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EXISTENCE OF FIXED POINTS IN QUASI METRIC SPACES

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Abstract. In this paper, we obtain some new fixed point theorems for two pairs of weakly compatible mappings in the framework of non-symmeterical quasi metric spaces. Several interesting corollaries are also deduced. The results obtained extend various well known results of the literature in the setting of quasi metric space. We also construct an example to demonstrate the usability of the proved results.

1. Introduction and Preliminaries

In 2002, Aamri and Moutaawakil [1] introduced the notion of property E.A. in metric spaces and proved various results in the area of fixed point theory. Later on, using the idea of property E.A., Liu et al. [19] defined common (E.A.) property and proved various common fixed point theorems under strict contractive conditions.

In 2006, Mustafa and Sims [21] introduced a new notion of generalized metric space, called G-metric space, by showing that most of the results concerning Dhage's D-metric spaces [10] are invalid. After then, many authors studied fixed and common fixed points in $G-$metric spaces, see [3, 4, 5, 7, 8, 11, 20, 21, 22, 23, 24].

Here, we give preliminaries and basic definitions which are helpful in the sequel.

Definition 1.1. (see [2, 18]) A quasi-metric on a non-empty set X is a function $q : X \times X \to [0, \infty)$ satisfying the following properties:

(q1) $q(x, y) = 0$ if and only if $x = y$;
(q2) $q(x, y) \leq q(x, z) + q(z, y)$, for all $x, y, z \in X$.

In such a case, the pair (X, q) is called a quasi-metric space.

For symmetry, convergence, Cauchy sequence, completeness, continuity in quasi-metric space see [2].

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Example 1.2. (see [2]) Let X be a subset of $\mathbb{R}$ containing $[0,1]$ and define, for all $x, y \in X$,

$$q(x,y) = \begin{cases} x - y, & \text{if } x \geq y, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, q) is a quasi-metric space.

Definition 1.3. (see [2, 21]) Let X be a nonempty set and $G : X \times X \times X \to [0, +\infty)$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$,
(G2) $0 < G(x,x,y)$, for all $x, y \in X$ with $x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

For more details about Symmetric G-metric, $G-$ Cauchy sequence, continuity of G function, $G-$ completeness, one may refers to paper [21].

Let (X, G) be a G-metric space. Then

(a) (X, G) is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x, y \in X$.
(b) The pair (S, T) of self mappings of a G-metric space (X, G) is said to be weakly compatible if they commute at their coincidence points.
(c) The pair (S, T) of self mappings of a G-metric space (X, G) is said to satisfy the property E.A if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Tx_n = t$, for some $t \in X$.
(d) Two pairs (A, S) and (B, T) of self mappings of a $G-$ metric space are said to satisfy the common (E.A.) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Ty_n = z$ for some $z \in X$.

Recently, Chandok et al. [6] used the concept of a C-class functions which cover a large class of contractive conditions.

Definition 1.4. A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called *C-class* function if for any $s, t \in [0, \infty)$, the following conditions hold:

(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0,0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ will denote the class of all C- functions.

Example 1.5. (see [6]) The following examples show that the class $\mathcal{C}$ is nonempty:

(1) $F(s,t) = s - t$.

(2) $F(s,t) = ms$, for some $m \in (0,1)$.
(3) $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,\infty)$.
(4) $F(s,t) = \log(t + a^s)/(1+t)$, for some $a > 1$.
(5) $F(s,t) = \ln(1 + a^s)/2$, for $e > a > 1$.
Indeed $F(s,t) = s$ implies that $s = 0$.

Throughout this paper, we suppose that Ψ denote the class of all real valued continuous non-decreasing functions $\psi : [0,\infty) \to [0,\infty)$ satisfying $\psi(t) = 0$ if and only if $t = 0$ and Φ denote the class of all real valued continuous non-decreasing functions $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) > 0$ if $t > 0$.

In this paper, by using C- class functions, we prove some new common fixed point theorems for two pairs of weakly compatible mappings in the framework of quasi metric spaces. Several interesting corollaries are also deduced. The results obtained extend various well known results of the literature in the setting of quasi metric space. We also construct an example to demonstrate the usability of the proved results.

2. Main results

Throughout this section, we assume that $\phi \in \Phi, \psi \in \Psi$ and F is a C-class function.

Also, we assume that (X,G) is a G-metric space and define $d_G : X \times X \to [0,\infty)$ by $d_G(x,y) = G(x,y,y)$. Using Lemma 3.3.1 of [2], every G-metric G induces a quasi-metric d_G in the sense of Kunzi [18] in such a way that $\tau(G) = \tau(d_G)$.

Now, we prove the main result of this section.

Theorem 2.1. *Let (X,d_G) be a quasi metric space and let A,B,S,T be four self-mappings on set X such that:*

(i) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*
(ii) *for all $x,y \in X$,*

$$\psi(d_G(Ax,By)) \le F(\psi(M(x,y)), \phi(M(x,y))),$$

where $M(x,y,y) = \max\{d_G(Sx,Ty), d_G(Sx,By), d_G(Ty,By)\}$;
(iii) *one of $A(X), B(X), S(X)$ or $T(X)$ is a closed subset of X.*

Further assume that one of the pairs (A,S) or (B,T) satisfies the property E.A. Then the pairs (A,S) and (B,T) have a coincidence point. Moreover, if (A,S) and (B,T) are weakly compatible then A,B,S and T have a unique common fixed point in X.

Proof. If the pair (B,T) satisfies the property E.A., then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} Bx_n = \lim_{n\to+\infty} Tx_n = t$, for some $t \in X$. Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n\to+\infty} Sy_n = t$. We shall show that $\lim_{n\to+\infty} Ay_n = t$. From (ii), take $x =$

$y_n, y = x_n$,
we have

$$\psi(d_G(Ay_n, Bx_n)) \leq F(\psi(M(y_n, x_n)), \phi(M(y_n, x_n))) \leq \psi(M(y_n, x_n))$$

where

$$M(y_n, x_n) = \max\{d_G(Sy_n, Tx_n), d_G(Sy_n, Bx_n), d_G(Tx_n, Bx_n)\}.$$

Taking the limit as $n \to +\infty$ (upper limit) and using the definition of F, ϕ and ψ, we have

$$\psi(\lim_{n\to+\infty} d_G(Ay_n, t)) \leq \psi(0) = 0.$$

So, $\psi(\lim_{n\to+\infty}(d_G(Ay_n, t))) = 0$. Thus, $\lim_{n\to+\infty} d_G(Ay_n, t) = 0$ and so $\lim_{n\to+\infty} Ay_n = t$. Thus we have

$$\lim_{n\to+\infty} Ay_n = \lim_{n\to+\infty} Bx_n = \lim_{n\to+\infty} Sy_n = \lim_{n\to+\infty} Tx_n = t.$$

Suppose that $S(X)$ is a closed subset of X. Then $t = Su$ for some $u \in X$. Now, we shall show that $Au = Su = t$. From (ii), take $x = u$, $y = x_n$, we have

$$\psi(d_G(Au, Bx_n)) \leq F(\psi(M(u, x_n)), \phi(M(u, x_n)))$$

where

$$M(u, x_n) = \max\{d_G(Su, Tx_n), d_G(Su, Bx_n), d_G(Tx_n, Bx_n)\}.$$

Taking the limit as $n \to +\infty$, and using the definition of F, ϕ and ψ, we have

$$\psi(d_G(Au, Su)) \leq F(\psi(d_G(t, t)), \phi(d_G(t, t))) \leq \psi(d_G(t, t)) = \psi(0) = 0.$$

This gives, $Au = Su$. Thus u is a coincidence point of the pair (A, S). The weak compatibility of A and S implies that $ASu = SAu$ and hence $AAu = ASu = SAu = SSu$. As $A(X) \subseteq T(X)$, there exists $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$. By (ii), take $x = u$, $y = v$, we have

$$\psi(d_G(Au, Bv)) \leq F(\psi(M(u, v)), \phi(M(u, v))),$$

where

$$M(u, v) = \max\{d_G(Su, Tv), d_G(Su, Bv), d_G(Tv, Bv)\};$$

or

$$M(u, v) = \max\{0, d_G(Au, Bv), d_G(Au, Bv)\} = d_G(Au, Bv).$$

Using the definition of F, ϕ and ψ, we have

$$\psi(d_G(Au, Bv)) \leq F(\psi(d_G(Au, Bv)), \phi(d_G(Au, Bv))) \leq \psi(d_G(Au, Bv)).$$

This gives, $\psi(d_G(Au, Bv)) = 0$ or $\phi(d_G(Au, Bv)) = 0$. This implies that $Au = Bv$ and hence $Tv = Bv$. It follows that also the pair (B, T) has a coincidence point. Thus we have $Au = Su = Tv = Bv$.

Now, if B and T are weakly compatible, then we obtain $BTv = TBv = TTv = BBv$ and this shows that Au is a common fixed point of A, B, S and T. Again from (ii), take $x = Au$, $y = v$, we have

$$\psi(d_G(AAu, Au)) = \psi(d_G(AAu, Bv)) \leq F(\psi(M(Au, v)), \phi(M(Au, v))),$$

where

$$M(Au, v) = \max\{d_G(SAu, Tv), d_G(SAu, Bv), d_G(Tv, Bv)\};$$

$$M(Au, v) = \max\{d_G(AAu, Bv), d_G(AAu, Bv), 0\} = d_G(AAu, Bv).$$

Again using the definition of F, ϕ and ψ, we have

$$\psi(d_G(AAu, Au)) \leq F(\psi(d_G(AAu, Au)), \phi(d_G(AAu, Au))) \leq \psi(d_G(AAu, Au)).$$

This implies $Au = AAu = Bv$. Therefore, $Au = AAu = SAu$ is a common fixed point of A and S. Similarly, one can prove that Bv is a common fixed point of B and T. Since $Au = Bv$, we deduce that Au is a common fixed point of A, B, S and T.

Now, we have to show that the common fixed point is unique. Suppose to the contrary that w and z $(w \neq z)$, are two common fixed points of A, B, S and T. Then, from (ii) and using the definition of F, ϕ, ψ, we have

$$\psi(d_G(Az, Bw)) \leq F(\psi(M(z, w)), \phi(M(z, w))),$$

where

$$\begin{aligned} M(z, w) &= \max\{d_G(Sz, Tw), d_G(Sz, Bw), d_G(Tw, Bw)\} \\ &= \max\{d_G(z, w), d_G(z, w), d_G(w, w)\} \\ &= d_G(z, w). \end{aligned}$$

Therefore, we have

$$\psi(d_G(z, w)) \leq F(\psi(d_G(z, w)), \phi(d_G(z, w))) \leq \psi(d_G(z, w)).$$

This gives, $w = z$. Therefore, A, B, S and T have a unique common fixed point.

Clearly proceeding on the foregoing lines, one can easily obtain the same conclusion in case (instead of $S(X)$) one of $A(X), B(X)$ or $T(X)$ is a closed subset of X, and in case (instead of (B, T)) (A, S) satisfies the property E.A. □

Corollary 2.2. *Let (X, d_G) be a quasi metric space and $A, B, S, T : X \to X$ be four mappings such that:*

(i) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*

(ii) *for all $x, y \in X$*

$$d_G(Ax, By) \leq F(\psi(M(x, y)), \phi(M(x, y))),$$

where $M(x, y) = \max\{d_G(Sx, Ty), d_G(Sx, By), d_G(Ty, By)\}$;

(iii) *one of $A(X), B(X), S(X)$ or $T(X)$ is a closed subset of X.*

Suppose that one of the pairs (A, S) or (B, T) satisfies the property E.A. Then the pairs (A, S) and (B, T) have a coincidence point. Further, if (A, S) and (B, T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

If we assume $S = T$ in the above Theorem 2.1, we deduce the following result involving three self-mappings.

Corollary 2.3. *Let (X, d_G) be a quasi metric space and $A, B, S : X \to X$ be three mappings such that:*

(i) *$A(X) \subseteq S(X)$ and $B(X) \subseteq S(X)$;*
(ii) *for all $x, y \in X$,*

$$\psi(d_G(Ax, By)) \leq F(\psi(M(x, y)), \phi(M(x, y))),$$

where $M(x, y) = \max\{d_G(Sx, Sy), d_G(Sx, By), d_G(Sy, By)\}$;
(iii) *one of $A(X), B(X)$ or $S(X)$ is a closed subset of X.*

Suppose that one of the pairs (A, S) or (B, S) satisfies the property E.A. Then the pairs (A, S) and (B, S) have a coincidence point. Further, if (A, S) and (B, S) are weakly compatible then A, B and S have a unique common fixed point in X.

Example 2.4. Let $F(s, t) = \frac{99s}{100}, X = [0, 2]$ and $G : X \times X \times X \to [0, +\infty)$ be defined by $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$, for all $x, y, z \in X$. Define also $A, B, S : X \to X$ by $Ax = 1$, $Bx = 2 - x$ and $Sx = x$ for all $x \in X$ and $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = 20t$ for all $t \geq 0$. Clearly, the hypotheses (i) and (iii) of Corollary 2.3 hold trivially. Moreover, the pair (A, S) satisfies the property E.A. by taking sequence $x_n = \frac{n+1}{n}$. Here we show only that the hypothesis (ii) of Corollary 2.3 holds. In fact for all $x, y \in X$ we have, $d_G(Ax, By) = G(Ax, By, By) = G(1, 2 - y, 2 - y) = |1 - y|$, $d_G(Sx, Sy) = G(Sx, Sy, Sy) = G(x, y, y) = |x - y|$, $d_G(Sx, By) = G(Sx, By, By) = G(x, 2 - y, 2 - y) = |2 - x - y|$, $d_G(Sy, By) = G(Sy, By, By) = G(y, 2 - y, 2 - y) = 2|1 - y|$ and consequently

$$\psi(d_G(Ax, By)) = \psi(G(Ax, By, By)) \leq F(\psi(M(x, y)), \phi(M(x, y))).$$

Further, it implies that

$$|y - 1| \leq \frac{99}{100} M(x, y),$$

where

$$M(x, y) = \max\{d_G(Sx, Sy), d_G(Sx, By), d_G(Sy, By)\} = \max\{|x-y|, |x+y-2|, 2|y-1|\},$$

which is true. Then, by the Corollary 2.3, the pairs (A, S) and (B, S) have a coincidence point, that is, $u = 1$. Moreover, since (A, S) and (B, S) are weakly compatible, $u = 1$ is the unique common fixed point of A, B and S in X.

Corollary 2.5. *Let (X, d_G) be a quasi metric space and $A, T : X \to X$ be two mappings such that:*

(i) *for all $x, y \in X$,*

$$d_G(Ax, Ty) \leq F(\psi(M(x, y)), \phi(M(x, y))),$$

where $M(x, y) = \max\{d_G(Tx, Ty), d_G(Tx, Ay), d_G(Ty, Ay)\}$;
(iii) *$T(X)$ is a closed subset of X.*

Suppose that the pair (A, T) satisfies the property E.A. Then the pair (A, S) has a coincidence point. Further, if pair (A, S) is weakly compatible then A and T have a unique common fixed point in X.

Remark 2.6. Theorem 2.1 still remains true if $M(x,y)$ in condition (ii) is replaced by:

$$M_1(x,y) = \max\{d_G(Sx,Ty), d_G(Ax,Sx), d_G(By,Ty)\}.$$

By taking, $F(s,t) = s - t$, we obtain the following corollary.

Corollary 2.7. *Let (X, d_G) be a quasi metric space and A, B, S, T be four self-mappings on set X such that:*

(i) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*

(ii) *for all $x, y \in X$,*

$$\psi(d_G(Ax,By)) \le \psi(M(x,y)) - \phi(M(x,y)),$$

where $M(x,y) = \max\{d_G(Sx,Ty), d_G(Sx,By), d_G(Ty,By)\}$;

(iii) *one of $A(X), B(X), S(X)$ and $T(X)$ is a closed subset of X.*

Suppose that one of the pairs (A,S) and (B,T) satisfies the property E.A. Then the pairs (A,S) and (B,T) have a coincidence point. Moreover, if (A,S) and (B,T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

Theorem 2.8. *Let A, B, S, T be four self-mappings on a quasi metric space (X, d_G) satisfying the condition (ii) of Theorem 2.1 and*

(a) *the pair (A,S) and (B,T) share the common (E.A.) property,*

(b) *$S(X)$ and $T(X)$ are closed subsets of X.*

Then the pairs (A,S) and (B,T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Proof. In view of (a), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Ty_n = z$ for some $z \in X$.

Since $S(X)$ is a closed subset of X, therefore, there exists a point $u \in X$ such that $z = Su$. We claim that $Au = z$. By (ii), take $x = u, y = y_n$,

$$\psi(d_G(Au,By_n)) \le F(\psi(M(u,y_n)), \phi(M(u,y_n))) \le \psi(M(u,y_n)),$$

where

$$M(u,y_n) = \max\{d_G(Su,Ty_n), d_G(Su,By_n), d_G(Ty_n,By_n)\}.$$

Taking the limit as $n \to +\infty$ (upper limit) and using the definition of F, ϕ and ψ, we have

$$\psi(d_G(Au,z)) \le F(0,0) \le \psi(0) = 0.$$

This gives, $Au = z = Su$ which shows that u is a coincidence point of the pair (A,S).

Since $T(X)$ is also a closed subset of X, therefore $\lim_{n\to+\infty} Ty_n = z$ in $T(X)$ and hence there exists $v \in X$ such that $Tv = z = Au = Su$. Now, we show that $Bv = z$. By using inequality (ii) of Theorem 2.1, take $x = u, y = v$, we have

$$\psi(d_G(Au,Bv)) \le F(\psi(M(u,v)), \phi(M(u,v))),$$

where

$$M(u,v) = \max\{d_G(Su,Tv), d_G(Su,Bv), d_G(Tv,Bv)\}.$$

Using the definition of F, ϕ and ψ, we have

$$\psi(d_G(z,Bv)) \leq F(\psi(d_G(z,Bv)), \phi(d_G(z,Bv)) \leq \psi(d_G(z,Bv)).$$

This gives, $\psi(d_G(z,Bv)) = 0$ or $\phi(d_G(z,Bv) = 0$. Hence $Bv = z = Tv$ which shows that v is a coincidence point of the pair (B,T).
Since the pairs (A,S) and (B,T) are weakly compatible and $Au = Su, Bv = Tv$, therefore, $Az = ASu = SAu = Sz, Bz = BTv = TBv = Tz$.

Next, we show that $Az = z$. Again, by using inequality (ii) of Theorem 2.1, take $x = z$ and $y = v$ we have

$$\psi(d_G(Az,Bv)) \leq F(\psi(M(z,v)), \phi(M(z,v))),$$

where

$$M(z,v) = \max\{d_G(Sz,Tv), d_G(Sz,Bv), d_G(Tv,Bv)\}.$$

Again by using the definition of F, ϕ and ψ, we have

$$\psi(d_G(Az,z)) \leq F(\psi(d_G(Az,z)), \phi(d_G(Az,z))) \leq \psi(d_G(Az,z)).$$

This gives, $Az = z = Sz$.
Similarly, one can prove that $Bz = Tz = z$. Hence, $Az = Bz = Sz = Tz$, and z is common fixed point of A, B, S and T.
For uniqueness, let z and w be two common fixed points of A, B, S and T. Then by using inequality (ii) of Theorem 2.1, we have

$$\psi(d_G(Az,Bw)) \leq F(\psi(M(z,w)), \phi(M(z,w))),$$

where

$$M(z,w) = \max\{d_G(Sz,Tw), d_G(Sz,Bw), d_G(Ty,Bw)\}.$$

Again by using the definition of F, ϕ and ψ, we easily get $z = w$. □

Remark 2.9. For different variants of $F(s,t)$ as in Example 1.5, we have various variants of our proved results for two or three or four self mappings with E.A. property or common (E.A.) property.

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