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A Comparative Study of Finite Difference Scheme for Burger's Equation

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ABSTRACT

This paper represents a comparative study of the Lax-Friedrich scheme and Lax-Wendroff's scheme for the numerical solution of Burger's equation. Performing the numerical computation of the Burger's equation by using the first order and second order schemes respectively, we verify the numerical features like accuracy, rate of convergence and efficiency of the schemes for given initial and boundary values.

Keywords: Burger's equation, Finite difference schemes, Numerical solution, initial value problem, Lax-Friedrich scheme, Lax-Wendroff's scheme.

1. INTRODUCTION

1.1. Lax-Friedrichs scheme for Burger's equation

A very useful scheme is the Lax-Friedrichs scheme; this will be used for the numerical solution of Burger's equation. For this purpose we consider the time dependent Cauchy problem in one space dimension [5]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \ , \ -\infty < x < \infty, \quad t \geq 0 \qquad (1)$$

$$u(x,0) = u_0(x) \qquad (2)$$

We discretize the x-t plane by choosing a mesh width $h = \Delta x$ *and a time step* $k = \Delta t$ and the discrete mesh points $(x_{j,} t_n)$ an approximation to a cell average of $u(x,t_n)$ is defined by

$$u_j^n = \frac{1}{h}\int_a^b u(x, t_n)\,dx \qquad (3)$$

Where $a = x_{j-\frac{1}{2}}, b = x_{j+\frac{1}{2}}$

We will primarily study the theory of numerical methods for the Cauchy problem as indicated in (2). In practice we must compute on a finite spatial domain say $a \leq x \leq b$ and we require appropriate boundary condition at a and or b from the initial data $u_0(x)$, we

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have defined data U^0 for our approximate solution. We use a time marching procedure to construct the approximation U^1 *from* U^0 *then* U^2 *from* U^1 and so on. There are wide variety of finite difference methods that can be used, many of these are derived simply by replacing the derivatives occurring in (1) by approximate finite difference approximations. For example replacing u_t by a forward in time approximation and u_x by a spatially centered approximation. We obtain the following difference approximation for U^{n+1}

$$\frac{U_j^{n+1}-U_j^n}{k}+A\left(\frac{U_{j+1}^n-U_{j-1}^n}{2h}\right)=0 \qquad (4)$$

This can be solved for U_j^{n+1} to obtain the Lax-Friedrichs scheme

$$U_j^{n+1}=U_j^1-\frac{1}{2h}A\left(U_{j+n}^n-U_{j-1}^n\right) \qquad (5)$$

Unfortunately despite the quite natural derivative of this methods, it suffers from severe stability problems and is useless in practice .A far more stable method is obtained by evaluating the centered approximation to u_x at time t_{n+1} rather than at time t_n

So the discrete version of the non-linear PDE formulates:

Giving $U_j^{n+1}=U_j^n-\frac{k}{2h}\left(U_{j+1}^{n+1}-U_{j-1}^{n+1}\right)$ (6)

The method (5) is explicit and the method (6) is implicit. In order to determine U^{n+1} *from the data* U^n in each step. But in any practical calculation we would use a bounded integral with N grid points and they would be a finite system the method (5) and (6) are explicit and implicit respectively. In this system a wide variety of methods can be devised for the linear system by using difference approximations. Most of these are based directly on finite difference approximation to the PDE. The Lax-Friedrich scheme is based on the Taylor series expansion

$$u(x,t+k))\approx u(x,t)+k\frac{\partial u}{\partial t}+\frac{k^2}{2}\frac{\partial^2 u}{\partial x^2}\approx u(x.t)+k(\mu\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x})+ \qquad (7)$$

Considering $u_t=-Au_x$ here we can compute

$$u_{tt}=-Au_{xt}=-Au_{tx}=-A(-Au_x)_x=A^2u_{xx} \qquad (8)$$

The equation (7) become in the form as

$$u(x,t+k)\approx u(x,\ t)-kAu_x(x,t)+\frac{1}{2}k^2A^2u_{xx}(x,\ t)+.. \qquad (9)$$

Since the scheme (9) is unstable so if we replaces U_J^N *by* $\frac{1}{2}\left[U_{J-1}^N+U_{j+1}^n\right]$ then the unstable method is stable provided $\frac{k}{h}$ is sufficiently small. Therefore the L-F scheme takes the form:

$$U_j^n=\frac{1}{2}\left(U_{j-1}^n+U_{j+1}^n\right)-\frac{k}{2h}A\left(U_{j+1}^n-U_{j-1}^n\right) \qquad (10)$$

This difference equation is the analogous with the Lax-Fridrich scheme.

We implement this scheme for our model.

We choose the viscid burgers equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2} \quad (11)$$

Now we will discretize the different derivatives of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ $and\ \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial x}\approx\frac{u_{i+1}^n-u_{i-1}^n}{2h} \quad (12)$$

$$\frac{\partial u}{\partial t}\approx\frac{u_i^{n+1}-u_i^n}{k} \quad (13)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{u_{i+1}^n-2u+u_{i-i}^{n\,n}}{h^2} \quad (14)$$

If $\mu\to 0$ then the viscid burgers' equation becomes in the form as:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \quad (15)$$

Now substituting the equation of (12), (13) and (14) in (11) then it is obtain:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}=\mu\frac{u_{i+1}^n-2u+u_{i-1\,i}^{n\ \ n}}{\Delta x^2} \quad (16)$$

1.2. Lax-Wendroff Scheme for Burger's Equation

A very popular scheme for general non-linear flux function "f" is the Lax–Wendroff scheme. In order to develop this scheme we will consider the numerical solution.

For the implementation of the numerical solution of the viscid Burgers' equation we would like to use this scheme [5].We discretize the x - t plane by choosing a mesh with $h=\Delta x\ and\ a\ time\ step\ k=\Delta t$ and define the mesh points (x_j, t_n).For simplicity we take a uniform mesh, with h and k constant, although most of the methods discussed can be extended to variable meshes .An approximation to a cell average of $u(x_n, t_n)$ is defined by

$$u_j^n=\frac{1}{h}\int_a^b u(x,t_n)\,dx \quad (17)$$

As initial data for the numerical method we use $u_0(x)$ to define U^0 either by point wise values. We will primarily study the theory of numerical methods for the Cauchy problem. In practice we must compute on a finite spatial domain, $a\le x\le b$ and we required appropriate boundary conditions at a or b.One simple case is obtained if we take periodic boundary conditions,

Where u (a,t) = u(b,t) $\forall t\ge 0$ (18)

We can use periodicity in applying the finite difference methods as well. This is equivalent to Cauchy problem with periodic initial conditions, since the solution remains periodic and we need compute over only one period .For linear equation the study of the Cauchy problem is particularly attractive .The study of the general initial boundary value problem is more complex. From the initial data $u_0(x)$ we have defined data U_0 for our approximate solution. A wide variety of methods can be devised for the linear system (27) by using different finite difference approximations. A few possibilities are listed along with their stencils. Most of these are based directly on finite difference approximations to the PDE. An exception is the Lax-Wendroff method which is based on the Taylor series expansion.

1.3. Lax-Wendroff Scheme

Expansion by Taylor series:

This scheme is very important for the numerical solution of Burger's equation so we need to check the performance of the scheme, for this purpose we would like to choose the viscid burger's equation as follows

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2} \quad (19)$$

Now we will discretize the different derivatives of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ for viscid burger's equation

$$\frac{\partial u}{\partial x}\approx\frac{u_{i+1}^n-u_{i-1}^n}{2h} \quad (20)$$

$$\frac{\partial u}{\partial t}=\frac{u_i^{n+1}-u_i^n}{k} \quad (21)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{h^2} \quad (22)$$

$$\frac{\partial^3 u}{\partial x^3}=\frac{u_{i+2}^n-2u_{i+1}^n+u_{i-2}^n}{2h^3} \quad (23)$$

$$\frac{\partial^4 u}{\partial x^4}=\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2}\right)$$
$$=\frac{\left(u_{i+1}^n-2u_{i+1}^n+u\right)-2\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)+\left(u_i^n-2u_{i-1}^n+u_{i-2}^n\right)}{h^4}$$
$$=\frac{u_{i+2}^n-4u_{i+1}^n+6u_l^n-4u_{i-1}^n+u_{i-2}^n}{h^4} \quad (24)$$

1.4. The Lax-Wendroff Scheme Expansion by Tailor's Series

$$u(x,t+k)\approx u(x,t)+k\frac{\partial u}{\partial t}+\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}$$
$$=u(x,t)+k(\mu\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x})+\frac{k^2}{2}\frac{\partial}{\partial t}(\mu\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x})-$$
$$=u(x,t)+k(\mu\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x})+\frac{k^2\mu}{2}\frac{\partial^2}{\partial x^2}(\frac{\partial u}{\partial t})-\frac{k^2}{2}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}-\frac{k^2 u}{2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$$
$$=u(x,t)+k(\mu\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x})+\frac{kv^2}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x}\right)-\frac{k^2}{2}\left(\mu\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}-\frac{k^2 u}{2}\frac{\partial}{\partial x}\left(\mu\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x}\right)$$
$$=u(x,t)+k\left(\mu\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x}\right)+\frac{k^2\mu^2}{2}\frac{\partial^4 u}{\partial x^4}-\frac{k^2\mu u}{2}\frac{\partial^3 u}{\partial x^3}-\frac{k^2\mu}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}-\frac{k^2\mu}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}+\frac{k^2 u}{2}\left(\frac{\partial u}{\partial x}\right)^2$$
$$-\frac{k^2 u\mu}{2}\frac{\partial^3 u}{\partial x^3}+k^2u^2\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\frac{k^2 u}{2}\left(\frac{\partial u}{\partial x}\right)^2$$
$$=u(x,t)+k\left(\mu\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x}\right)+\frac{k^2\mu^2}{2}\frac{\partial^4 u}{\partial x^4}-k^2\mu u\frac{\partial^3 u}{\partial x^3}-k^2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}+k^2u\left(\frac{\partial u}{\partial x}\right)^2+\frac{k^2u^2}{2}\frac{\partial^2 u}{\partial x^2}$$

If $\mu\to 0$ then the viscid burgers' equation becomes in the form as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \quad (25)$$

Now inserting (20), (21) and (22) in (19) then the discrete version formulates in the form:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}=\mu\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2} \quad (26)$$

1.5. Comparisons Between the Lax-Friedrich Scheme and Lax-Wendroff Scheme

(i) The Lax-Friedrichs scheme has one degree precision along time and two degree precision along space
(ii) The Lax-Wendroff scheme has two degree precision along both time and space.
(iii) Since Lax-Wendroff scheme has two degree precision along time whether Lax-Wendroff scheme has one two degree precision along time, so Lax-Wendroff scheme gives more accurate solution than that of Lax-Friedrich scheme.
(iv) Since in implementing Lax-Wendroff scheme, we need to calculate derivatives up to 4^{th} order w.r.to x whether in implementing Lax-Friedrichs scheme, we need to calculate derivatives up to 2^{nd} order w.r.to x, so it needs more computational time in implementing Lax-Wendroff scheme than that of Lax-Friedrich scheme.

1.6. Numerical implementation of Lax-Friedrichs scheme

In this chapter we implement our numerical methods by choosing the same initial condition and the boundary conditions and using computer programming. In implementation of our scheme, we consider the spatial domain over the space $[0,\ 2\pi]$ and the maximum time step T=5

We consider the initial condition

$$u(x\ ,0) = u_0(x) = \sin x \quad (27)$$

We would like to consider the Homogeneous Dirichlet boundary conditions

$$u(0\ ,t) = 0 = u(2\pi, t) \quad (28)$$

For this purpose we need to provide the stability condition is given below:

$$\frac{h}{2} \le \nu \le \frac{h^2}{2k} \quad (29)$$

We see that the discrete values of h and k depend on ν but for very small values of ν, there we are able to perform the numerical solutions of Burger's equation using E.F.D.S.

For $\nu = 0.1$, we get the stability condition.

From (29) we have

$$\frac{h}{2} \le 0.1 \le \frac{h^2}{2k}$$

$$or\ \frac{h}{2} \le 0.1 \Rightarrow h \le 0.2\ and\ \frac{h^2}{2k} \ge 0.1 \Rightarrow \frac{(0.1)^2}{2\times .1} \ge k \Rightarrow k \le .05$$

Since $h \le 0.2\ \ and\ \ k \le 0.05$, so let h = 0.1 and k =0 .01

For h=.15 and k=.02
For h=0.1and k=.03
For h=.12 and k=.04
For h=0.1and k=.02
For h=0.1and k=.04

If we will continue this process, then we will get several numerical solutions of this scheme.

1.7. Solution of Burger's equation using Lax-Friedrichs for $\nu = 0.1\ ,\Delta x = 0.1\ ,\Delta t = 0.03$

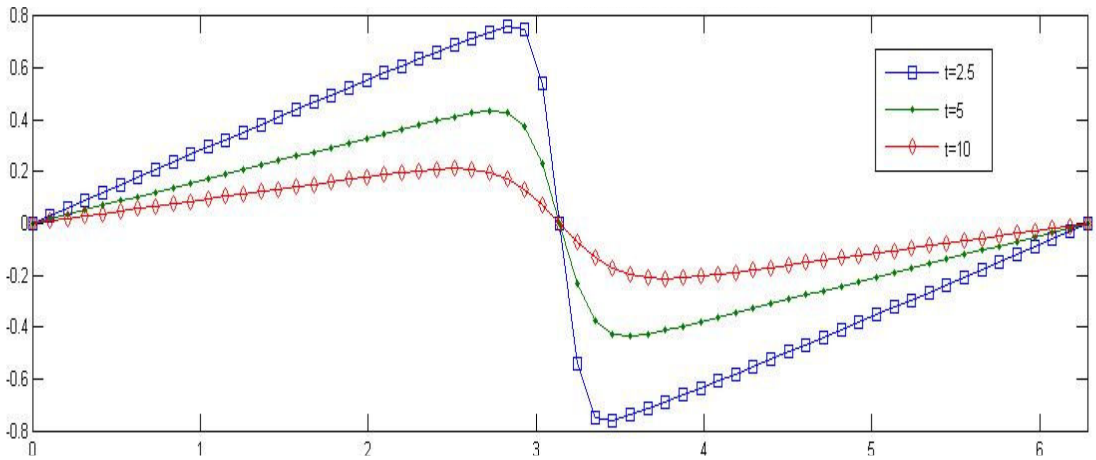

Figure 1. Solution of Burger's equation using Lax-Friedrichs for $\nu = 0.1\ ,\Delta x = 0.1\ ,\Delta t = 0.03$

1.8. Numerical Implementation of Lax-Wendroff Scheme

For the implementation of the numerical solution of the viscid Burgers' equation we would like to use this scheme [5].

We discretize the x-t plane by choosing a mesh with $h = \Delta x$ and a time step $k = \Delta t$ and define the mesh points (x_j, t_n).For simplicity we take a uniform mesh, with h and k constant, although most of the methods discussed can be extended to variable meshes

An approximation to a cell average of $u(x_n, t_n)$

In implementation of our scheme, we consider the spatial domain over the space $[0,\ 2\pi]$ and the maximum time step $T = 5$

We consider the initial condition:

$$u(x\ ,0) = u_0(x) = \sin x$$

Numerical implementation of Lax-Wendroff sheme at $t = 0,1,3,5$

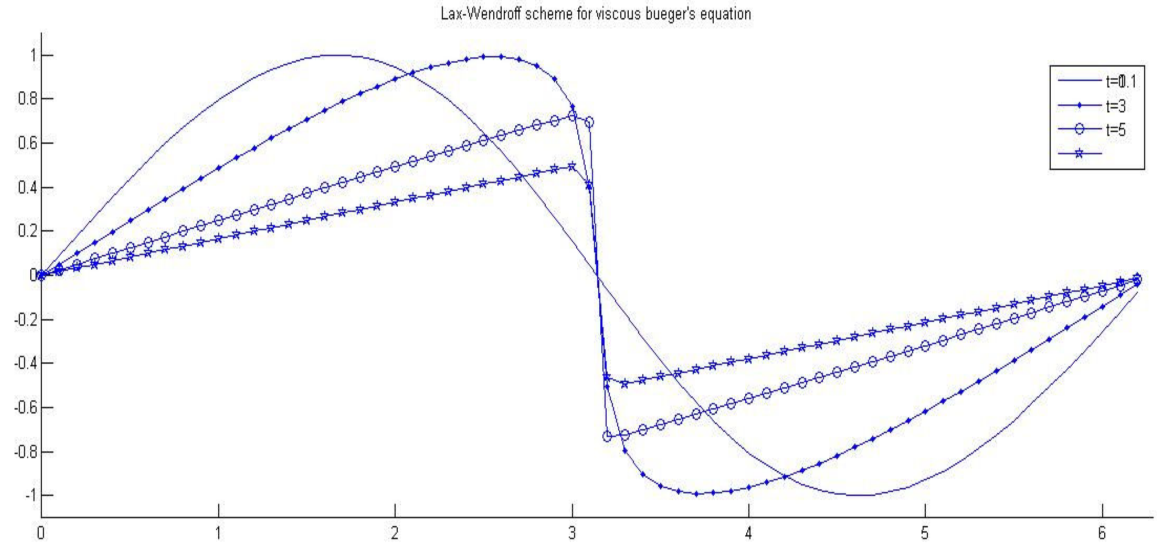

Figure 2. Numerical implementation of Lax-Wendroff scheme at $t = 0,1,3,5$.

In figure 2, we observe that the curve is very steeper while the time is progresses.

1.9. Relative Error for Lax-Friedriches Scheme

We perform our numerical scheme for $\nu = 0.1$ *up to time* $t = 5$ *in* $\Delta t = 0.01$ time steps in spatial domain $[0\ ,2\pi]$ with $\Delta x = 0.1$ which guarantees the stability condition $\frac{h}{2} \le 0.1 \le \frac{h^2}{2}$ and the obtained graph of relative error is shown in figure 3;

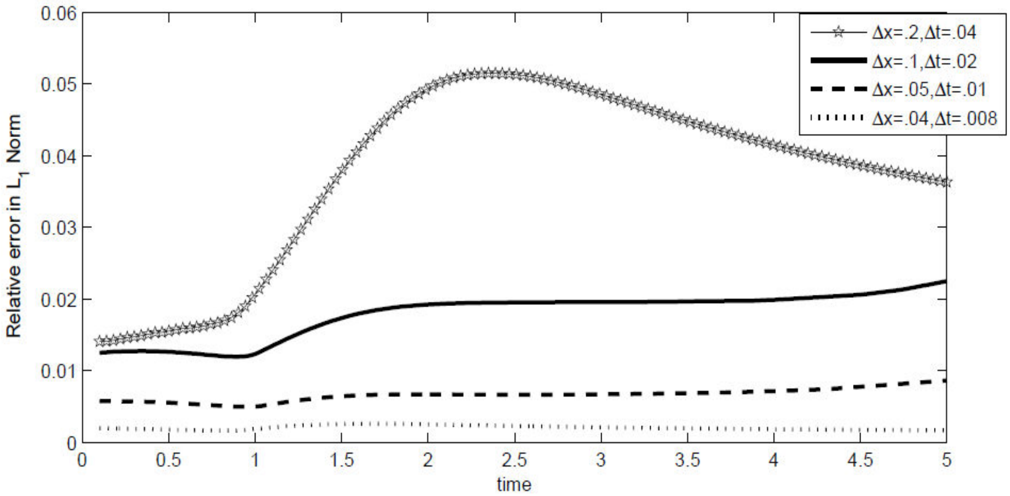

Figure 3: Relative error for Lax-Friedrichs scheme

Convergences of relative errors to zero when we use Lax-Friedrich scheme as shown in Figure 3. Convergence of relative error curve for explicit finite difference scheme to x-axis that actually describes the convergence of the numerical solution to analytical solution

1.10. Relative Error for Lax-Wendroff Scheme

We perform our numerical scheme for $\nu = 0.1$ up to time t =5 in $\Delta t = 0.01$ time steps in spatial domain $[0, 2\pi]$ with mesh width $\Delta x = 0.1$ .We also perform the numerical operation for some other pairs (h,k) each for $\nu = 0.1$ to the rate of convergence of numerical solution to analytical one.

Relative error for Lax-Wendroff scheme:

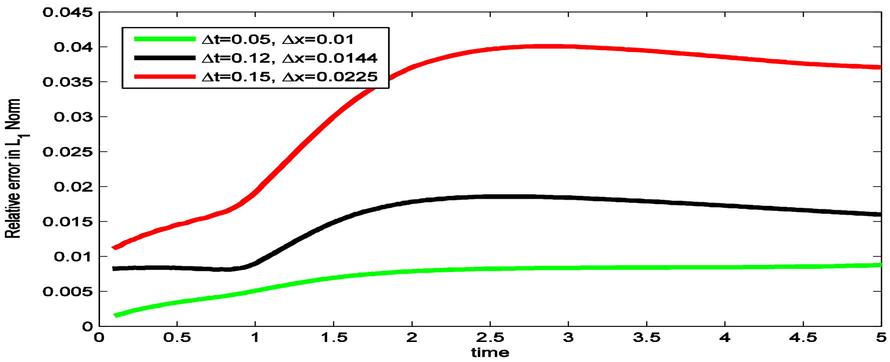

Figure 4: Relative error for Lax-Wendroff scheme

Convergence of relative errors to zero when we use Lax-Wendroff's scheme shown in Figure 4.

CONCLUSION

The Lax-Friedrichs scheme has one degree precision along time and two degree precision along space. The Lax-Wendroff scheme has two degree precision along both time and space. Since Lax-Wendroff scheme has two degree precision along time whether Lax-Wendroff scheme has one two degree precision along time,so Lax-Wendroff scheme gives more accurate solution than that of Lax-Friedrich scheme. Since in implementing Lax-Wendroff scheme, we need to calculate derivatives up to 4^{th} order w.r.to x whether in implementing Lax-Friedrichs scheme, we need to calculate derivatives up to 2^{nd} order w.r.to x, so it needs more computational time implementing Lax-Wendroff scheme than that of Lax-Friedrich scheme. We obtain the numerical solution at different time steps and obtained the better result of theme scheme.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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