\to+\infty} F^k(u_0^n, u_0^1, u_0^2, \dots\dots, u_0^{n-1}) = u^n \end{aligned} \tag{61}$$

Without loss of generality, we may assume that,

$$\begin{aligned} &u^1 = F(u^1, u^2, u^3, \dots\dots, u^n) < F(v^1, v^2, v^3, \dots\dots, v^n) = v^1, \\ &u^2 = F(u^2, u^3, \dots\dots, u^n, u^1) \geq F(v^2, v^3, \dots\dots, v^n, v^1) = v^2, \\ &\vdots \\ &u^n = F(u^n, u^1, u^2, \dots\dots, u^{n-1}) < F(v^n, v^1, v^2, \dots\dots, v^{n-1}) = v^n, (if\ n\ is\ odd) \\ &u^n = F(u^n, u^1, u^2, \dots\dots, u^{n-1}) \geq F(v^n, v^1, v^2, \dots\dots, v^{n-1}) = v^n, (if\ n\ is\ even). \end{aligned} \tag{62}$$

By the definition of ρ_n and Remark 7, we have

$$\begin{aligned} \rho_n(A, A^\star) &= \rho_n\big((u^1, u^2, u^3, \dots\dots, u^n), (v^1, v^2, v^3, \dots\dots, v^n)\big) \\ &= max\{d(u^1, v^1), d(u^2, v^2), d(u^3, v^3), \dots\dots, d(u^n, v^n)\} \\ &= max\left\{\begin{array}{c} d\big(F(u^1, u^2, u^3, \dots\dots, u^n), F(v^1, v^2, v^3, \dots\dots, v^n)\big), \\ d\big(F(u^2, u^3, \dots\dots, u^n, u^1), F(v^2, v^3, \dots\dots, v^n, v^1)\big), \\ \vdots \\ \vdots \\ d\big(F(u^n, u^1, u^2, \dots\dots, u^{n-1}), F(v^n, v^1, v^2, \dots\dots, v^{n-1})\big) \end{array}\right\} \\ &< max\left\{\begin{array}{c} max\{d(u^1, v^1), d(u^2, v^2), d(u^3, v^3), \dots\dots, d(u^n, v^n)\}, \\ max\{d(u^2, v^2), d(u^3, v^3), \dots\dots, d(u^n, v^n), d(u^1, v^1)\}, \\ \vdots \\ \vdots \\ max\{d(u^n, v^n), d(u^1, v^1), d(u^2, v^2), \dots\dots, d(u^{n-1}, v^{n-1})\} \end{array}\right\} \\ &= max\{d(u^1, v^1), d(u^2, v^2), d(u^3, v^3), \dots\dots, d(u^n, v^n)\} \\ &= \rho_n\big((u^1, u^2, u^3, \dots\dots, u^n), (v^1, v^2, v^3, \dots\dots, v^n)\big) \\ &= \rho_n(A, A^\star) \end{aligned}$$

This is a contradiction, therefore must be $A = A^\star$.

Case: 2. If $(u^1, u^2, u^3, \dots\dots, u^n)$ is not comparable to $(v^1, v^2, v^3, \dots\dots, v^n)$. By assumption, there exists $B = (w^1, w^2, w^3, \dots\dots, w^n) \in X^n$ such that

$$(F(w^1, w^2, w^3, \dots, w^n), F(w^2, w^3, \dots, w^n, w^1), \dots\dots, F(w^n, w^1, w^2, \dots, w^{n-1}))$$

is comparable with

$$(F(u^1, u^2, u^3, \dots\dots, u^n), F(u^2, u^3, \dots\dots, u^n, u^1), \dots\dots, F(u^n, u^1, u^2, \dots\dots, u^{n-1}))$$

and

$$(F(v^1, v^2, v^3, \dots\dots, v^n), F(v^2, v^3, \dots\dots, v^n, v^1), \dots\dots, F(v^n, v^1, v^2, \dots\dots, v^{n-1})).$$

Define sequences $(w_k^1), (w_k^2), (w_k^3), \dots\dots, (w_k^n)$ such that

$$(w^1, w^2, w^3, \dots\dots, w^n) = (w_0^1, w_0^2, w_0^3, \dots\dots, w_0^n)$$

and for any $k \geq 1$,

$$\begin{aligned} w_k^1 &= F(w_{k-1}^1, w_{k-1}^2, w_{k-1}^3, \dots\dots, w_{k-1}^n) = F^k(w_0^1, w_0^2, w_0^3, \dots\dots, w_0^n), \\ w_k^2 &= F(w_{k-1}^2, w_{k-1}^3, \dots\dots, w_{k-1}^n, w_{k-1}^1) = F^k(w_0^2, w_0^3, \dots\dots, w_0^n, w_0^1), \\ w_k^3 &= F(w_{k-1}^3, \dots\dots, w_{k-1}^n, w_{k-1}^1, w_{k-1}^2) = F^k(w_0^3, \dots\dots, w_0^n, w_0^1, w_0^2), \\ &\vdots \\ w_k^n &= F(w_{k-1}^n, w_{k-1}^1, w_{k-1}^2, \dots\dots, w_{k-1}^{n-1}) = F^k(w_0^n, w_0^1, w_0^2, \dots\dots, w_0^{n-1}). \end{aligned} \tag{63}$$

Since (59) and (60) comparable with (58), we may assume that

$$(u^1, u^2, u^3, \dots\dots, u^n) \geq (w^1, w^2, w^3, \dots\dots, w^n) = (w_0^1, w_0^2, w_0^3, \dots\dots, w_0^n)$$

By using (40), we get that for all k,

$$(u^1, u^2, u^3, \dots\dots, u^n) \geq (w_k^1, w_k^2, w_k^3, \dots\dots, w_k^n) \tag{64}$$

From (64), we have

$$u^1 < w_k^1, u^2 \geq w_k^2, u^3 < w_k^3, \dots\dots, u^n < w_k^n, (if\ n\ is\ odd), u^n \geq w_k^n \text{ (if n is even)} \tag{65}$$

Similarly,

$$v^1 < w_k^1, v^2 \geq w_k^2, v^3 < w_k^3, \dots\dots, v^n < w_k^n, (if\ n\ is\ odd), v^n \geq w_k^n \text{ (if n is even)} \tag{66}$$

By Proposition 8, and using (65), (66) we have

$$\rho_n\left(\widetilde{F^k}(A), \widetilde{F^k}(B)\right) \to 0ask \to +\infty. \tag{67}$$

and

$$\rho_n\left(\widetilde{F^k}(A^\star), \widetilde{F^k}(B)\right) \to 0ask \to +\infty. \tag{68}$$

By the triangle inequality, we get

$$\begin{aligned} \rho_n(A, A^\star) &= \rho_n\left(\widetilde{F^k}(A), \widetilde{F^k}(A^\star)\right) \\ &\leq \rho_n\left(\widetilde{F^k}(A^\star), \widetilde{F^k}(B)\right) + \rho_4\left(\widetilde{F^k}(B), \widetilde{F^k}(A^\star)\right) \to 0\ as\ k \to +\infty. \end{aligned}$$

This implies that $A = A^\star$.

Corollary17 In addition to the hypotheses of Theorem 17, assume that for all $(u^1, u^2, u^3, \dots\dots, u^n), (v^1, v^2, v^3, \dots\dots, v^n) \in X^n$, there exists $(w^1, w^2, w^3, \dots\dots, w^n) \in X^n$. such that

$$(F(w^1, w^2, w^3, \dots, w^n), F(w^2, w^3, \dots, w^n, w^1), \dots\dots, F(w^n, w^1, w^2, \dots, w^{n-1})$$

is comparable to

$$(F(u^1, u^2, u^3, \dots\dots, u^n), F(u^2, u^3, \dots\dots, u^n, u^1), \dots\dots, F(u^n, u^1, u^2, \dots\dots, u^{n-1}))$$

and

$$(F(v^1, v^2, v^3, \dots\dots, v^n), F(v^2, v^3, \dots\dots, v^n, v^1), \dots\dots, F(v^n, v^1, v^2, \dots\dots, v^{n-1})).$$

Then F has a unique n-tuplet fixed point.

Remark: 18 In view of theorem 16, the mapping F, in examples 12, 13, has a unique k-tuplet fixed point, a unique 5-tuplet fixed point, respectively.

Theorem 19 Let $(X, d, \leq)$ be a partially ordered complete metric space and $F: X^n \to X$ be a given mapping having the mixed monotone property on X. If the condition

$$d\big(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)\big) \leq a_1 d(x_1, y_1) + a_2 d(x_2, y_2) + \cdots \dots + a_n d(x_n, y_n)$$

is satisfied, where $a_i \in [0,1)\ with \sum_{i=1}^{n} a_i = h < 1$. Then F is a generalized Meir-Keeler type contraction.

Proof: Assume that

$$d\big(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)\big) \leq a_1 d(x_1, y_1) + a_2 d(x_2, y_2) + \cdots \dots + a_n d(x_n, y_n)$$

is satisfied, where $a_i \in [0,1)\ with\ \sum_{i=1}^{n} a_i = h < 1$. For all $\varepsilon > 0$, one can check that (8) is satisfied with $\delta(\varepsilon) = \left(\frac{1}{h-1} - 1\right)\varepsilon$.

8. APPLICATIONS

Motivated by Suzuki [16] and on the same lines of theorem 3.11 of [8], one can prove the following result.

Theorem: 20 Let $(X, d, \leq)$ be a partially ordered complete metric space and $F: X^n \to X$be a given mapping. Assume that there exists a function θ from $[\theta, +\infty)$ into itself satisfying the following:

1. $\theta(0) = 0$ and $\theta(t) > 0$ for every $t > 0$,
2. θ is a non decreasing and right continuous,
3. For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \theta(\max\{d(x_1, y_1), d(x_2, y_2), \dots \dots d(x_n, y_n)\}) < \varepsilon + \delta(\varepsilon)$$
$$\Rightarrow \theta\Big(d\big(F(x_1, x_2, x_3, , \dots, x_n), F(y_1, y_2, y_3, , \dots, y_n)\big)\Big) < \varepsilon \quad (69)$$

for all $x_1, x_2, x_3, x_4, \dots \dots, x_n, y_1, y_2, y_3, y_4, \dots \dots, y_n \in X$ with $x_1 \geq y_1, x_2 \leq y_2, \dots, x_n \geq y_n$ (if n is odd),$x_n \leq y_n$ (if n is even) or $x_1 < y_1, x_2 \geq y_2, \dots, x_n < y_n$(if n is odd),$x_n \geq y_n$ (if n is even). Then F is a generalized Meir-Keeler type contraction.

The following result is an immediate consequence of Theorems 10, 16 and 20.

Corollary:**21** Let $(X, d, \leq)$ be a partially ordered complete metric space and $F: X^n \to X$be a given mapping satisfying the following hypotheses:

1. F has the mixed strict monotone property.
2. For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \int_0^{max\{d(x_1,y_1),d(x_2,y_2),\dots\dots d(x_n,y_n)\}} \varphi(t)dt < \varepsilon + \delta(\varepsilon)$$
$$\Rightarrow \int_0^{d(F(x_1,x_2,x_3,,\dots,x_n),F(y_1,y_2,y_3,,\dots,y_n))} \varphi(t)dt < \varepsilon \quad (70)$$

for all $x_1, x_2, x_3, x_4, \dots \dots, x_n, y_1, y_2, y_3, y_4, \dots \dots, y_n \in X$ with $x_1 \geq y_1, x_2 \leq y_2, \dots, x_n \geq y_n$(if n is odd),$x_n \leq y_n$ (if n is even), where $\varphi: [0, +\infty) \to [0, +\infty)$ is a locally integrable function satisfying

$$\int_0^s \varphi(t)dt > 0 \text{for all } s > 0. \quad (71)$$

3. There exist $x_0^1, x_0^2, x_0^3, \dots \dots, x_0^n \in X$ be as in (11).

Assume, either F is continuous or $(X, d, \leq)$ is regular. Then F has an n-tuplet fixed point.

To end this paper, we give the following corollary.
Corollary:**22** Let $(X, d, \leq)$ be a partially ordered complete metric space and $F: X^n \to X$ be a given mapping satisfying the following hypotheses:

1. F has the mixed strict monotone property,
2. For all $x_1, x_2, x_3, x_4, \dots \dots, x_n, y_1, y_2, y_3, y_4, \dots \dots, y_n \in X$ with $x_1 \geq y_1, x_2 \leq y_2, \dots, x_n \geq y_n$, (if n is odd), $x_n \leq y_n$(if n is even),

$$\int_0^{d(F(x_1,x_2,x_3,,\ldots\ldots,x_n),F(y_1,y_2,y_3,,\ldots\ldots,y_n))} \varphi(t)dt \leq c\int_0^{\max\{d(x_1,y_1),d(x_2,y_2),\ldots\ldots d(x_n,y_n)\}} \varphi(t)dt \quad (72)$$

where $c \in (0,1)$ and $\varphi: [0,+\infty) \to [0,+\infty)$ is a locally integrable function satisfying

$$\int_0^s \varphi(t)dt > 0 \text{forall } s > 0, \quad (73)$$

3. There exist $x_0^1, x_0^2, x_0^3, \ldots \ldots, x_0^n \in X$ be as in (11).

Assume, either F is continuous or $(X, d, \leq)$ is regular. Then F has an n-tuplet fixed point.

Proof: For all $\varepsilon > 0$, we take $\delta(\varepsilon) = \left(\frac{1}{k} - 1\right)\varepsilon$ and we apply Corollary 21.

9. CONCLUSION

In this paper, we introduced the concept of mixed strict monotone property, generalized Meir-Keeler type contraction and established an n-tuplet fixed point theorem for continuous mapping $F: X^n \to X$ under a generalized Meir-Keeler contraction in the context of partially ordered complete metric spaces. Also established these results are still valid for $F: X^n \to X$, not necessarily continuous, assuming $(X, d, \leq)$ is regular. We proved the uniqueness of an n-tuplet fixed point for such mappings in this setup. These results are extensions of results in [8, 15] to the case n-tuplet fixed points. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

COMPETING INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

AUTHOR'S CONTRIBUTIONS

Both authors contributed equally to this work. Both authors read and approve the final manuscript.

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