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Properties of Pre A*-Functions

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ABSTRACT

This manuscript is a study on various properties of Pre A*-functions. The concept of equivalent Pre A*-function has been introduced. It is shown that a Pre A*-expression in n variables is a Pre A*-expression on $\mathbf{3}^n$and every Pre A*-expression represents a unique Pre A*-function. The concepts of dual Pre A*-function, sum-of-products expansion and implicants of the Pre A*-function have been initiated. It is observed that, a min term of a Pre A*-variables is a product of n literals, which is one literal for each variable.

Key words: Pre A*-algebra, Pre A*-function, Pre A*-variables, Pre A*-expressions, duality, Sum-of-Products expansion, Product-of-Sums expansion and implicants.

1. INTRODUCTION

Koteswara Rao [1] introduced the concept of A*-algebra $(A, \wedge, \vee, *, (-)^{\sim}, (-)_{\pi}, 0, 1, 2)$. He studied the equivalence of A*-algebra with Ada, C-algebra, Ada's connection with 3-Ring, stone type representation and also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra.

Venkateswara Rao [2] initiated the concept of Pre A*-algebra $(A, \vee, \wedge, (-)^{\sim})$ analogous to C-algebra as a reduct of A*-algebra. Venkateswara Rao and Srinivasa Rao [3] identified a congruence relation on Pre A*-algebra. Venkateswara Rao and Srinivasa Rao [4] obtained the well known Cayley's theorem on centre of Pre A*-algebras. Further, Venkateswara Rao and Srinivasa Rao [5] instigated a ternary operation on Pre-A*-algebra. Venkateswara Rao et al. [6] a studied on Pre A* - functions.

Based on the connection between Pre A*-algebras and Boolean algebras, analogous to Boolean functions, a Pre A*-function defined as a mapping $f: \mathbf{3}^n \to \mathbf{3}$ (where $\mathbf{3} = \{0, 1, 2\}$). Furthermore, identified results on Pre A*-functions such as the dominance property of 2 and the order relation $\leq$. Further, properties such as representations and implicants of Pre A*-functions are studied.

This manuscript is divided into two sections. The first section is devoted to the introduction of Pre A*-functions (as mapping$f: \mathbf{3}^n \to \mathbf{3}$) and various results of Pre A*-functions.

The second section is concerned with properties of Pre A*-functions. The duality property, representations (Sum-of-Products expansion and Product-of-Sums expansion) and implicants of Pre A*-functions are studied and examples about these properties are specified.

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2. INTRODUCTION TO PRE A*-FUNCTIONS

This section deals with the basic definition of Pre A*-Algebras and Pre A*-Functions.

Definition 2.1: An algebra $(A, \vee, \wedge, (-)^{\sim})$ where A is non-empty set with $\vee$, $\wedge$ are binary operations and $\sim$ is a unary operation satisfying the following axioms:

i) $(x^{\sim})^{\sim} = x \; \forall x \in A$,
ii) $x \wedge x = x \; \forall x \in A$,
iii) $x \wedge y = y \wedge x, \forall x, y \in A$
iv) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$, $\forall x, y \in A$
v) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in A$
vi) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$,
vii) $x \wedge y = x \wedge (x^{\sim} \vee y)$, $\forall x, y \in A$

is called a Pre A*-algebra.

Example 2.1: $\mathbf{3} = \{0, 1, 2\}$ with operations $\wedge$, $\vee$, $(-)^{\sim}$ defined as below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

Note 2.1: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$ (b) $1 \wedge x = x$ for all x in **3** (c) $0 \vee x = x$ for all x in **3** (d) $2 \wedge x = 2 = 2 \vee x$ for all x in **3**.

Example 2.2: $\mathbf{2} = \{0, 1\}$ with operations $\wedge$, $\vee$, $(-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

Note 2.2 (i) $(\mathbf{2}, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

(ii) Axioms (i) and (iv) imply that the varieties of Pre A*-algebras satisfy all the dual statements of (i) to (vii).
(iii) Here the binary operations juxta position, $+$, respectively $\wedge$, $\vee$ considered as in Boolean algebra.

Note 2.3[6]: A Pre A*-variable is a variable which assumes only the values 0, 1 and 2. That is, it takes value from the set **3**. Two Pre A*-variables are said to be independent variables if they assume values from **3** independent of each other.

Definition 2.2[6]: 1) A mapping $f: \mathbf{3} \to \mathbf{3}$ is called a Pre A*-function of one variable.
2) A mapping f: $\mathbf{3}^{\mathrm{n}} \to \mathbf{3}$ is said to be a Pre A*-function of n variables.
Note that, by the counting principle of products, the total number of Pre A*-functions (f: $\mathbf{3}^{n} \to \mathbf{3}$**)** is $3^{(3^n)}$.

Theorem 2.1 (Dominance Property of 2**) [6]:** If any Pre A*-variable assumes the value 2 in its Pre A*-function (that is in its functional value), then the function has the value 2.

Note 2.4[6]: Variables of a Boolean function can be taken as propositional variables. Because, Boolean algebra itself is the study of logic, and a proposition is a declarative sentence which has a truth value of true or false but not both. Similarly, each Boolean variable has the value 0 or 1 but not both and we can associate the truth value true by 1 and the truth value false by 0. But a Pre A*-function is an extension of this function, and introduces another proposition with undefined truth value that can be represented by the value 2.

3. PROPERTIES OF PRE A*-FUNCTIONS

This section deals with some basic properties of Pre A*-functions analogous to the basic properties of Boolean functions.

Definition 3.1[6]:

1. A Pre A*-expression in the variables $x_1, x_2, \ldots, x_n$ are defined recursively as $0, 1, 2, x_1, x_2, \ldots, x_n$ are Pre A*-expressions.
2. If E_1 and E_2 are Pre A*-expressions in $x_1, x_2, \ldots, x_n$ variables then $E_1^{\sim}$, $(E_1 + E_2)$ and $(E_1 E_2)$ are also Pre A*-expressions in $x_1, x_2, \ldots, x_n$ variables.
3. Any Pre A*-expression is formed by finitely many applications of the rules (1) and (2) of this definition.

Definition 3.2: We say that two Pre A*-expressions E_1 and E_2 are equivalent if they represent the same Pre A*-function.When this is the case, we write $E_1 = E_2$.

Example 3.1: The Pre A*-expressions $E_1 = xyz + xyx^{\sim}z + xyzy^{\sim}$ and $E_2 = xyz$ are equivalent expressions.

Since; $E_1 = xyz + xyx^{\sim}z + xyzy^{\sim} = xyz + xx^{\sim}yz + xyy^{\sim}z$ (since $xy = yx$)
$= (x + xx^{\sim})yz + xyy^{\sim}z = xyz + xyzy^{\sim}$ (since $x + xx^{\sim} = x$)
$= x(y + yy^{\sim})z = xyz$

Therefore these two expression represent the same Pre A*-function.

Note 3.1: We also show that a Pre A*-expression in n variables, $x_1, x_2, \dots, x_n$ is a Pre A*-expression on $\mathbf{3}^n$. Every Pre A*-expression E represents a unique Pre A*-function.

Note 3.2. There are 3^{3^n} Pre A*-functions of n variables, there are infinitely many Pre A*-expressions of n variables. These remarks motivate the distinction that we draw between Pre A*- functions and Pre A*-expressions.

3.1. Duality of Pre A*-Functions

With every Pre A*-function f, the following definition associates another Pre A*-function f^d called the dual of f.

Definition 3.1.1[6]: The dual of a Pre A*-function f denoted by f^d is the function f^d defined by; $f^d(X) = [f(X^{\sim})]^{\sim}$ for all $X = (x_1, x_2, \dots, x_n) \in \mathbf{3}^n$, where

$X^{\sim} = (x_1{}^{\sim}, x_2{}^{\sim}, \dots, x_n{}^{\sim})$.

Note 3.1.1[6]: The dual of a Pre A*-function f is represented by a Pre A*-expression is a function represented by the dual of this expression.

Definition 3.1.2[6]: The dual of a Pre A*-expression is obtained by interchanging Pre A*-sums and Pre A*-products, interchanging 0's and 1's and interchanging of 2 with itself.

Example 3.1.1. [6]: The dual of the Pre A*-expression $x(y+0)$ is $x + (y \cdot 1)$ which is also Pre A*-expression. The dual of $x^{\sim} \cdot 2 + (y^{\sim} + z)$ is $x^{\sim} + 2 \cdot (y^{\sim} \cdot z)$.

Theorem 3.1.1[6]: If f and g are two Pre A*-functions of n variables, then the following holds.

a) $(f^d)^d = f$ (Involution: the dual of the dual is the function itself)
b) $(f^{\sim})^d = (f^d)^{\sim}$
c) $(f+g)^d = f^d g^d$
d) $(fg)^d = f^d + g^d$

Theorem 3.1.2: If the expression E represents the Pre A*-function f, then E^d represents the Pre A*-function f^d.

Proof: Let t denotes the total number of Pre A*-sum $(+)$, Pre A*-product $(\cdot)$ and Pre A*-negation $(\sim)$ operators in the Pre A*-expression E. We prove this theorem by induction on t. If $t = 0$, then E is either a constant or a literal and the statement is easily seen to be hold. Assume that $t > 0$. Then by the above definition 3.1, the Pre A*-expression E takes either the form $E = E_1 + E_2$ or the form $E = E_1 E_2$ or the form $(E_1)^{\sim}$. Assume for instance that, $E = E_1 + E_2$ (the other cases are similar). Then by definition 3.2, $E^d = E_1{}^d E_2{}^d$. Let g be the function represented by E_1 and let h be the function represented by E_2. Then by the principle of induction, $E_1{}^d$ and $E_2{}^d$ represent g^d and h^d respectively. So, E^d represents $g^d h^d$ which is equal to f^d by theorem 3.1.1.

Note 3.1.2: A literal of a Pre A*-function is a Pre A*-variable x or its Pre A*-complement $x^{\sim}$.

Corollary 3.1.1[6]: If we define the Pre A*-function 2 by $2(X) = 2, \forall X \in \mathbf{3}^n$,
then $(f+2)^d = 2 = (f \cdot 2)^d$.

3.2. Representation of Pre A*-Functions: Sum of Product Expressions

Definition 3.2.1: Min term of a Pre A*-variables $x_1, x_2, \dots, x_n$ is a Pre A*-product $y_1 y_2 \dots y_n$ where $y_i = x_i$ or $y_i = x_i{}^{\sim}$.
Hence, a min term of a Pre A*-variables is a product of n literals, in which one literal for each variable.
If one of the Pre A*-variables has the value 2, then the min term has the value 2. The min term has the value 1 for one and only one combination of values of its variables. More precisely, the min term $y_1, y_2, \dots, y_n$ is 1 if and only if each y_i is 1 and this occurs if and only if $x_i = 1$ when $y_i = x_i$ and $x_i = 0$ when $y_i = x_i{}^{\sim}$.

Example 3.2.1: Find a min term that equals 1 if $x_1 = x_3 = 0$ and $x_2 = x_4 = x_5 = 1$ and equals 0 otherwise.

Solution: The min term $x_1{}^{\sim} x_2 x_3{}^{\sim} x_4 x_5$ has the correct set of values. Thus the min term $x_1{}^{\sim} x_2 x_3{}^{\sim} x_4 x_5$ has the value 1.

Example 3.2.2: The min term $xyzx^{\sim}y^{\sim}$ has the value 2 if and only if any one of them these three Pre A*-variables has the value 2 otherwise it has the value 0. In other words this min term cannot have the value 1.

By taking sum of distinct min terms we can build up a Pre A*-expression with a specified set of values. In particular, a Pre A*-sum of min terms has the value 2 when exactly one of the min terms in the sum has the value 2. If all the min terms in the sum has the value different from 2, then the Pre A*-sum of min terms has the value 1 when any one of the min terms has the value 1othewise it has the value 0.

Definition 3.2.2: The sum of min terms that represents a Pre A*-function is called the sum-of-products expansion (SPE) of the Pre A*-function.

Theorem 3.2.1: If any one of the Pre A*-variables in any min term has the value 2, then the sum of min terms containing that min term has the value 2.

Proof: Let the min term be $y_1 y_2 \dots y_n$ where $y_i = x_i$ or $y_i = x_i{}^{\sim}$. Let the Pre A*-variable x_i for $i = 1,2, \dots n$ has the value 2 (that is $x_i = 2$), then by the above remark, the min term $y_1 y_2 \dots y_n$ has the value 2 (since in Pre A*-

algebra, $x+2=2=x\cdot 2, \forall x \in \mathbf{3}$ and $2^{\sim}=2$). Also, by the above remark the sum of min terms has the value 2 if and only if any one of the min terms has the value 2.

Therefore if any one of the Pre A*-variables in any min term has the value 2, then the sum of min terms containing that min term has the value 2.

Note 3.2.1: The min term $y_1 y_2 \dots y_n$ where $y_i = x_i$ or $y_i = \overline{x_i}$ in Boolean variables $x_1, x_2, \dots, x_n$ has the value 1 if and only if each y_i is 1and this occurs if and only if $x_i = 1$ when $y_i = x_i$ and $x_i = 0$ when $y_i = \overline{x_i}$. Otherwise it has the value 0. Hence the above does not hold in case of Boolean variables. That is, the min term of Boolean variables may not have the value 1 whenever any one of the Boolean variables has the value 1.

Note 3.2.1: It is also possible to find a Pre A*-expression that represents a Pre A*-function by taking a Pre A*-product of Pre A*-sums. The resulting expansion is called the Product-of-Sums expansion (PSE) of the Pre A*-function. These expansions can be found from Sum-of-Products expansion by taking the duals.

Theorem 3.2.2: Every Pre A*-function can be represented by a Sum-of-Products expansion (SPE) or by Products-of-sums expansion (PSE).

Proof: Let f be a Pre A*-function on $\mathbf{3}^n$.

Case 1: If $f(X)=2$ for all X in $\mathbf{3}^n$, then the proof is trivial. That is f can be represented as SPE. Since, any Pre A*-function has the value 2 whenever any one of the Pre A*-variables takes the value 2 in its functional value. And hence it can be represented as Sum-of-Products expansion (SPE). And the same is for Product-of-Sums expansion (PSE), since PSE can be found from SPE by duality principle.

Case 2: $f(X) \neq 2$, for all X in $\mathbf{3}^n$, clearly either $f(X) = 1$ or $f(X)=0$, for all X in $\mathbf{3}^n$. For simplicity of our notation, let us denote " $+$ " by " $\vee$ " (meet) and " $\cdot$ " by " $\wedge$ " (join). Let T be the set of value 1 of f, and consider SPE

$$E_f(x_1, x_2, \dots, x_n) = \vee_{Y \in T}(\wedge_{i/y_i=1} x_i \wedge_{j/y_j=0} x_j^{\sim}) \qquad (1)$$

If we interprate E_f as a Pre A*-function on $\mathbf{3}^n$, then E_f has the value 1 at the point $X^* \in \mathbf{3}^n$ if and only if there exists

$Y=(y_1, y_2, \dots, y_n) \in T$ such that

$$\wedge_{i/y_i=1} x_i^{*} \wedge_{j/y_j=0} x_j^{*\sim} = 1 \qquad (2)$$

But condition (2) simply means that $x_i^{*}=1$ whenever $y_i=1$ and $x_i^{*}=0$ whenever $y_i=0$. That is $X^*=Y$.

Hence, E_f has the value 1 at the point X^* (that is $E_f(X^*)=1$) if and only if $X^* \in T$, and we conclude that E_f represents f.

A similar reasoning establishes that f can also represented by the SPE. Or simply this can be done by the dual of the first part (that is the proof of the above) of this theorem.

3.3. Implicants of Pre A*-Functions

Definition 3.3.1: Given two Pre A*-functions f and g on $\mathbf{3}^n$, we say that f implies g (or that f is a minorant of g, or that g is a majorant of f) if

$f(X)=2$ implies $g(X)=2$ for all X in $\mathbf{3}^n$.

When this is the case, we write $f \le g$.

This definition extends in a straightforward way to Pre A*-expressions, since every Pre A*-expression can be regarded as a Pre A*-function.

Theorem 3.3.1[6]: Let f *and* g be two Pre A*-functions on $\mathbf{3}^n$, then the following holds.

a) $f \le f+g$ b) $fg \le f$

Note 3.3.1[6]: From the above theorem 3.3.1, it is also true that $g \le f+g$ *and* $fg \le g$.

Theorem 3.3.2: For all Pre A*-functions f and g on $\mathbf{3}^n$, the following statements are equivalent.

1. $f \le g$
2. $f+g=g$
3. $f^{\sim}+g = f+g^{\sim} = f+g$
4. $fg=f$

Proof: (1) implies (2).

Let $f \le g$. Then $f(X)=2$ implies that $g(X)=2$ for all X in $\mathbf{3}^n$.

Then $f(X)+g(X)=2+2=2$, which implies that $f+g=2$ for all X in $\mathbf{3}^n$.

This implies that, $f+g=2$ implies that $g=2$. That is $f+g \le g$. (I)

Conversely, suppose that $g(X)=2$ for all X in $\mathbf{3}^n$

Since by condition (1), $f \le g$, then

$$f+g(X) = f(X)+g(X) = 2+2 = 2$$

This means that, $g=2$. This leads to $f+g=2$. That is $g \le f+g$. (II)

Therefore, from (I) and (II) we have $f+g=g$.

Similar reasoning proves the remaining conditions.

Properties of implicants

Theorem 3.3.3: For all Pre A*-functions f, g and h on $\mathbf{3}^n$, we have the following.

i. $0 \leq f \leq 2$
ii. $fg \leq f \leq f + g$
iii. $f = g$ if and only if $f \leq g$ and $g \leq f$
iv. $f \leq h$ and $g \leq h$ if and only if $f + g \leq h$
v. $f \leq g$ and $f \leq h$ if and only if $f \leq gh$
vi. If $f \leq g$ then $fh \leq gh$
vii. If $f \leq g$ then $f + h \leq g + h$.

Proof: All these properties can be easily verified from the definition of implicants.

To see that, let us prove the fifth property. Let $f \leq g$ and $f \leq h$, $f = 2$ implies that $g = 2$ and $h = 2$ for all X in $\mathbf{3}^n$. Then, $gh = 2 \cdot 2 = 2$. That is $f = 2$ implies that $gh = 2$ and hence $f \leq gh$.
Conversely, suppose that $f \leq gh$. Then $f = 2$ implies that $gh = 2$. And hence,
$gh = 2$ implies that either $g = 2$ of $h = 2$ or both equals 2. Therefore, $f \leq gh$ implies that $f \leq g$ and $f \leq h$ and hence the proof.

Definition 3.3.2: Let f be a Pre A*-function and C be an elementary join. We say that C is an implicant of f if C implies f.

Example 3.3.1: Let $f(x, y, z) = xy + xy^{\sim}z + x^{\sim}yz^{\sim}$ be a Pre A*-function. Then the elementary joins xy, $xy^{\sim}z$, $x^{\sim}yz^{\sim}$ are implicants of f. Since, if any one of these elementary joins (min terms) has the value 2, then automatically f will have the value 2.

Theorem 3.3.4: If E is a Sum-of-Products (SPE) representation of the Pre A*-function f, then every term of E is an implicant of f. Moreover, if C is an imlicant of f, then the SPE $E + C$ also represents f.

Proof: For the first statement, notice that, if any term of E takes the value 2, then E, and hence f, take the value 2.

For the second part of this theorem, we just successively check that

$E + C \leq f$ and $f \leq E \leq E + C$.

To do this, suppose that $E + C = 2$ then either E or C or both equals 2. Since E is the SPE representation of f and C is an implicant of f, then clearly f has the value 2. That is,

$E + C = 2$ implies that $f = 2$.

Therefore $E + C \leq f$. And let $f = 2$. Since E is the SPE representation of f, then $E = 2$. Which means that $f \leq E$ and $f = 2$ also implies $E + C = 2$. That is $f \leq E + C$.

Hence the SPE $E + C$ represents f.

Example 3.3.2: By the above theorem, the Pre A*-function $f(x, y, z) = xyz + xz^{\sim}$ admits the Sum-of-Products expansion $xyzz + xyz^{\sim}z + xyzy^{\sim} + xz^{\sim}$.
So $xyzz + xyz^{\sim}z + xyzy^{\sim} + xz^{\sim} = xyz + xyzz^{\sim} + xyy^{\sim}z + xz^{\sim}$ (since $zz = z$)
$= xy(z + zz^{\sim}) + xyy^{\sim}z + xz^{\sim} = xyz + xyy^{\sim}z + xz^{\sim}$ (as $z + zz^{\sim} = z$)
$= x(y + yy^{\sim})z + xz^{\sim} = xyz + xz^{\sim}$ (as $y + yy^{\sim} = y$)

Definition 3.3.3: Let f be a Pre A*-function and C_1, C_2 be implicants of f. We say that C_1 absorbs C_2 if $C_1 + C_2 = C_1$ or equivalently $C_2 \leq C_1$.

Note 3.3.2: In the case of Pre A*-functions, a Pre A*-variable can be implicant of a Pre A*-function because, if a Pre A*-variable has the value 2 in its functional value, then immediately the Pre A*-function will have the value 2. But this is not generally true in the case of Boolean functions.

4. CONCLUSION

In this manuscript, it is noticed that, the total number of Pre A*-functions $f: \mathbf{3}^n \rightarrow \mathbf{3}$ is $3^{(3^n)}$. It is also observed that if any Pre A*-variable assumes the value 2 in its Pre A*-function, then the function has the value 2 (the dominance property of 2). The principle of duality and its properties of Pre A*-functions is identified. The min term of a Pre A*-variables $x_1, x_2, \dots, x_n$ is obtained as a Pre A*-product $y_1 y_2 \dots y_n$ where $y_i = x_i$ or $y_i = x_i^{\sim}$. It is also noticed that, if any one of the Pre A*-variables in any min term has the value 2, then the sum of min terms containing that min term has the value 2. Every Pre A*-function can be represented by a Sum-of-Products expansion (SPE) or by Sum-of-Products expansion (PSE). It is observed that, a Pre A*-variable can be an implicant of a Pre A*-function but this is not generally true in the case of Boolean functions.

In general one can observe that, many common properties are obeyed by both Pre A*-functions and Boolean functions.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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