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Characterization of Congruence Kernel Ideals in Sectionally Pseudo Complemented Semilattices

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ABSTRACT

This manuscript illustrates several principal results concerning congruence kernels of pseudo complemented semilattices will also hold in sectionally pseudo complemented semi lattices. Also it presents necessary and sufficient conditions such that any subset of sectionally pseudo complemented semi lattice which satisfies these conditions is kernel of some congruence, and it institutes the notion of * ideal in sectionally pseudo complemented semilattice and demonstrates that every kernel ideal is a * ideal. As well it establishes a condition for smallest * congruence of sectionally pseudo complemented semilattice with kernel ideal.

Keywords: Semi lattice, Pseudo complemented semi lattice, congruence kernel, * congruence, ideal.

AMS Mathematics subject classification numbers: 06A06, 06A12, 06D15.

1. INTRODUCTION

Cornish [6] investigated a congruence relation on pseudo complemented distributive lattices and identified those ideals that are congruence kernels. Blyth [5] showed results concerning congruence kernels and co-kernels hold in semilattice and therefore they do not depend on distributivity nor on the existence of unions.

In this paper we show that many of foremost results concerning congruence kernels of pseudo complemented semilattices will also hold in sectionally pseudo complemented semilattices. Also it is supplied that the necessary and sufficient conditions such that any subset of sectionally pseudo complemented semilattice which satisfies these conditions is a kernel of some congruence. Further we introduced the notion of * ideal in sectionally pseudo complemented semilattice and proved that every kernel ideal is a * ideal. We also established a condition for smallest * congruence of sectionally pseudo complemented semilattice with kernel ideal.

2. PRELIMINARIES

2.1. Gratzer [5] Definition: An element a^* is pseudo complement of an element a if and only if $a \wedge a^* = 0$ and $a \wedge x = 0 \Rightarrow x \leq a^*$.

2.2. Gratzer [5] Theorem: Let L be a pseudo complemented semi lattice S (L) = $\{a^* / a \in L\}$ (be set of all pseudo complements). Then the partial ordering of L partially orders S (L) and makes S (L) into a Boolean lattice for a, b ϵ S(L), we have $a \wedge b \in S(L)$ and join in S(L) is described by $a \vee b = (a^* \wedge b^*)^*$.

2.3. Gratzer [5] Result: Let L be a pseudo complemented semi lattice and let a, b ϵ L then $(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$.

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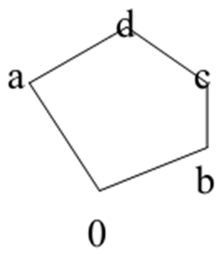
2.4. Blyth [6] Definition: If (S , Λ, * , 0) is a pseudo complemented semi lattice then by a * congruence on (we shall mean) is a semi lattice congruence ≡ that satisfies the additional condition x ≡ y implies x* ≡ y * for x, y ϵ S.

3. SECTIONALLY PSEUDO COMPLEMENTED SEMI LATTICE

3.1. Definition: Sectionally pseudo complemented semi lattice: A meet semi lattice S with 0
is called a sectionally pseudo complemented semilattice if and only if for every a ϵ S, the interval [0 , a] is pseudo complemented semi lattice.

S= {0,a,b,c,d; ≤ } is sectionally pseudo complemented semi lattice.

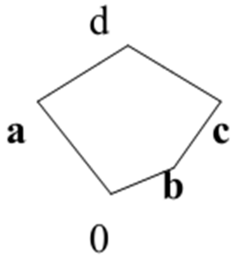


In this diagram the interval [0,a], [0,b] and [0,c] and [0,d] are all sectionally pseudo complemented semi lattices.

3.2. Theorem: Every pseudo complemented semi lattice is a sectionally pseudo complemented semi lattice, but converse need not be true.

Proof: Let S be the pseudo complemented semi lattice, then for a ϵ S, a^1 is a pseudo complement of a in S , i.e., x Λ a =0 if and only if x ≤ a^1 . Since a^1 is a pseudo complement of a, a Λ a^1 =0.

Let y ϵ [0,a] then 0 ≤ y≤a, thus yΛ0=0 and yΛa =y, as a^1 is pseudo complement of a in S, then y Λ a =0 for y ≤ a^1 . Let y ≤ a^1 , then y Λ a^1 = y, implies y Λa^1 Λa= y Λa , implies yΛ0=yΛa, implies yΛa =0. Therefore for yϵ[0,a] there exists a^1 ϵ[0,a] such that y ≤ a^1 . Therefore [0,a] is pseudo complemented semi lattice. Hence every pseudo complemented semi lattice is sectionally pseudo complemented semi lattice.



Conversely , the above semilattice S= {0,a,b,c,d; ≤ } is sectionally pseudo complemented semilattice as the intervals [0,a]; [0,b]; [0,c] and [0,d] are all pseudo complemented but S is not pseudo complemented because a Λ b =0 and aΛc =0 but c ≰ b, thus b is not pseudo complement of a. Therefore S is not pseudo complemented semilattice.

3.3. Definition: Let (S, Λ,*, 0) be a sectionally pseudo complemented semi lattice and R be a binary relation on S denoted by $[0]_R$ = {x ϵ S; < x,0 > ϵ R }. Especially, if R is a congruence on S, $[0]_R$ is called a congruence kernel (of R).

3.4. Example: Let S be a sectionally pseudo complemented semi lattice, for x, y and z in S, define a relation R as follows: xRy if and only if xΛy =0. Now we verify the relation is a congruence relation. Since 0Λ0=0 implies 0R0, the R is reflexive. Let xRy, implies xΛy=0, implies yΛx=0 , implies yRx, then R is symmetric. Let xRy and yRz, then xΛy=0 and yΛz=0.
Now 0 = (xΛz)= (xΛ0) Λ(0Λz) =(xΛy) Λ(yΛz) for y=0. Then xΛz =0, implies xRz. Therefore R is transitive. Hence the relation R is congruence.

3.5. Theorem: If S is sectionally pseudo complemented semi lattice, then a semi lattice congruence ≡ on S is a * congruence if and only if x ≡ 0 ⇒ x* ≡ 1

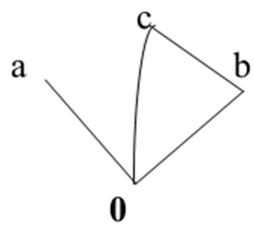
Proof: Let S be sectionally pseudo complemented semi lattice, then for x, y ϵ [0,1] ⊆ S, sectionally pseudo complement semi lattice is * congruence if x≡ y ⇒ x* ≡ y* on S. put y = 0 then x≡0 ⇒ x*≡ 1 in S. Thus condition is necessary.

Suppose, conversely that the condition hold and let x≡ y for x, y ϵ S. Then 0= x Λ x* ≡ y Λ x* ⇒ x*Λ y ≡ 0. Thus (x*Λ y)* ≡ 1.

Using the identity x Λ (x Λ y)* = x Λ y* and since 1 ≡ x*, we have
1≡ x* = x*Λ x* ≡ x* Λ 1 ≡ x* Λ (x*Λ y)* = x* Λ y* (by using identity).
Therefore x* ≡ x* Λ y*; similarly y* ≡ x*Λ y*.
Hence x* ≡ y*.

3.6. Definition: Ideal- A non empty subset I of Sectionally pseudo complemented semi lattice S is called an ideal if (i) for x, y in I , x Λ y ϵ I (ii) for x ϵ I, t ϵ S such that x ≤ t ⇒ t ϵ I.

3.7. Example: Let S= { 0,a,b,c} be a semilattice , which is given in following figure.



Clearly the semilattice S is sectionally pseudo complemented semilattice as the intervals [0,a] [0,b], [0,c] are all pseudo complemented semilattices. Let I= {0,a,b} be subset of S,

(i) for a, b ϵ I , a $\wedge$ b = 0 ϵ I.

(ii) for 0 ϵ I, a ϵ S such that $0 \leq a$, implies a ϵ I. Thus I is an ideal of S.

3.8. Definition: An ideal I of sectionally pseudo complemented semi lattice S will be called a kernel ideal if I is the kernel of a * congruence on S.

3.9. Note: A variety V is subtractive if and only if for each A ϵ V and every θ, φ ϵ Con A it holds that $[0]_{\theta}.\varphi = [0]\varphi._{\theta}$, these conditions are usually called permutable at 0.

We have the following Lemma by chajda, I Langer H [2]

3.10. Lemma: Let V be a subtractive variety, A = (A, F) ϵ V and R be a reflexive and compatible binary relation on A. Let $\theta_{(R)}$ be the least congruence on A containing R, then $[0]_R = [0]_{\theta(R)}$.

4. CHARACTERIZING CONGRUENCE KERNELS IN SECTIONALLY PSEUDO COMPLEMENTED SEMI LATTICE

4.1. Theorem: Let S be a sectionally pseudo complemented semilattice and a non empty subset I of S. Then I is kernel of some congruence on S if and only if I satisfies the following two conditions (i) If x in I and a in S then x$\wedge$ a ε I. (ii) If x, y ϵ I then $(x^* \wedge y^*)^*$ ϵ I.

Proof: If I = [0] θ for some θ ϵ Con A is the kernel of some congruence on S and for x, y ϵ I and a ϵ S, we have $x \equiv 0 \Rightarrow x\wedge a \equiv 0\wedge a = 0$; implies $x\wedge a \equiv 0 \Rightarrow x\wedge a$ ϵ I. More over $x \equiv 0$ and $y \equiv 0$
then $x^* \equiv 1$ and $y^* \equiv 1$ in S.
Then $x^* \wedge y^* \equiv 1$, $\Rightarrow (x^*\wedge y^*)^* \equiv 0$, implies $(x^*\wedge y^*)^*$ ϵ I.
Thus (i) yields 0 ϵ I, x ϵ S, $a \leq x \Rightarrow a = a\wedge x \equiv 0, \Rightarrow a \equiv 0$, then a ϵ I.
Also (ii) yields x ϵ I implies x^{**} ϵ I (taking x = y).
Conversely, suppose that I is a nonempty subset of S satisfying the conditions (i) and (ii) and also define a relation R on S as follows: x Ry if and only if $x \wedge y^*$ ϵ I and $y\wedge x^*$ ϵ I.

Since $x \wedge x^* = 0$ ϵ I $\Rightarrow$ x R x , thus the relation R is reflexive.
To prove compatibility of R, define, If xRy then $x \wedge y^*$ ϵ I and $y \wedge x^*$ ϵ I. By the condition (ii) we have $(x \wedge y^*)^{**}$ ϵ I and $(y \wedge x^*)^{**}$ ϵ I, also $x^{**}\wedge y^{**}$ ϵ I and $y^{**} \wedge x^*$ ϵ I, $\Rightarrow x^*$ R y^*.
Suppose x Ry and y R z then $x \wedge y^*$ ϵ I and $y \wedge x^*$ ϵ I and $y\wedge z^*$ ϵ I & $z\wedge y^*$ ϵ I, thus by condition (ii) we have
$[(x\wedge y^*)^* \wedge (y\wedge z^*)^*]^*$ ϵ I and $[(y\wedge x^*)^* \wedge (z\wedge y^*)^*]^*$ ϵ I .---------------(P)
Then $x\wedge y\wedge [(x\wedge y^*)^* \wedge (y\wedge z^*)^*] \wedge y^* = (x\wedge y^*) \wedge (x\wedge y^*)^* \wedge (y\wedge z^*)^* \wedge y = 0 \wedge (y\wedge z^*)^* \wedge y = 0$.
Similarly $x\wedge y \wedge [(x\wedge y^*)^* \wedge (y\wedge z^*)^*] \wedge z^* = 0$.
Thus $x\wedge y \wedge [(x\wedge y^*)^* \wedge (y\wedge z^*)^*] \leq y^{**}$ and $x\wedge y \wedge [(x\wedge y^*)^* \wedge (y\wedge z^*)^*] \leq z^{**}$.
Since $x\wedge y \wedge [(x\wedge y^*)^* \wedge (y\wedge z^*)^*] \leq y^{**}\wedge z^{**} = (y \wedge z)^{**}$, which yields
$x\wedge y \wedge [(x\wedge y^*)^* \wedge (y\wedge z^*)^*] \wedge (y\wedge z)^* = 0$, it gives $x\wedge y \wedge (y\wedge z)^* \leq [(x\wedge y^*)^* \wedge (y\wedge z^*)^*]^*$
Which together with (P) and (i) yields $x\wedge y \wedge (y\wedge z)^*$ ϵ I.
Similarly we show that $(y\wedge z) \wedge (x\wedge y)^*$ ϵ I. Therefore $(x\wedge y)$ R $(y\wedge z)$.
Hence R is reflexive and compatibility relation on S.
If a ϵ I, then $a \wedge 0^* = a \wedge 1 = a$ ϵ I and $a \wedge 1^* = a \wedge 0 = 0$ ϵ I.
Thus a R 0, i.e.. a ϵ $[0]_R$ implies I $\subseteq$ $[0]_R$. If a ϵ $[0]_R$ then a R 0 and hence $a = a \wedge 0^*$ ϵ I.,i.e., $[0]_R \subseteq$ I. Thus we have $[0]_R$ = I. Therefore by lemma 2.10, I= $[0]_{\theta (R)}$.
Therefore I is a congruence kernel.

4.2. Result: An ideal I of sectionally pseudo complemented semi lattice S is a kernel ideal of S if and only if i, j ϵ S implies $(i^* \wedge j^*)^*$ ϵ I.

Proof: If I is kernel of a * congruence $\equiv$ and if i,j ϵ I , then $i \equiv 0$, implies $i^* \equiv 1$; similarly $j \equiv 0$ implies $j^* \equiv 1$. Therefore $i^* \equiv j^* \equiv 1$, then $i^* \wedge j^* \equiv 1$, implies $(i^* \wedge j^*)^* \equiv 0$ ϵ I.

Conversely, suppose that the condition holds, consider the relation "~" defined on S by
$x \sim y \Leftrightarrow$ there exist iϵ I such that $x \wedge i^* = y \wedge i^*$. Since $x \wedge i^* = x\wedge i^*$ for iϵI $\Leftrightarrow x \sim x$, thus "~ " is reflexive. Let $x\sim y$, then $x\wedge i^* = y\wedge i^*$, implies $y\wedge i^* = x\wedge i^*$, implies $y\sim x$. Therefore the relation "~" is symmetric. Let $x\sim y$ and $y \sim z$, then $x \wedge i^* = y\wedge i^*$ and $y\wedge j^* = z\wedge j^*$ for i,j ϵ I, as condition holds for i,j ϵ I, $(i^* \wedge j^*)^*$ ϵ I. Let $k= (i^* \wedge j^*)^*$ ϵ I. Now $x\wedge k^* = x \wedge (i^* \wedge j^*)^{**} = x \wedge i^* \wedge j^* = y \wedge i^* \wedge j^* = z \wedge i^* \wedge j^* = z\wedge (i^* \wedge j^*) = z \wedge ((i^*\wedge j^*)^*)^* = z \wedge k^*$.
Therefore $x \sim z$. Hence the relation "~" is transitive. It is clear that the equivalence relation "~" is a semilattice congruence on S. Now by taking i=j in the condition, we obtain iϵ I, implies $(i^* \wedge i^*)^* = i^{**}$ ϵ I. Thus we have $x\sim 0$ $\Leftrightarrow$ there exist iϵ I, such that $x \wedge i^* = 0\wedge i^* = 0$ $\Leftrightarrow$ there exists iϵI, such that $x \leq i^{**}$ $\Leftrightarrow$ x ϵ I., i.e., $x \equiv 0$. So, the kernel of "~" is I. Also $x \sim 1$ $\Leftrightarrow$ there exists iϵ I such that $x\wedge i^* = 1\wedge i^* = i^*$ $\Leftrightarrow$ there exists iϵ I such that $x \geq i^*$; so that $x \sim 0$ implies their exists iϵ I such $x \wedge i^* = 0$, implies $i^* \leq x^*$ implies $x^* \sim 1$ i.e., $x^* \equiv 1$. Therefore, if S is

sectionally pseudo complemented semi lattice then a semi lattice congruence ≡ on S is a * congruence if and only if x ≡0 implies x* ≡ 1. Hence the relation "~" is a * congruence on S.

4.3. Corollary: I is a kernel ideal if and only if (i) i ϵ I ⇒ i** ϵ I and (ii) for all i,j ϵ I, there exist k ϵ I such that i* Λ j* = k*.

Proof: By above result, If I is a kernel ideal then for i ϵ I, implies i** ϵ I and for all i,j ϵ I, there exists k ϵ I such that i* Λ j* = k*, where k = (i* Λ j*)*. Conversely, suppose that the conditions (i) and (ii) holds then for all i,j ϵ I, there exists k ϵ I such that i* Λ j* = k*, implies (i* Λ j*)* = k**, then by condition (i) for k ε I, k** ϵ I. Therefore by above result, I is kernel.

4.4. Definition: An ideal I of sectionally pseudo complemented semi lattice S will be called
* Ideal if i ϵ I, ⇒ i** ϵ I.

Example: as i* is the pseudo complement of i then i** = iϵ I.

4.5. Theorem: Every kernel ideal is a * ideal.

Proof: Let I be a kernel ideal of sectionally pseudo complemented semi lattice S then by corollary 3.3, we have I is kernel ideal if and only if i ϵI, implies i** ϵ I and for all i,j ϵ I there exist k ϵ I such that i* Λ j* = k* . Thus I is a * ideal.

4.6. Theorem: Let S be sectionally pseudo complemented semi lattice S and let I be a kernel ideal of S, then the smallest * congruence on S with kernel I is given by a relation R as x R y if and only if there exists i ϵ I, x Λ i* = y Λ i*.

Proof: If I is a kernel of a * congruence ≡ and if i,j ϵ I, then i≡0 ⇒ i* ≡ a ,
Similarly j≡ 0 ⇒ j* ≡ a in S (i.e., in an interval [0, a]) .
Therefore i*≡j*≡ a, then i*Λj* ≡ a.
Hence (i* Λ j*)* ≡ 0 ϵ I. Conversely, suppose that the condition holds. Define a relation R on S as follows: x R y if and only if there exists i ϵ I such that xΛi* = yΛ i*.
Since xΛi* = xΛ i* for i ϵ I if and only if x R x, thus R is reflexive.
Let x R y then xΛi* = yΛ i*, ⇒yΛi* = xΛ i*, ⇒ y R x. Thus R is symmetric.
Let x R y and y R z, then xΛi* = yΛ i* and yΛj* = zΛ j* for i, j ϵ I.
As condition holds, for i,j ϵ I, (i* Λ j*)* ϵ I. Let k = (i* Λ j*)* ϵ I,
Now xΛ k* = x Λ(i* Λ j*)** = x Λ i* Λ j* = y Λ i* Λ j*= z Λ i* Λ j*=z Λ(i* Λ j*)* = z Λk*.
Thus x R z. Hence R is transitive. It is clear that the equivalence relation R is semi lattice congruence on S. Now by taking i= j in the condition obtains (i*Λj*)* = i** ϵ I. Thus we have
x R 0 if and only if there exists i ϵ I, x Λ i* = 0 Λ i* = 0 if and only if there exists i ϵ I ,x ≤ i**
if and only if x ϵ I , i.e., x ≡ 0. So the kernel of R is I. Also x R a if and only if there exist i ϵ I
xΛi* = a Λ i* = i* if and only if there exists i ϵ I, x ≥ i*, so that x R 0 implies there exists i ϵ I,
xΛi* = 0, implies there exists i ϵ I such that i* ≤ x* ⇒ x* R a, i.e.,, x* ≡ a.
We know that if S is sectionally pseudo complemented semi lattice then a semi lattice congruence ≡ on S is a smallest * congruence if and only if x ≡ 0 implies x* ≡ a.
Hence the relation R is a * congruence on S.
4.7. Note: The relation R^{I_o} is defined as follows

xR^{I_o} y ⇔ ((x Λ y*) Λ (yΛx*)) ϵ I_o , where I_o = { x ϵ S / x Λ i=0 for all i}

4.8. Corollary: x R^{I_o} y if and only if [(xΛy*)*Λ (x* Λ y)*]* ϵ I_o

Proof: If I is an ideal of sectionally pseudo complemented semi lattice S then define
I_o = {x ϵ S; (for all i ϵ I) xΛ i = 0}, the set of disjoint elements from I. Let x, y ϵ I_o , then
xΛ i = 0, y Λ i = 0, ⇒ i≤ x* and i≤ y*, i≤ x*Λ y* ⇒ (x* Λ y*)* ≤ i*
⇒ (x*Λ y*)* Λ i= 0, implies (x* Λ y*)* ϵ I_o . Thus for x, y ϵ I_o , we have (x* Λ y*)* ϵ I_o .Hence I_o is kernel ideal .
As we define a relation R^{I_o} as follows
x R^{I_o} y ⇔ xΛ y* ϵ I_o and y Λ x* ϵ I_o . Then we have (xΛy*) Λi =0 and (yΛx*) Λi =0
⇔ i≤ (x Λy*)* and i≤ (yΛ x*)*; ⇔ (xΛy*)*Λ (yΛ x*)* ≤ i*
⇔ [(x Λy*)* Λ (yΛ x*)*]* Λ I =0; ⇔ [(x Λy*)*Λ (yΛ x*)*]* ϵ I_o.

4.9. Theorem: If I is an ideal of sectionally pseudo complemented semi lattice S then x R^{I_o} y if and only if (x**] ∩ I = (y**] ∩ I.

Proof: Let ≡ be the equivalence relation defined by x ≡y ⇔ (x**] ∩ I = (y**] ∩ I.
Since (xΛy]**=(x*] ∩ (y**] , it is clear that ≡ is compatible with Λ. Now x ≡ 0 ⇔ (x**] ∩ I = 0 ⇔ x** ϵ I_o ⇔ x ϵ I_o .Since I_o is kernel ideal, and x ≡ a ⇔ (x**] ∩ I = (a**] ∩ I = I
⇔ I ⊆ (x**]. Thus we have x ≡ 0 implies x ϵ I_o , implies x Λ i= 0 for i ϵI, implies i≤ x*
⇒I ⊆(x*] = (x***] ⇒ (x***] ∩ I = I ⇒ (x*] ∩ I =I= (a**] ∩ I ⇒x* ≡ a. Thus for x ≡ 0, x* ≡a.

Hence by know theorem 3.6 ≡ is a * congruence with kernel I_o. Suppose that x R^{I_o} y then by corollary 3.8 we have [(xΛy*)*Λ (x* Λ y)*]* ϵ I_o . Hence for all i ϵ I, [(xΛy*)*Λ (x* Λ y)*]* Λ i=0. Since [(xΛy*)*Λ (x* Λ y)*]* ≥ (x Λ y*)** = x** Λ y*, it follows that for i ϵ I (x** Λy*) Λi =0, implies i≤ (x** Λ y*)*. If then i ϵ (x**]∩ I, we have y* Λ i=0, so i≤ y**. Hence i ϵ(y**]∩ I.

Thus (x**] ∩ I ⊆ (y**] ∩ I. Similarly we can prove that (y**] ∩ I ⊆ (x**] ∩ I.
Hence (x**] ∩ I = (y**] ∩ I. Whence we obtain x≡y. Since R^{Io} is the largest * congruence with kernel I_o it follows that R^{Io} and ≡ coinside.

5. CONCLUSION

This manuscript exemplifies many of primary results disquieting congruence kernels hold in sectionally pseudo complemented semi lattices. It is observed that a necessary and sufficient condition for a subset of sectionally pseudo complemented semilattice to be kernel of some congruence and verified that every kernel ideal is a * ideal. Also it has been established a condition for smallest * congruence of sectionally pseudo complemented semi lattice with kernel ideal.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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