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Common Fixed Point Theorems for Single Valued Weakly Compatible Maps in Metric Space

Saurabh MANRO[1,♠], Anita TOMAR[2]

[1]*School of Mathematics and Computer Applications, Thapar University, Patiala (Punjab)*
[2]*V.S.K.C. Government Degree College Dakpathar, Dehradun (Uttarakhand) India*

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ABSTRACT
The aim of this paper is to establish common fixed point theorems for single valued maps satisfying general contractive conditions of integral type using weak compatibility wherein the conditions of completeness of the underlying subspaces and containment of ranges amongst involved maps is not needed. Moreover, an example and an application are also given to illustrate the usability of the obtained results.

Keywords: Weakly compatible, CLR_S property, property (*E.A.*), common property (*E.A.*), $JCLR_{ST}$ property, fixed point.

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1. INTRODUCTION

The fixed point tells us which parts of the space are pinned, i.e., not moved, by single valued or multivalued map. Fixed points and fixed point theorems have always been useful in the theory of ordinary differential equation, integral equation, partial differential equation, game theory and in other related areas. The importance of fixed point may be understood from the fact that most of the functional equations $y = fx$ may be transformed to a fixed point problem $x = fx$ and then applied a fixed point theorem to get information on the existence or existence and uniqueness of the fixed point, that is, of a solution for the original equation.

Aamri et al. [1] introduced the notion of property (*E.A.*) which contain the class of compatible as well as noncompatible maps and this is the motivation to use the property (*E.A.*) instead of compatibility or non-compatibility. Liu et al. [7] further extended it to common property (*E.A.*). It was pointed out that the property (*E.A.*) and common property (*E.A.*) allows replacing the completeness requirement of the space to more natural condition of closedness of range and relaxes the continuity of maps. Also the notion of common property (*E.A.*) relaxes containment of range of one map into the range of other which is utilized to construct the sequence of joint iterates (also containment of range of maps is not needed for a pair of maps which satisfy property (*E.A.*)) besides minimizing the commutativity conditions of the maps to the commutativity just at their points of coincidence. However property (*E.A.*) and common property (*E.A.*) always require closedness of subspace for the existence of coincidence and common fixed point. Recently, Sintunavarat et al. [9] introduced the notion of common limit in the range (*CLRg)* property which is more general than property (*E.A*). On the other hand, Chauhan et al. [4] introduced $JCLR_{ST}$ property for two pairs of self maps which are more general than common property (*E.A*) and even relaxes the closedness requirements of the underlying subspaces. For more detail, one can refers to papers[1,4,7,8,9].

Altun et al. [3, Theorem 2.1] proved common fixed point theorems for multivalued maps satisfying some general contractive conditions of integral type using weak compatibility, containment of underlying subspaces and completeness of one of the subspaces. The aim of this paper is to establish similar result for single valued maps wherein the conditions of completeness of the underlying subspaces and containment of ranges amongst involved maps is not needed. Moreover, an example and an application are also given to illustrate the usability of the obtained results.

♠Corresponding author, e-mail: sauravmanro@yahoo.com

2. PRELIMINARIES

Definition 2.1. A pair of self maps (*S, T*) of a metric space (X,d) is

(i) compatible [5] if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

(ii) non-compatible if there exists at least one sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$ but either $\lim_{n\to\infty} d(STx_n, TSx_n) \neq 0$ or non-existent.

(iii) weakly compatible [6] if *S* and *T* commute at coincidence points, that is, $STx = TSx$ whenever $Sx = Tx$.

(iv) satisfy the property (*E*.A) [1] if there exist a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

(v) satisfies the common limit in the range property *(CLR_T)* [9] if there exist a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tz$ for some $z \in X$.

Definition 2.2. Two pairs of self maps (*A, S*) and (*B, T*) of metric space (X,d) is

(i) satisfy the common property (*E.A*) [7] if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$.

(ii) satisfy the (*$JCLR_{ST}$*) property (with respect to maps *S* and *T*) [4] if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Sz = Tz$ for some $z \in X$.

3. MAIN RESULTS

The main result of the paper is the following theorem.

Theorem 3.1: Let *A*, *B*, *S* and *T* be self maps of a metric space (*X*, *d*) satisfying the following:

(3.1)

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \alpha \int_0^{\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Ax,Ty)}{2}} \varphi(t)dt + b\int_0^{\frac{d(By,Sx)}{2}} \varphi(t)dt \right]$$

for all $x, y \in X$, where $0 \leq \alpha < 1$, $a \geq 0$, $b \geq 0$, $a+b<1$ and $\varphi : R^+ \to R^+$ is a Lebesgue integrable function which is summable, non-negative, and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$;

(3.2) $AX \subseteq TX$ and $BX \subseteq SX$;

(3.3) pair (*A, S*) or (*B, T*) satisfies the property (*E.A*)

(3.4) the range of one of the maps *A, B, S* or *T* is a closed subset of *X*.

Then pairs (*A, S*) and (*B, T*) have coincidence point. Further if (*A, S*) and (*B, T*) be weakly compatible pairs of self maps of metric space (*X*, *d*) then *A, B, S* and *T* have a unique common fixed point in *X*.

Proof: If the pair (*B, T*) satisfies the property (*E.A.*), then there exist a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Since, $BX \subseteq SX$, therefore, there exist a sequence $\{y_n\}$ in *X* such that $Bx_n = Sy_n$. Hence, $\lim_{n\to\infty} Sy_n = z$. Also, since $AX \subseteq TX$, there exist a sequence $\{z_n\}$ in *X* such that $Tx_n = Az_n$. Hence, $\lim_{n\to\infty} Az_n = z$.

Suppose that SX is a closed subset of X. Then $z = Su$ for some $u \in X$. Therefore, $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z = Su$.

We first claim that $Au = z$. If $Au \neq z$, then by using (3.1), take $x = u$, $y = x_n$, we get

$$\int_0^{d(Au,Bx_n)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Su,Tx_n),d(Au,Su),d(Bx_n,Tx_n)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Au,Tx_n)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Bx_n,Su)}{2}} \varphi(t)dt\right]$$

taking $n \to \infty$, we get

$$\begin{aligned}
\int_0^{d(Au,z)} \varphi(t)\,dt &\le \alpha \int_0^{\max\{d(z,z),d(Au,z),d(z,z)\}} \varphi(t)\,dt + (1-\alpha)\left[a\int_0^{\frac{d(Au,z)}{2}} \varphi(t)\,dt + b\int_0^{\frac{d(z,z)}{2}} \varphi(t)\,dt\right] \\
&= \alpha \int_0^{d(Au,z)} \varphi(t)\,dt + (1-\alpha)\left[a\int_0^{\frac{d(Au,z)}{2}} \varphi(t)\,dt\right] \\
&\le \alpha \int_0^{d(Au,z)} \varphi(t)\,dt + (1-\alpha)\left[a\int_0^{d(Au,z)} \varphi(t)\,dt\right] \\
&= \left[\alpha + (1-\alpha)\mathrm{a}\right]\left[\int_0^{d(Au,z)} \varphi(t)\,dt\right] \\
&< \left[\alpha + (1-\alpha)\right]\left[\int_0^{d(Au,z)} \varphi(t)\,dt\right] \\
&= \int_0^{d(Au,z)} \varphi(t)\,dt
\end{aligned}$$

which gives contradiction, hence $Au = z$.

Therefore, $Au = z = Su$ which shows that u is a coincidence point of the pair (A, S). As A and S are weakly compatible. Therefore, $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $AX \subseteq TX$, there exist v in X such that $Au = Tv$.
Now, we show that $Bv = z$. If $Bv \neq z$, then again by using (3.1), take $x = u$, $y = v$, we have

$$\int_0^{d(Au,Bv)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Su,Tv),d(Au,Su),d(Bv,Tv)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Au,Tv)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Bv,Su)}{2}} \varphi(t)dt\right]$$

$$\begin{aligned}
\int_0^{d(z,Bv)} \varphi(t)dt &\le \alpha \int_0^{\max\{d(z,z),d(z,z),d(Bv,z)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(z,z)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Bv,z)}{2}} \varphi(t)dt\right] \\
&= \alpha \int_0^{d(Bv,z)} \varphi(t)dt + (1-\alpha)\left[b\int_0^{\frac{d(Bv,z)}{2}} \varphi(t)dt\right] \\
&\le \alpha \int_0^{d(Bv,z)} \varphi(t)dt + (1-\alpha)\left[b\int_0^{d(Bv,z)} \varphi(t)dt\right] \\
&= \left[\alpha + (1-\alpha)b\right]\left[\int_0^{d(Bv,z)} \varphi(t)dt\right] \\
&< \left[\alpha + (1-\alpha)\right]\left[\int_0^{d(Bv,z)} \varphi(t)dt\right] = \int_0^{d(Bv,z)} \varphi(t)dt
\end{aligned}$$

which gives contradiction, hence $Bv = z$.

Therefore, $Bv = z = Au = Tv$ which shows that $Bv = Tv$, i.e., v is a coincidence point of the pair (B, T). As B and T are weakly compatible, therefore, $BTv = TBv$ and hence, $BTv = TBv = TTv = BBv$.

Next, we show that $AAu = Au$, if not, then again by using (3.1), take $x = Au$, $y = v$, we have

$$\int_0^{d(AAu,Bv)} \varphi(t)dt \le \alpha \int_0^{\max\{d(SAu,Tv),d(AAu,SAu),d(Bv,Tv)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(AAu,Tv)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Bv,SAu)}{2}} \varphi(t)dt \right]$$

$$\int_0^{d(AAu,z)} \varphi(t)dt \le \alpha \int_0^{\max\{d(AAu,z),0,d(z,z)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(AAu,z)}{2}} \varphi(t)dt + b\int_0^{\frac{d(z,AAu)}{2}} \varphi(t)dt \right]$$

$$\le \alpha \int_0^{d(AAu,z)} \varphi(t)dt + (1-\alpha)\left[a\int_0^{d(AAu,z)} \varphi(t)dt + b\int_0^{d(AAu,z)} \varphi(t)dt \right]$$

$$= \left[\alpha + (1-\alpha)(a+b)\right]\left[\int_0^{d(AAu,z)} \varphi(t)dt\right]$$

$$< \left[\alpha + (1-\alpha)\right]\left[\int_0^{d(AAu,z)} \varphi(t)dt\right] = \int_0^{d(AAu,z)} \varphi(t)dt$$

which gives contradiction, hence $AAu = Au$. Therefore, $AAu = Au = SAu$ and Au is a common fixed point of A and S. Similarly, we can prove that Bv is a common fixed point of B and T. As $Au = Bv$, we conclude that Au is a common fixed point of A, B, S and T.

The proof is similar when TX is assumed to be a closed subset of X. The cases in which AX or BX is a closed subset of X are similar to the cases in which TX or SX respectively, is closed since $AX \subseteq TX$ and $BX \subseteq SX$.

For uniqueness, let z and w be two fixed points of A, B, S and T. Then by (3.1), we have

$$\int_0^{d(Az,Bw)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Sz,Tw),d(Az,Sz),d(Bw,Tw)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Az,Tw)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Bw,Sz)}{2}} \varphi(t)dt \right]$$

$$\int_0^{d(z,w)} \varphi(t)dt \le \alpha \int_0^{\max\{d(z,w),d(z,z),d(w,w)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(z,w)}{2}} \varphi(t)dt + b\int_0^{\frac{d(w,z)}{2}} \varphi(t)dt \right]$$

$$\le \alpha \int_0^{d(z,w)} \varphi(t)dt + (1-\alpha)\left[a\int_0^{d(z,w)} \varphi(t)dt + b\int_0^{d(z,w)} \varphi(t)dt \right]$$

$$= \left[\alpha + (1-\alpha)(a+b)\right]\left[\int_0^{d(z,w)} \varphi(t)dt\right]$$

$$< \left[\alpha + (1-\alpha)\right]\left[\int_0^{d(z,w)} \varphi(t)dt\right] = \int_0^{d(z,w)} \varphi(t)dt$$

a contradiction, hence, $w = z$. It implies that A, B, S and T have unique common fixed point in X. Hence the result.

Now we attempt to drop containment of subspaces by replacing property (*E.A.*) by a weaker condition common property (*E.A.*) in Theorem 3.1.

Theorem 3.2: Let *A*, *B*, *S* and *T* be self mappings of a metric space (*X*, *d*) satisfying condition (3.1) of Theorem 3.1 and the following:

(3.5) the pair (*A*, *S*) and (*B*, *T*) share the common (*E.A.*) property;
(3.6) *SX* and *TX* are closed subsets of *X*.

Then the pairs (*A*, *S*) and (*B*, *T*) have a point of coincidence each. Further if (*A, S*) and (*B, T*) be weakly compatible pairs of self maps of metric space (*X*, *d*) then *A, B, S* and *T* have a unique common fixed point in *X*.

Proof: In view of (3.2), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$.

Since *SX* is a closed subset of *X*, therefore, there exists a point *u* in *X* such that $z = Su$.

We claim that $Au = z$. If $Au \neq z$, then by using (3.1), take $x = u$, $y = y_n$, we get

$$\int_0^{d(Au,By_n)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Su,Ty_n),d(Au,Su),d(By_n,Ty_n)\}} \varphi(t)dt$$
$$+(1-\alpha)\left[a\int_0^{\frac{d(Au,Ty_n)}{2}} \varphi(t)dt + b\int_0^{\frac{d(By_n,Su)}{2}} \varphi(t)dt \right]$$

taking $n \to \infty$, we get

$$\int_0^{d(Au,z)} \varphi(t)dt \le \alpha \int_0^{\max\{d(z,z),d(Au,z),d(z,z)\}} \varphi(t)dt$$
$$+(1-\alpha)\left[a\int_0^{\frac{d(Au,z)}{2}} \varphi(t)dt + b\int_0^{\frac{d(z,z)}{2}} \varphi(t)dt \right]$$
$$= \alpha \int_0^{d(Au,z)} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Au,z)}{2}} \varphi(t)dt \right]$$
$$\le \alpha \int_0^{d(Au,z)} \varphi(t)dt + (1-\alpha)\left[a\int_0^{d(Au,z)} \varphi(t)dt \right]$$
$$= \left[\alpha + (1-\alpha)\mathrm{a}\right]\left[\int_0^{d(Au,z)} \varphi(t)dt\right]$$
$$< \left[\alpha + (1-\alpha)\right]\left[\int_0^{d(Au,z)} \varphi(t)dt\right]$$
$$= \int_0^{d(Au,z)} \varphi(t)dt$$

which gives contradiction, hence $Au = z$.

Therefore, $Au = z = Su$ which shows that *u* is a coincidence point of the pair (*A*, *S*).

Since *TX* is also a closed subset of *X*, therefore $\lim_{n\to\infty} Ty_n = z$ in *TX* and hence there exists *v* in *X* such that $Tv = z = Au = Su$. Now, we show that $Bv = z$.

If not, then by using (3.1), take $x = u$, $y = v$, we have

$$\int_0^{d(Au,Bv)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Su,Tv),d(Au,Su),d(Bv,Tv)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Au,Tv)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Bv,Su)}{2}} \varphi(t)dt \right]$$

$$\int_0^{d(z,Bv)} \varphi(t)dt \le \alpha \int_0^{\max\{d(z,z),d(z,z),d(Bv,z)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(z,z)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Bv,z)}{2}} \varphi(t)dt \right]$$

$$= \alpha \int_0^{d(Bv,z)} \varphi(t)dt + (1-\alpha)\left[b\int_0^{\frac{d(Bv,z)}{2}} \varphi(t)dt \right]$$

$$\le \alpha \int_0^{d(Bv,z)} \varphi(t)dt + (1-\alpha)\left[b\int_0^{d(Bv,z)} \varphi(t)dt \right]$$

$$= \left[\alpha + (1-\alpha)b\right]\left[\int_0^{d(Bv,z)} \varphi(t)dt \right]$$

$$< \left[\alpha + (1-\alpha)\right]\left[\int_0^{d(Bv,z)} \varphi(t)dt \right] = \int_0^{d(Bv,z)} \varphi(t)dt$$

which gives contradiction, hence $Bv = z$.

Therefore, $Bv = z = Tv$ which shows that v is a coincidence point of the pair (B, T).
Since the pairs (A, S) and (B, T) are weakly compatible and $Au = Su$, $Bv = Tv$, therefore, $Az = ASu = SAu = Sz$, $Bz = BTv = TBv = Tz$.

Next, we claim that $Az = z$. If not, then again using (3.1), take $x = z$, $y = v$, we have

$$\int_0^{d(Az,Bv)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Sz,Tv),d(Az,Sz),d(Bv,Tv)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Az,Tv)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Bv,Sz)}{2}} \varphi(t)dt \right]$$

$$\int_0^{d(Az,z)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Az,z),0,0\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Az,z)}{2}} \varphi(t)dt + b\int_0^{\frac{d(z,Az)}{2}} \varphi(t)dt \right]$$

$$\le \left[\alpha + (1-\alpha)(a+b)\right]\left[b\int_0^{d(Az,z)} \varphi(t)dt \right]$$

$$< \left[\alpha + (1-\alpha)\right]\left[\int_0^{d(Az,z)} \varphi(t)dt \right] = \int_0^{d(Az,z)} \varphi(t)dt$$

which gives contradiction, hence $Az = z$. Therefore, $Az = z = Sz$.

Similarly, one can prove that $Bz = Tz = z$. Hence, $Az = Bz = Sz = Tz$, and z is common fixed point of A, B, S and T.
Uniqueness easily follows by the use of inequality (3.1). Hence the result.

Now we attempt to drop containment of subspaces by using weaker condition $JCLR_{ST}$ property in Theorem 3.2.

Theorem 3.3: Let A, B, S and T be four selfmaps in metric space (X, d) satisfying condition (3.1) of Theorem 3.1 and (3.7) (A, S) and (B, T) shares the $JCLR_{ST}$ property.

Then pairs (A, S) and (B, T) have coincidence point. Further if (A, S) and (B, T) be weakly compatible pairs of self maps of X then A, B, S and T have a unique common fixed point in X.

Proof: The pairs (*A, S*) and (*B*, *T*) satisfy the ($JCLR_{ST}$) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Su = Tu \text{ for some } u \in X .$$

First we claim that $Tu = Bu$. Suppose not, then again by using (3.1), take $x = x_n$, $y = u$, we easily get, a contradiction, hence $Tu = Bu$.

Next, we show that $Au = Tu$. Suppose not, then again as done above, by using (3.1), take $x = u$, $y = y_n$, we get a contradiction, hence $Au = Tu$. Hence, $Au = Bu = Su = Tu = z$ (say). Since the pair (*A, S*) is weakly compatible, $ASu = SAu$ and then $Az = Sz$. Similarly, as the pair (*B, T*) is weakly compatible, $BTu = TBu$ and then $Tz = Bz$.

Next, we claim that $Az = z$, suppose not. Then again as done above, by taking take $x = z$, $y = u$, in (3.1), we can easily get a contradiction, hence, $Az = Bz = z$. Therefore, *z* is a common fixed point of *A* and *B*. Similarly, we prove that $Sz = Tz = z$ by taking $x = u$, $y = z$ in (3.1). Therefore, we conclude that $z = Az = Bz = Sz = Tz$ this implies that *A, B, S* and *T* have common fixed point in *X*.

Uniqueness easily follows by the use of inequality (3.1).

On taking $A = B$ and $S = T$ in Theorem 3.1 then we get the following interesting result:

Corollary 3.1: Let *A* and *S* be self maps of a metric space (*X*, *d*) satisfying the following:
(3.8)

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Sx,Sy),d(Ax,Sx),d(Ay,Sy)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Ax,Sy)}{2}} \varphi(t)dt + b\int_0^{\frac{d(Ay,Sx)}{2}} \varphi(t)dt \right]$$

for all $x, y \in X$, where $0 \le \alpha < 1$, $a \ge 0$, $b \ge 0$, $a + b < 1$ and $\varphi : R^+ \to R^+$ is a Lebesgue integrable function which is summable, non-negative, and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$;

(3.9) pair (*A, S*) satisfies the property (*E.A*)
(3.10) *S*(*X*) is a closed subset of *X*.

Then pair (*A, S*) has a coincidence point. Further if (*A*, *S*) be weakly compatible pair of self maps of metric space (*X*, *d*) then *A* and *S* has a unique common fixed point in *X*.

On taking $A = B$ and $S = T$ in Theorem 3.4 then we get the following interesting result:

Corollary 3.2: Let *A* and *S* be self maps of a metric space (*X*, *d*) satisfying the following:

(3.14) (*A, S*) satisfies the CLR_S property.

Then pair (*A, S*) has a coincidence point. Further if (*A*, *S*) be weakly compatible pair of self maps of metric space (*X*, *d*) then *A* and *S* has a unique common fixed point in *X*.

Finally, we conclude this paper by furnishing example to demonstrate Theorem 3.3 besides exhibiting its superiority over earlier relevant results.

Example 3.2. Let (*X*, *d*) be a metric space where $X = [3, 19)$. Let $\phi : (0,1] \to (0,1]$ be defined as $\phi(t) = t$ for all $t \in R^+$. Define *A, B, S* and *T* by

$$Ax = \begin{cases} 1, & x \in \{1\} \cup (3,15) \\ x+11, & x \in (1,3] \end{cases}, \quad Bx = \begin{cases} 1, & x \in \{1\} \cup (3,15) \\ x+5, & x \in (1,3] \end{cases}$$

$$Sx = \begin{cases} 1, & x=1 \\ 6, & x \in (1,3] \\ \dfrac{x+1}{4}, & x \in (3,15) \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, & x=1 \\ 11, & x \in (1,3] \\ x-2, & x \in (3,15). \end{cases}$$

Take $\{x_n\} = \{y_n\} = \left\{3 + \frac{1}{n}\right\}$, clearly

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 = S1 = T1$, where $1 \in X$ .Thus, (*A, S*) and (*B, T*) satisfies $JCLR_{ST}$ property. Also, $AX = \{1\} \cup (12,14]$, $BX = \{1\} \cup (6,8]$,

$SX = [1,4) \cup \{6\}$, $TX = (1, 13)$ and condition (3.1) is satisfied by maps *A, B, S* and *T*. Thus, the self maps *A, B, S* and *T* satisfy all the conditions of Theorem 3.3. Hence *A, B, S* and *T* have a unique common fixed point $x = 1$.Moreover it should be noted that *AX*, *BX*, *SX* and *TX* are not closed subsets of *X*. Also, $AX \not\subset TX$ and $BX \not\subset SX$. Also, *A*, *B*, *S* and *T* are all discontinuous maps at fixed point $x = 1$. Careful examination of example reveals that it cannot be covered by those coincidence and common fixed point theorems in which involved maps are continuous or underlying subspaces are complete/closed or containment of ranges amongst involved subspaces is essential.

Remark 3.1. The conclusions of Theorem 3.1- 3.4 remain true if we replace the inequality (3.1) by any one of the following (besides retaining the rest of hypotheses):

(A)

$$\int_0^{d(Ax,By)} \varphi(t)dt \le \alpha \int_0^{L(x,y)} \varphi(t)dt$$

where $L(x,y) = \max\left\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \dfrac{d(Ax,Ty)}{2}, \dfrac{d(By,Sx)}{2}\right\}$

for all $x, y \in X$, where $0 \le \alpha < 1$, and $\varphi : R^+ \to R^+$ is a Lebesgue integrable function which is summable, non-negative, and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$.

(B)

$$\int_0^{d(Ax,By)} \varphi(t)dt \le \psi \left(\int_0^{L(x,y)} \varphi(t)dt\right)$$

where $L(x,y) = \max\left\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \dfrac{d(Ax,Ty)}{2}, \dfrac{d(By,Sx)}{2}\right\}$

for all $x, y \in X$, where $0 \le \alpha < 1$, $\varphi : R^+ \to R^+$ is a Lebesgue integrable function which is summable, non-negative, and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$ and $\psi : R^+ \to R^+$ is non-decreasing function such that $\psi(0) = 0,\ \psi(t) < t$ for $t > 0$.

Remark 3.2: Theorem 3.1, 3.2, 3.3 and 3.4 generalizes the result of Aliouche [2, Theorem 3.5] and references there in. On taking $\varphi(t) = 1$ in Theorem 3.1, we get the following corollary:

Corollary 3.3: Let *A*, *B*, *S* and *T* be self maps of a metric space (*X*, *d*) satisfying the following conditions (3.2), (3.3), (3.4) and:

(3.1)

$$d(Ax,By) \le \alpha \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty)\} + (1-\alpha)\left[a\frac{d(Ax,Ty)}{2} + b\frac{d(By,Sx)}{2}\right]$$

for all $x, y \in X$, where $0 \le \alpha < 1$, $a \ge 0$, $b \ge 0$, $a + b < 1$.

Then pairs (*A, S*) and (*B, T*) have coincidence point. Further if (*A, S*) and (*B, T*) be weakly compatible pairs of self maps of metric space (*X*, *d*) then *A, B, S* and *T* have a unique common fixed point in *X*.

Similarly, on taking $\varphi(t) = 1$ in Theorem 3.2, 3.3 and 3.4, we get various corollaries and of which several fixed point theorems in the literature are special cases.

Remark 3.3: The conclusions of Theorem 3.1- 3.4 remain true if we replace the inequality (3.1) by any one of the following:

(i) $\int_0^{d(Ax,By)} \varphi(t)dt \le \int_0^{\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty)\}} \varphi(t)dt$ for all $x, y \in X$;

(ii) $\int_0^{d(Ax,By)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty)\}} \varphi(t)dt$

for all $x, y \in X$, where $0 \le \alpha < 1$;

(iii)

$$\int_0^{d(Ax,By)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty)\}} \varphi(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(Ax,Ty)}{2}} \varphi(t)dt \right]$$

for all $x, y \in X$, where $0 \le \alpha < 1$, $0 \le a < 1$.

4. APPLICATION

Definition 4.1 Two families of self maps $\{A_i\}_{i=1}^m$ and $\{B_j\}_{j=1}^n$ are said to be pairwise commuting if

(i) $A_i A_j = A_j A_i$, $i, j \in \{1,2,3,...m\}$,

(ii) $B_i B_j = B_j B_i$, $i, j \in \{1,2,3,...n\}$,

(iii) $A_i B_j = B_j A_i$, $i \in \{1,2,3,...n\}$, $j \in \{1,2,3,...n\}$.

As an application of Theorem 3.2, we prove a common fixed point theorem for four finite families of maps on metric spaces. While proving our result, we utilize Definition 3.1, which is a natural extension of commutativity condition to two finite families.

Theorem 3.5: Let $\{A_1, A_2, ..., A_m\}$, $\{B_1, B_2, ..., B_n\}$, $\{S_1, S_2, ..., S_p\}$ and $\{T_1, T_2, ..., T_q\}$ be four finite families of self maps of a metric space (*X*, *d*) such that $A = A_1.A_2.....A_m$, $B = B_1.B_2.....B_n$, $S = S_1.S_2.....S_p$ and $T = T_1.T_2.....T_q$ satisfying the conditions (3.1), (3.5), (3.6) and

(3.12) the pairs of families $(\{A_i\},\{S_k\})$ and $(\{B_r\},\{T_t\})$ commute pairwise.

Then the pairs (A,S) and (B,T) have a point of coincidence each. Moreover, $\{A_i\}_{i=1}^m, \{S_k\}_{k=1}^p$, $\{B_r\}_{r=1}^n$ and $\{T_t\}_{t=1}^q$ have a unique common fixed point.

Proof: By using (3.15), we first show that $AS = SA$ as

$AS = (A_1A_2 \ldots A_m)(S_1S_2 \ldots S_p) = (A_1A_2 \ldots A_{m-1})(A_m S_1S_2 \ldots S_p)$
$= (A_1A_2 \ldots A_{m-1})(S_1S_2 \ldots S_p A_m) = (A_1A_2 \ldots A_{m-2})(A_{m-1} S_1S_2 \ldots S_p A_m)$
$= (A_1A_2 \ldots A_{m-2})(S_1S_2 \ldots S_p A_{m-1}A_m) = \ldots = A_1(S_1S_2 \ldots S_p A_2 \ldots A_m)$
$= (S_1S_2 \ldots S_p)(A_1A_2 \ldots A_m) = SA$.

Similarly one can prove that $BT = TB$. And hence, obviously the pair (*A*, *S*) is compatible and (*B*, *T*) is weakly compatible. Now using Theorem 3.2, we conclude that *A*, *S*, *B* and *T* have a unique common fixed point in *X*, say *z*.

Now, one needs to prove that *z* remains the fixed point of all the component mappings.

For this consider

$A(A_i z) = ((A_1A_2 \ldots A_m)A_i)z = (A_1A_2 \ldots A_{m-1})(A_m A_i)z$
$= (A_1A_2 \ldots A_{m-1})(A_i A_m)z = (A_1A_2 \ldots A_{m-2})(A_{m-1} A_i A_m)z$
$= (A_1A_2 \ldots A_{m-2})(A_i A_{m-1} A_m)z = \ldots = A_1(A_iA_2 \ldots A_m)z$
$= (A_1A_i)(A_2 \ldots A_m)z$
$= (A_iA_1)(A_2 \ldots A_m)z = A_i(A_1A_2 \ldots A_m)z = A_i Az = A_i z$.

Similarly, one can prove that

$A(S_k z) = S_k (Az) = S_k z$, $S(S_k z) = S_k(Sz) = S_k z$,
$S(A_i z)= A_i(Sz) = A_i z$, $B(B_r z) = B_r(Bz) = B_r z$,
$B(T_t z) = T_t(Bz) = T_t z$, $T(T_t z) = T_t(Tz) = T_t z$ and $T(B_r z) = B_r(Tz) = B_r z$,
which show that (for all i, r, k and t) $A_i z$ and $S_k z$ are other fixed point of the pair (A, S) whereas $B_r z$ and $T_t z$ are other fixed points of the pair (B, T). As A, B, S and T have a unique common fixed point, so, we get

$$z = A_i z = S_k z = B_r z = T_t z, \quad \text{for all } i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, p, \quad r = 1, 2, \ldots, n, \quad t = 1, 2, \ldots, q$$

which shows that z is a unique common fixed point of $\{A_i\}_{i=1}^{m}$, $\{S_k\}_{k=1}^{p}$, $\{B_r\}_{r=1}^{n}$ and $\{T_t\}_{t=1}^{q}$.

Remark 4.1: Theorem 4.1 is a slight but partial generalization of Theorem 3.2 as the commutativity requirements in this theorem are slightly stronger as compared to Theorem 3.2.

Remark 4.2: From the above results, it is asserted that for the existence of common fixed point of two pairs of self maps in metric spaces satisfying $JCLR_{ST}$ property the following conditions are never required:

(a) the containment of ranges amongst the involved maps;
(b) the completeness of the whole space/subspace;
(c) the closedness of space/subspaces;
(d) continuity requirement amongst the involved maps.
(e)

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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