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Temporal Differential Transform and Spatial Finite Difference Methods for Unsteady Heat Conduction Equations with Anisotropic Diffusivity

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ABSTRACT

Three unsteady heat conduction problems with anisotropic diffusivity and time-dependent heating or heat flux and/or heat source are considered in showing the utility of a hybrid method involving a combination of temporal differential transform and spatial finite difference methods. The segregation of time from the spatial component is the greatest advantage of the hybrid method that exhibits no instability of finite difference methods generally seen with parabolic equations. The easy-to-implement algorithm that is essentially a Poisson solver works with both linear and non-linear heat transport problems without any difficulty of sorts. To gain confidence in the results some simulation results are also presented of problems that have an Adomian solution. The method can be used in practical heat transfer problems concerning non-uniform materials like composites, alloys, heterogeneous porous media with thermal equilibrium or non-equilibrium, multi-layered media and such other problems.

Keywords: Differential Transform Method, Finite-difference approximation, Heat conduction, anisotropic diffusivity, time-dependent heating.

1. INTRODUCTION

Thermal engineering applications in homogeneous materials are mentioned in many engineering books dealing with heat transfer. The parabolic heat transport equations in these cases have generally constant coefficients. Over the years with the introduction of new materials like alloys, composite, fluid-saturated porous media, multi-layer media and such other heterogeneous working media, heat transport equations began to have variable coefficients. These mathematical models due to practical interest could no longer be solved analytically by classroom mathematical tools. The numerical methods of finite difference that began to be used with these suffered many problems of instability and thereby of convergence. Obtaining stability criterion for these problems was so difficult due to the variable coefficient in them.

With the introduction of the method of differential transforms proposed by Zhou [1] for electrical circuits dawned a new horizon of possibilities. Several different engineering problems were then solved by these methods [2-8]. Then many researchers began to use the hybrid methods which are a combination of two or more methods.

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The present paper is one such hybrid method that combines the best features of the differential transform and finite difference methods. Three practical examples are chosen for illustration of the easy-to-implement method that does not suffer from instability and non-convergence. We now consider the three examples one by one.

2. UNSTEADY, ONE-DIMENSIONAL HEAT CONDUCTION EQUATION WITH VARIABLE DIFFUSIVITY, TIME-DEPENDENT DIRICHLET BOUNDARY CONDITIONS AND SPACE-DEPENDENT INITIAL CONDITION

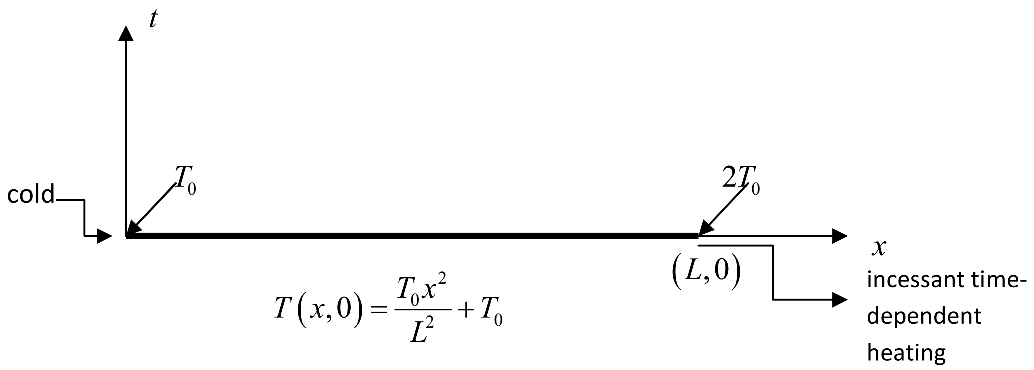


Figure 1: Schematic of the one-dimensional, unsteady heat conduction problem.

A heterogeneous, one-dimensional rod of length L is considered as shown in Figure 1. It is assumed that the thermal diffusivity varies as $\frac{\chi_o}{2}\left(\frac{x}{L}\right)^2$, where χ_o is a constant. One end of the rod is assumed to be maintained at ambient temperature while the other end is assumed to be incessantly and increasingly heated as time progresses. The heating is assumed to be an exponential function of time t, namely, $T_o e^{\frac{\chi_o}{L^2}\tau}$, where T_o is the ambient temperature. The deviation of the initial temperature distribution from T_o in the rod is assumed to vary as the square of x. It is further assumed that at all other points of the rod there is thermal insulation. The mathematical model using the Fourier second law thus is:

$$\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\frac{\chi_o}{2}\left(\frac{x}{L}\right)^2\frac{\partial T}{\partial x}\right),\ 0<x<L,\ \tau>0, \tag{2.1}$$

subject to

$$\left.\begin{aligned} &T=T_o && \text{at}\ \ x=0,\ \forall\tau>0,\\ &T=T_o+T_o e^{\frac{\chi_o}{L^2}\tau} && \text{at}\ \ x=L,\ \forall\tau>0,\\ &T=T_o+T_o\left(\frac{x}{L}\right)^2 && \text{at}\ \ \tau=0,\ \forall x\in[0,L]. \end{aligned}\right\} \tag{2.2}$$

The following non-dimensional variables are now defined to render the equations (2.1) and (2.2) dimensionless:

$$t=\frac{\tau\chi_o}{L^2},\ \ X=\frac{x}{L},\ u=\frac{T-T_o}{T_o}. \tag{2.3}$$

Using equation (2.3) in equations (2.1) and (2.2), the following equation is obtained on completing the differentiation:

$$\frac{\partial u}{\partial t}=X\frac{\partial u}{\partial X}+\frac{X^2}{2}\frac{\partial^2 u}{\partial X^2},\ \ 0<X<1,\ t>0, \tag{2.4}$$

subject to

$$\left.\begin{aligned} &u(0,t)=0,\ \ \forall t>0,\\ &u(1,t)=e^t,\ \ \forall t>0,\\ &u(X,0)=X^2,\ \ \forall X\in[0,1]. \end{aligned}\right\} \tag{2.5}$$

It must be noted here that there is no analytical solution to the above initial-boundary value problem (IBVP).

Taking the temporal differential transform of equation (2.4) results in the following differential-difference equation:

$$(k+1)U[X,k+1] = X\frac{\partial U}{\partial X}[X,k] + \frac{X^2}{2}\frac{\partial^2 U}{\partial X^2}[X,k], \quad (k=0,1,2,...), 0<X<1. \tag{2.6}$$

After having explicitly segregated the time derivative from the spatial component, the interval $[0,1]$ is now discretized using the following:

$$X_i = i\Delta X = \frac{i}{N}, \ i = 0(1)N. \tag{2.7}$$

Denoting the value of $U[X_i,k]$ by $U_i[k]$, the central difference approximation to the derivatives in equation (2.6) gives rise to the following difference equation:

$$(k+1)U_i[k+1] = \frac{i}{2}\{U_{i+1}[k]-U_{i-1}[k]\} + \frac{i^2}{2}\{U_{i+1}[k]-2U_i[k]+U_{i-1}[k]\}, \tag{2.8}$$
$$(k=0,1,2,...), (1<i<N-1).$$

Taking the temporal differential transform of the equations in (2.5), the following equations are obtained:

$$U_0[k]=0, \quad (k=0,1,2,...), \tag{2.9}$$

$$U_N[k]=\frac{1}{k!}, \quad (k=0,1,2,...), \tag{2.10}$$

$$U_i[0]=\frac{i^2}{N^2}, \quad (0<i<N). \tag{2.11}$$

At each time step, i.e., for each value of k, up to the desired value of time, the algebraic equation (2.8) is solved to obtain the $U_i[k]$'s. This means that at each discretized time we get the solution for $U[x_i,k]$ and the same may be used in obtaining the solution of $u(x_i,t)$ at the spatially discretized points using the inverse differential transform that gives us:

$$u[x_i,t]=\sum_{k=0}^{\infty}U[x_i,k]t^k, \ 0<i<N. \tag{2.12}$$

In the computation of results, 15 terms were taken in the time series (2.12) and with 40 internal spatially discretized points, i.e., N=40.

We note here that there is no restriction on reason of stability in choosing the time step and spatial step length in the hybrid method. One, however, has to note that the radius of convergence may be looked into. This problem can also be overcome in the hybrid method by dividing the time-domain into a number of sub-domains as suggested by Yu and Chen [9] in the context of the differential transform solution of the Blasius equation and used subsequently by Odibat et al. [10] extensively in his works. Further, Jang et al. [5] proved quite rigorously that the differential transform method has errors within bounds and is thereby stable and convergent (by Lax equivalence theorem). Let us consider a sample computation now and say we need to find the solution at $t=5$. Let us assume N=40, i.e., 40 spatially discretized points $x_0, x_1, x_2, x_3, ..., x_{40}$. Further, let us divide the time interval into 20 sub-intervals, say, [0,0.25], [0.25,0.5], [0.5,0.75], [0.75,1.0], ………., [4.75,5.0]. We first use the hybrid method and obtain the solution $u[x_i,t_k]$ at $t=0.25$ and at all the spatially discretized points. Using these values as the initial values in equations (2.9)-(2.11), we now compute the values at the next time step $t=0.5$ at $x_1, x_2, x_3, ..., x_{40}$. The above procedure used for one-dimensional heat conduction equation is generalized and used in two- and three-dimensional problems discussed below.

3. UNSTEADY, TWO-DIMENSIONAL HEAT CONDUCTION EQUATION WITH ANISOTROPIC DIFFUSIVITY AND WITH TIME- DEPENDENT NEUMANN BOUNDARY CONDITIONS AND SPATIALLY - DEPENDENT DIRICHLET INITIAL CONDITION

A heterogeneous, rectangular plate of length L and breadth B is considered as shown in Figure 2.

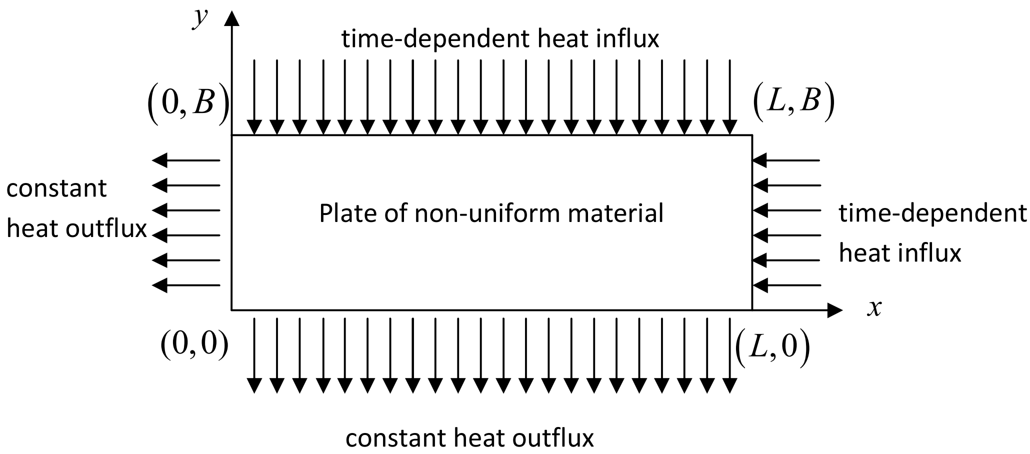


Figure 2: Schematic of the two-dimensional, unsteady heat conduction problem.

It is assumed that the thermal diffusivity is a tensor given by

$$\underline{\chi} = \begin{bmatrix} \frac{\chi_0}{2}\left(\frac{x}{L}\right)^2 & 0 \\ 0 & \frac{\chi_0}{2}\left(\frac{y}{L}\right)^2 \end{bmatrix}. \tag{3.1}$$

Thermal insulation is maintained on two sides and time-dependent heat influx and outflux are maintained at the other two sides, as shown in the Figure 2. The initial heat flux is assumed to vary quadratically with y as explained further on. The mathematical model for the problem using Fourier second law thus is

$$\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\frac{\chi_o}{2}\left(\frac{x}{L}\right)^2 \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\chi_o}{2A^2}\left(\frac{y}{B}\right)^2 \frac{\partial T}{\partial y}\right),\ 0 < x < L,\ 0 < y < B,\ \tau > 0, \tag{3.2}$$

subject to

$$-\chi_0 \frac{\partial T}{\partial x} = Q_0 \qquad \text{at} \quad x = 0,\ \forall\, y \in [0, B],\ \forall\, \tau > 0, \tag{3.3}$$

$$-\chi_0 \frac{\partial T}{\partial y} = -Q_0 \qquad \text{at} \quad y = 0,\ \forall\, x \in [0, L],\ \forall\, \tau > 0, \tag{3.4}$$

$$-\chi_0 \frac{\partial T}{\partial x} = Q_0 - 2Q_0 \sinh\left(\frac{\chi_0}{L^2}\tau\right) \qquad \text{at} \quad x = L,\ \forall\, y \in [0, B],\ \forall\, \tau > 0, \tag{3.5}$$

$$-\chi_0 \frac{\partial T}{\partial y} = -AQ_0 + 2AQ_0 \cosh\left(\frac{\chi_0}{L^2}\tau\right) \quad \text{at} \quad y = B,\ \forall x \in [0, L],\ \forall \tau > 0, \tag{3.6}$$

$$T = \frac{Q_o L}{\chi_0} + \frac{Q_o L}{2\chi_0}\left(\frac{y}{B}\right)^2 \quad \text{at} \quad \tau = 0,\ \forall x \in [0, L], \forall y \in [0, B], \tag{3.7}$$

where $A = \dfrac{L}{B}$ is the aspect ratio.

The following non-dimensional variables are now defined to render the equations (3.2)-(3.7) dimensionless:

$$t = \frac{\tau \chi_o}{L^2},\ X = \frac{x}{L},\ Y = \frac{y}{B},\ u = \frac{T\chi_o}{Q_o L}. \tag{3.8}$$

Using equation (3.8) in equations (3.2)-(3.7), the following equations are obtained on completing the differentiation:

$$\frac{\partial u}{\partial t} = X\frac{\partial u}{\partial X} + Y\frac{\partial u}{\partial Y} + \frac{X^2}{2}\frac{\partial^2 u}{\partial X^2} + \frac{Y^2}{2}\frac{\partial^2 u}{\partial Y^2},\ 0 < X < 1,\ 0 < Y < 1, t > 0, \tag{3.9}$$

subject to

$$\frac{\partial u}{\partial X} = -1 \quad \text{at} \quad X = 0,\ \forall Y \in [0,1],\ \forall t > 0, \tag{3.10}$$

$$\frac{\partial u}{\partial Y} = 1 \quad \text{at} \quad Y = 0,\ \forall X \in [0,1],\ \forall t > 0, \tag{3.11}$$

$$\frac{\partial u}{\partial X} = -1 + 2\sinh(t) \quad \text{at} \quad X = 1,\ \forall Y \in [0,1],\ \forall t > 0, \tag{3.12}$$

$$\frac{\partial u}{\partial Y} = 1 - 2\cosh(t) \quad \text{at} \quad Y = 1,\ \forall X \in [0,1],\ \forall t > 0, \tag{3.13}$$

$$u = 1 + Y^2 \quad \text{at} \quad t = 0,\ \forall X \in [0,1], \forall Y \in [0,1]. \tag{3.14}$$

It must be noted here that there is no analytical solution to the above IBVP. The hybrid method as explained in the previous section is used to solve the IBVP of equations (3.9)-(3.14).

An unsteady three-dimensional heat conduction problem will now be considered with thermal anisotropy tensor, volumetric heat source and space- and time-dependent initial and boundary conditions of Dirichlet type.

4. UNSTEADY, THREE-DIMENSIONAL HEAT CONDUCTION EQUATION WITH ANISOTROPIC DIFFUSIVITY TENSOR, VOLUMETRIC HEAT SOURCE AND TIME- AND SPACE - DEPENDENT INITIAL AND BOUNDARY CONDITIONS

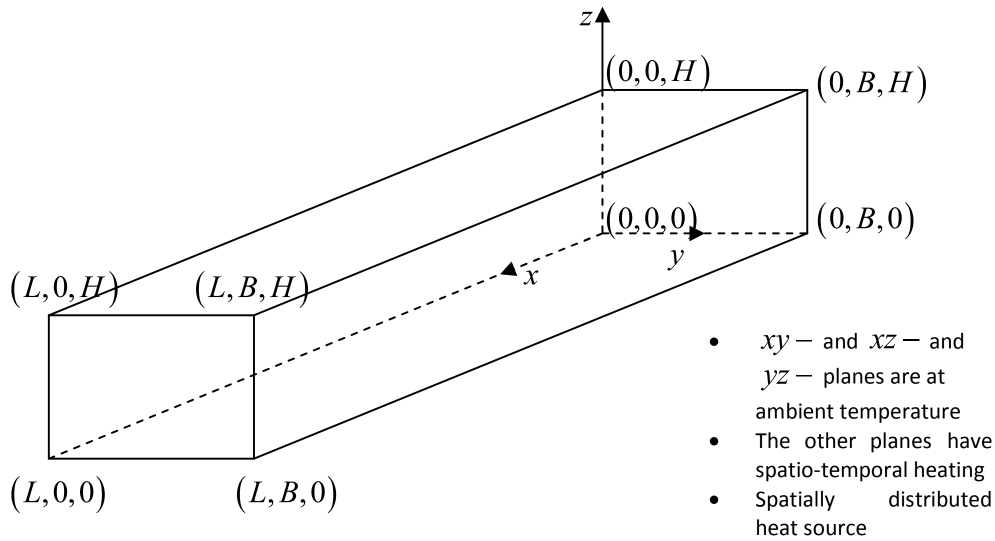


Figure 3: Schematic of the three-dimensional, unsteady heat conduction problem.

A heterogeneous rod of rectangular cross-section is considered as shown in Figure 3. It is assumed that the thermal diffusivity is a tensor given by:

$$\underline{\chi} = \begin{bmatrix} \frac{\chi_0}{2}\left(\frac{x}{L}\right)^2 & 0 & 0 \\ 0 & \frac{\chi_0}{2}\left(\frac{y}{L}\right)^2 & 0 \\ 0 & 0 & \frac{\chi_0}{2}\left(\frac{z}{L}\right)^2 \end{bmatrix}. \tag{4.1}$$

The mathematical model for this problem using Fourier second law is:

$$\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\frac{\chi_o}{2}\left(\frac{x}{L}\right)^2 \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\chi_o}{2A_1^2}\left(\frac{y}{L}\right)^2 \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\chi_o}{2A_2^2}\left(\frac{z}{L}\right)^2 \frac{\partial T}{\partial z}\right) + \frac{S\chi_o T_0}{2L^2}\left(\frac{x}{L}\frac{y}{B}\frac{z}{H}\right)^2,$$
$$0 < x < L,\ 0 < y < B, 0 < z < H,\ \tau > 0, \tag{4.2}$$

subject to the following initial and boundary conditions whose description is similar to that in the one-dimensional problem:

$$T = T_o \quad \text{at} \quad x = 0,\ \forall\ y \in [0, B],\ \forall\ z \in [0, H],\ \forall \tau > 0, \tag{4.3}$$

$$T = T_o \quad \text{at} \quad y = 0,\ \forall\ x \in [0, L],\ \forall\ z \in [0, H],\ \forall \tau > 0, \tag{4.4}$$

$$T = T_o \quad \text{at} \quad z = 0,\ \forall\ x \in [0, L],\ \forall\ y \in [0, B],\ \forall \tau > 0, \tag{4.5}$$

$$T = T_o + T_o\left(\frac{yz}{BH}\right)^4\left(e^{\frac{\chi_o}{L^2}\tau} - 1\right) \text{ at } x = L, \mathrm{y} \in [0, B],\ \mathrm{z} \in [0, H], \forall \tau > 0, \tag{4.6}$$

$$T = T_o + T_o\left(\frac{xz}{LH}\right)^4\left(e^{\frac{\chi_o}{L^2}\tau} - 1\right) \text{ at } y = B,\ \mathrm{x} \in [0, L], \mathrm{z} \in [0, H], \forall \tau > 0, \tag{4.7}$$

$$T = T_o + T_o\left(\frac{xy}{LB}\right)^4\left(e^{\frac{\chi_o}{L^2}\tau} - 1\right) \text{ at } z = H, x \in [0, L],\ y \in [0, B], \forall \tau > 0, \tag{4.8}$$

$$T = T_o \text{ at } \tau = 0,\ \forall x \in [0, L], \forall y \in [0, B], \forall z \in [0, H], \tag{4.9}$$

where $A_1 = \dfrac{B}{L}$, $A_2 = \dfrac{H}{L}$ and S are the aspect ratios and a non-dimensional constant.

The following non-dimensional variables are now defined to render the equations (4.2) to (4.9) dimensionless:

$$t = \frac{\tau\chi_o}{L^2},\ X = \frac{x}{L},\ X = \frac{y}{B},\ Z = \frac{z}{H},\ u = \frac{T - T_o}{T_o}. \tag{4.10}$$

Using equation (4.10) in equations (4.2) to (4.9), the following equations are got on completing the differentiation:

$$\frac{\partial u}{\partial t} = X\frac{\partial u}{\partial X} + Y\frac{\partial u}{\partial Y} + Z\frac{\partial u}{\partial Z} + \frac{X^2}{2}\frac{\partial^2 u}{\partial X^2} + \frac{Y^2}{2}\frac{\partial^2 u}{\partial Y^2} + \frac{Z^2}{2}\frac{\partial^2 u}{\partial Z^2} + S(XYZ)^2, \quad 0 < X < 1,\ 0 < Y < 1,\ 0 < Z < 1,\ t > 0, \tag{4.11}$$

subject to

$$u(0, Y, Z, t) = 0,\ Y \in [0,1],\ Z \in [0,1],\ \forall t > 0, \tag{4.12}$$

$$u(X, 0, Z, t) = 0,\ X \in [0,1],\ Z \in [0,1], \forall t > 0, \tag{4.13}$$

$$u(X, Y, 0, t) = 0,\ X \in [0,1],\ Y \in [0,1], \forall t > 0, \tag{4.14}$$

$$u(1, Y, Z, t) = (YZ)^4\left(e^t - 1\right),\ Y \in [0,1],\ Z \in [0,1], \forall t > 0, \tag{4.15}$$

$$u(X, 1, Z, t) = (XZ)^4\left(e^t - 1\right),\ X \in [0,1],\ Z \in [0,1], \forall t > 0, \tag{4.16}$$

$$u(X, Y, 1, t) = (XY)^4\left(e^t - 1\right),\ X \in [0,1],\ Y \in [0,1], \forall t > 0, \tag{4.17}$$

$$u(X, Y, Z, 0) = 0,\ \forall X, Y, Z \in [0,1]. \tag{4.18}$$

It must be noted here that there is no analytical solution to the above IBVP. The hybrid method as explained in a previous section is used to solve the IBVP of equations (4.11)-(4.18). Before we move on to the discussion of results we first note the existence in literature of the Adomian solution for some particular forms of the three IBVPs considered in our study. These IBVPs are documented in the next section.

5. NUMERICAL EXAMPLES

IBVP1

$$\frac{\partial u}{\partial t} = \frac{X^2}{2}\frac{\partial^2 u}{\partial X^2},\ 0 < X < 1,\ t > 0, \tag{5.1}$$

subject to

$$\left.\begin{aligned} &u(0,t)=0,\ \forall t>0,\\ &u(1,t)=e^t,\ \forall t>0,\\ &u(X,0)=X^2,\ \forall X\in[0,1].\end{aligned}\right\} \tag{5.2}$$

IBVP2

$$\frac{\partial u}{\partial \tau}=\frac{X^2}{2}\frac{\partial^2 u}{\partial X^2}+\frac{Y^2}{2}\frac{\partial^2 u}{\partial Y^2},\ 0<X<1,\ 0<Y<1,\tau>0, \tag{5.3}$$

subject to

$$\left.\begin{aligned} &\frac{\partial u}{\partial X}=0 && \text{at } X=0,\ \forall Y\in[0,1],\ \forall t>0,\\ &\frac{\partial u}{\partial Y}=0 && \text{at } Y=0,\ \forall X\in[0,1],\ \forall t>0,\\ &\frac{\partial u}{\partial X}=2\sinh(t) && \text{at } X=1,\ \forall Y\in[0,1],\ \forall t>0,\\ &\frac{\partial u}{\partial Y}=2\cosh(t) && \text{at } Y=1,\ \forall X\in[0,1],\ \forall t>0,\\ &u=Y^2 && \text{at } t=0,\ \forall X\in[0,1],\forall Y\in[0,1].\end{aligned}\right\} \tag{5.4}$$

IBVP3

$$\frac{\partial u}{\partial t}=\frac{X^2}{36}\frac{\partial^2 u}{\partial X^2}+\frac{Y^2}{36}\frac{\partial^2 u}{\partial Y^2}+\frac{Z^2}{36}\frac{\partial^2 u}{\partial Z^2}+(XYZ)^4, \tag{5.5}$$

$$0<X<1,\ 0<Y<1,\ 0<Z<1,\ t>0,$$

subject to

$$\left.\begin{aligned} &u(0,Y,Z,t)=0,\ Y\in[0,1],\ Z\in[0,1],\ \forall t>0,\\ &u(X,0,Z,t)=0,\ X\in[0,1],\ Z\in[0,1],\forall t>0,\\ &u(X,Y,0,t)=0,\ X\in[0,1],\ Y\in[0,1],\forall t>0,\\ &u(1,Y,Z,t)=(YZ)^4\left(e^t-1\right),\ Y\in[0,1],\ Z\in[0,1],\forall t>0,\\ &u(X,1,Z,t)=(XZ)^4\left(e^t-1\right),\ X\in[0,1],\ Z\in[0,1],\forall t>0,\\ &u(X,Y,1,t)=(XY)^4\left(e^t-1\right),\ X\in[0,1],\ Y\in[0,1],\forall t>0,\\ &u(X,Y,Z,0)=0,\ \forall X,Y,Z\in[0,1].\end{aligned}\right\} \tag{5.6}$$

In Tables 1 we have compared the Wazwaz and Gorguis [11] Adomian solution of the above three problems with those obtained by the methodology of our study. After convincing ourselves of the validation of our results we next handled the three heat conduction problems with thermal anisotropy reported in this paper. We now move on to discuss the results of the study and make some general conclusions.

one- dimensional			two- dimensional			three- dimensional		
t	Adomian	Hybrid	t	Adomian	Hybrid	t	Adomian	Hybrid
0	0.0625000	0.0625000	0	0.2500000	0.2500000	0.001	0.00000000095	0.00000000095
0.05	0.0657044	0.0657044	0.001	0.2500626	0.2500626	0.002	0.00000000191	0.00000000191
0.10	0.0690731	0.0690731	0.002	0.2501255	0.2501255	0.003	0.00000000287	0.00000000287
0.15	0.0726146	0.0726149	0.003	0.2501886	0.2501886	0.004	0.00000000382	0.00000000383
0.20	0.0763376	0.0763391	0.004	0.2502520	0.2502520	0.005	0.00000000478	0.00000000478
0.25	0.0802515	0.0802561	0.005	0.2503156	0.2503156	0.006	0.00000000574	0.00000000574
0.30	0.0843661	0.0843767	0.006	0.2503795	0.2503795	0.007	0.00000000670	0.00000000671
0.35	0.0886917	0.0887123	0.007	0.2504436	0.2504436	0.008	0.00000000766	0.00000000767
0.40	0.0932390	0.0932749	0.008	0.2505080	0.2505080	0.009	0.00000000862	0.00000000863
0.45	0.0980195	0.0980770	0.009	0.2505726	0.2505726	0.010	0.00000000958	0.00000000960

Table1: Comparison of Adomian solution with hybrid solution for the one, two and three-dimensional problems of Wazwaz and Gorguis [11].

6. RESULTS AND DISCUSSION

The examples of heat transfer problems such as non-uniform materials like composites, alloys, heterogeneous porous media with thermal equilibrium or non-equilibrium, multi-layered media and such other problems can be studied with a combination of the temporal differential transform and finite difference method. This work is proposed for three heat conduction problems with variable coefficients. The thermal diffusivity being an anisotropic tensor leads to the variable coefficients in these equations. Further, constant/time-dependent heating or heat flux conditions are chosen for obtaining the particular solution of the equations in the absence/presence of heat source. The solution procedure first converts the parabolic type heat equation to an elliptic heat equation using the central difference method. The temporal differential transform method as used in the paper takes care of stability and the finite difference method on the resulting equation results in a system of diagonally dominant linear algebraic equations. The Gauss-Siedel iterative procedure then used to solve the linear system thus has assured convergence. The convergence is optimized in the paper computationally by proper selection of time step in the differential transform part of the algorithm and then the spatial step size in the finite difference part of the algorithm. Quite conveniently we may point to the excellent work of Jang et al. (2000) on the differential transform method of solution whose work clearly showed that the error remains bounded if we choose the time step-size to satisfy:

$$\Delta t < \left[\frac{\varepsilon}{U_i[16]}\right]^{1/16} \quad \text{(for 15 terms in the time-series).}$$

We have actually been governed by this choice of step-size in our calculations and as a result we had a convergent solution. We now discuss the results of the computation obtained by a combination of two methods.

In the one dimensional heat conduction problem there is time-dependent heating at one end of the rod and the entire length of the rod is thermally insulated. The other end of the rod is comparatively cold. The heat is thus transported along the length of the rod from the hot to the cold. This is also clearly shown by Figure 4. As time progresses one sees that temperature remains higher near regions of the hotter end of the rod, though decreasing with time.

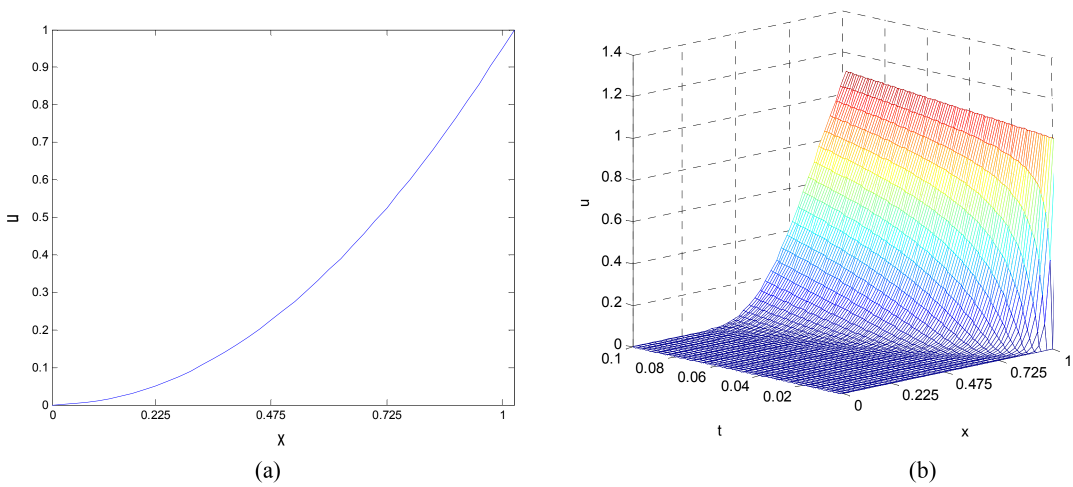
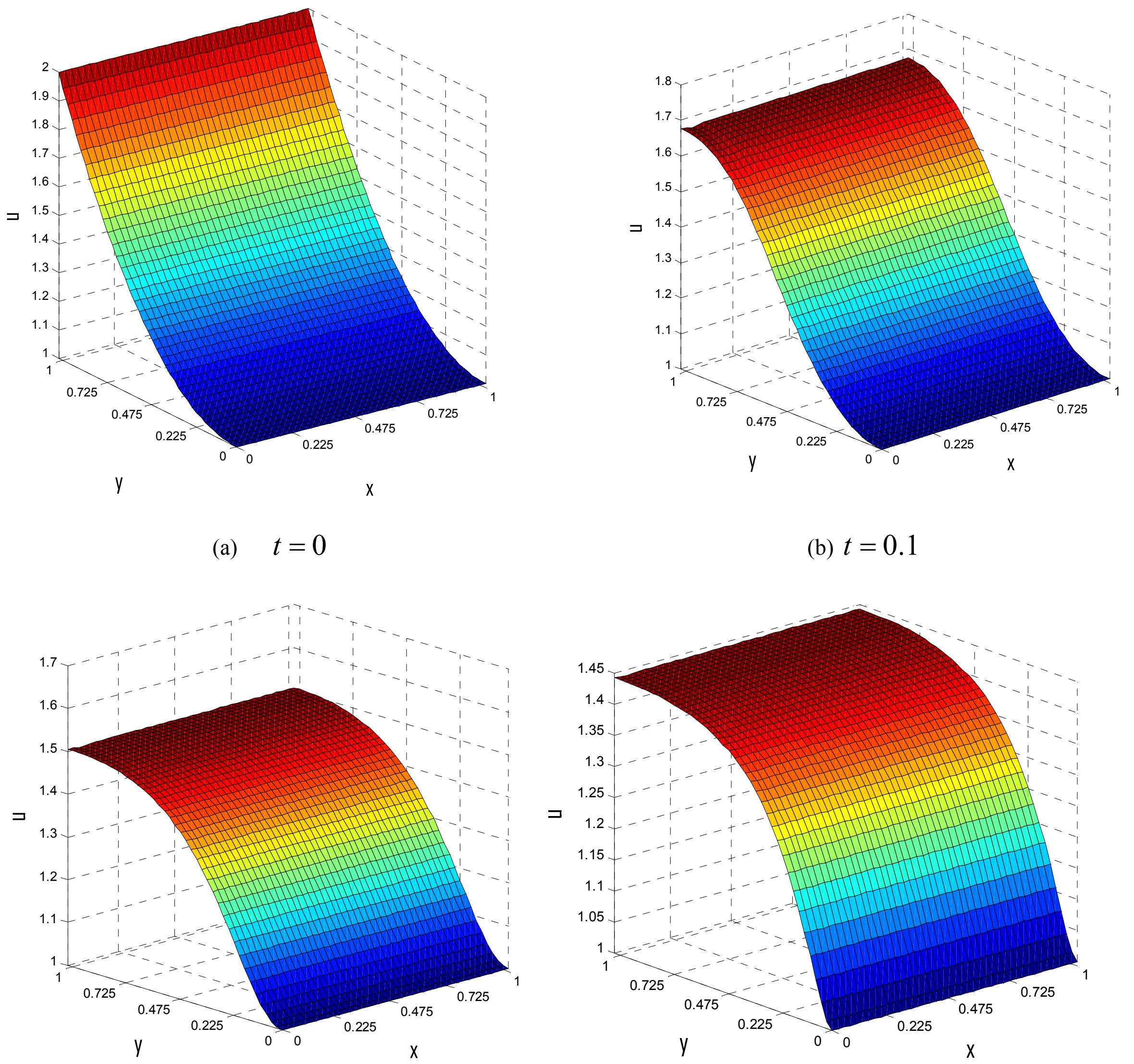


Figure 4: Three dimensional plot of the temperature distribution $u(x,t)$ for (a) $t = 0$ and (b) all time.

Figure 5 is a plot of the temperature distribution of the two-dimensional problem at different times at different points in space. The temperature is the highest at points where the time-dependent heating faces are near. The temperature is the least at the points towards the end where the faces are opposite to the time-dependent heating faces. Overall the temperature decreases with time at all points of the plate and the plots have been shown for those times wherein concave portions of the surface become convex – an overturning of sorts seen in these plots.

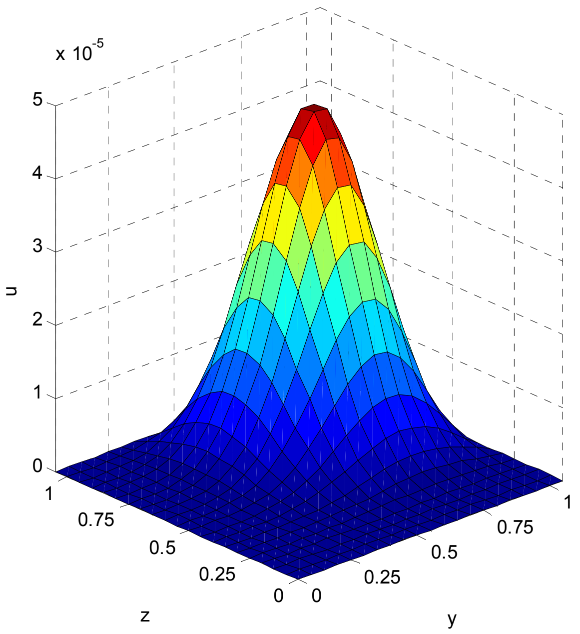


(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.5$ (d) $t = 1.0$

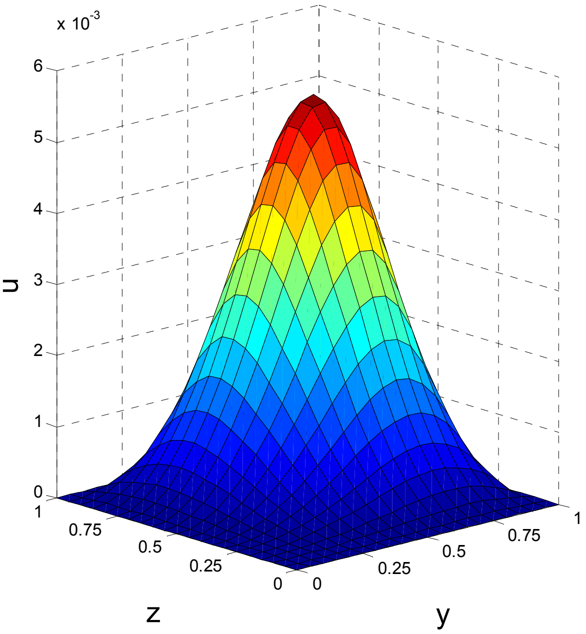
Figure 5: Plot of *u* versus *x* and *y* for different values of *t*.

The last plot in the paper shows the effect of heat source on the temperature distribution in the case of the three-dimensional problem. Clearly the heat source effect is more enhanced at points in the neighborhood of (1,1,1) compared to the other points. The points in the neighborhood of (0,0,0) are the ones that are least affected by the heat source.

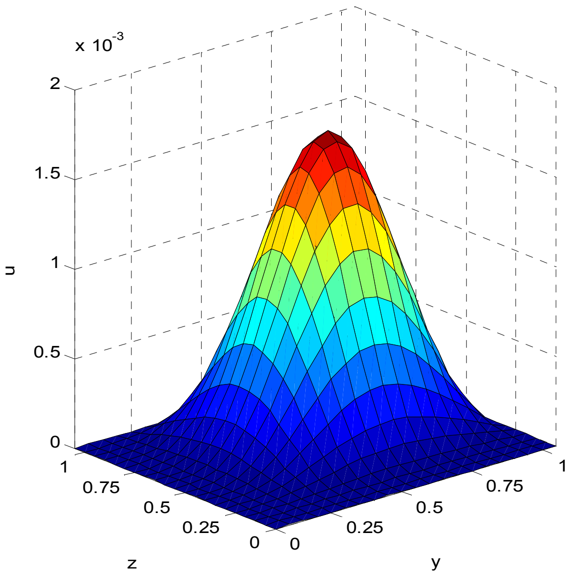
At any given point when the heat source is present the temperature is higher than when the heat source is absent. Therefore the rate of temperature increases when the heat source is present.



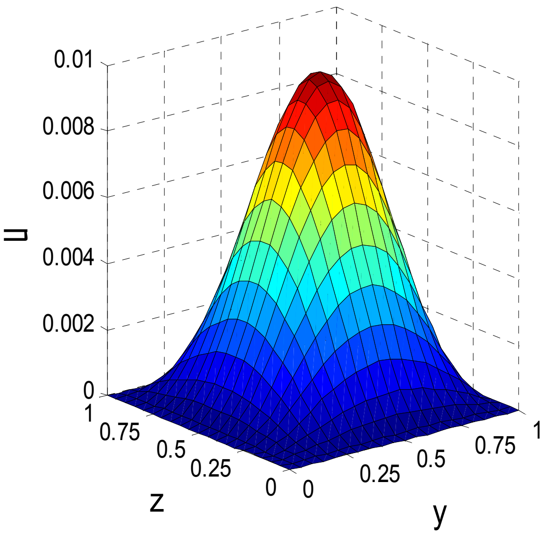
(a) $t=0.1 \quad ,x=0.5 \quad ,S=0$



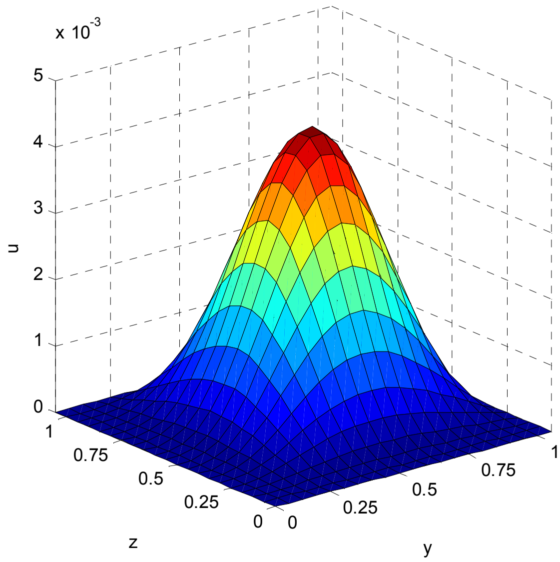
(b) $t=0.1 \quad ,x=0.5 \quad ,S=1$



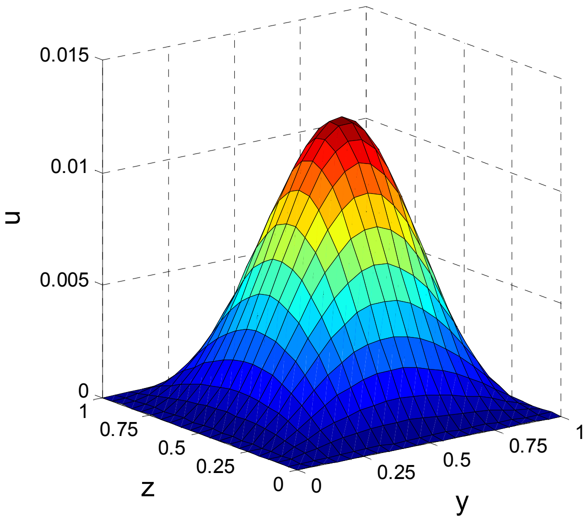
(c) $t=0.5 \quad ,x=0.5 \quad ,S=0$



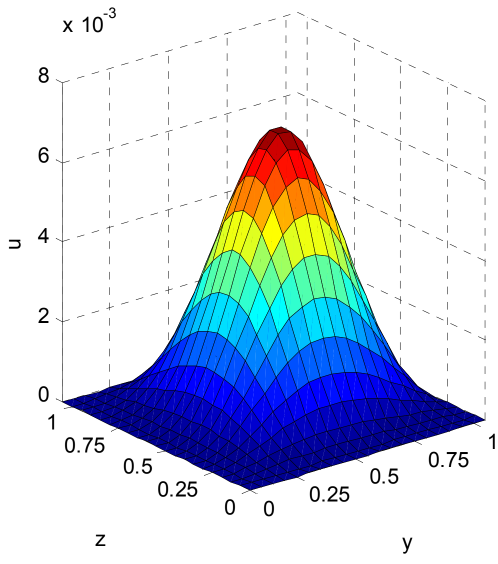
(d) $t=0.5 \quad ,x=0.5 \quad ,S=1$



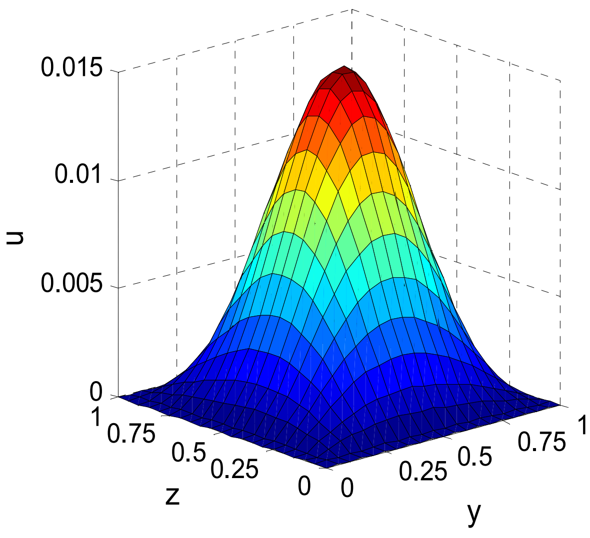
(e) $t = 1.0 \quad , x = 0.5 \quad , S = 0$



(f) $t = 1.0 \quad , x = 0.5 \quad , S = 1$



(g) $t = 1.5 \quad , x = 0.5 \quad , S = 0$



(h) $t = 1.5 \quad , x = 0.5 \quad , S = 1$

Figure 6: Plot of u versus y and z at different times and different point along the length of the rod, with each row of figures having plots without/with heat source.

The application of a combination of the temporal differential transform and finite difference method is thus shown to give satisfactory results in a physical problem. It must be emphasized that the method does not suffer from the instability problems usually seen in the case of parabolic problems. The methodology further allows one to go ahead with computation without having to look into stability considerations.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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