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Determinants of Circulant Matrices with Some Certain Sequences

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ABSTRACT

Let $\{a_k\}$ be a sequence of real numbers defined by an mth order linear homogenous recurrence relation. In this paper we obtain a determinant formula for the circulant matrix $A = circ(a_1, a_2, \cdots, a_n)$, providing a generalization of determinantal results in papers of Bozkurt [2], Bozkurt and Tam [3], and Shen, et al. [8].

Keywords: circulant matrix, determinant, Fibonacci sequence, Lucas sequence, tribonacci sequence.
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1. INTRODUCTION

The circulant matrix $V = circ(v_1, v_2, \dots, v_n)$ associated to real numbers $v_1, v_2, \dots, v_n$ is the $n \times n$ matrix

$$V = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ v_n & v_1 & \cdots & v_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ v_2 & v_3 & \dots & v_1 \end{pmatrix}.$$

Circulant matrices are one of the most interesting members of matrices. They have elegant algebraic properties. For example, $Circ(n)$ is an algebra on $\mathbb{C}$. Let ϵ be a primitive n^{th} root of unity. For each $0 \le k \le n-1$, $\lambda_k = \sum_{j=1}^{n} v_j\, \epsilon^{k(j-1)}$ is an eigenvalue of $V = circ(v_1, v_2, \dots, v_n)$ and the corresponding eigenvector is $x_k = \frac{1}{\sqrt{n}}(1, \epsilon^k, \epsilon^{2k}, \dots, \epsilon^{(n-1)k}) \in \mathbb{C}^n$. Indeed, all circulant matrices have the same ordered set of orthonormal eigenvectors $\{x_k\}$. Besides, $\det V = \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} v_j\, \epsilon^{kj}\right)$. The reader can consult the text of Davis [4] for further properties of circulant matrices. On the other hand, circulant matrices have a widespread applications in many parts of mathematics. The excellent survey paper [6] includes many applications of circulant matrices in various areas of mathematics. Also, they have applications in signal processing, the study of cyclic codes for error corrections [5] and in quantum mechanics [1].

Recently, many authors have investigated some properties of circulant matrices associated to so famous integer sequences, for example, the Fibonacci sequence and the Lucas sequence. Let $a, b, p, q \in \mathbb{Z}$. Define a sequence (U_n) by the second order recurrence relation

$$U_n = pU_{n-1} + qU_{n-2} \qquad (1)$$

$(n \ge 3)$ with initial conditions $U_1 = a$ and $U_2 = b$. Taking $(p, q, a, b) = (1,1,1,1)$, $(1,1,1,3)$, $(1,2,1,1)$ and $(1,2,1,3)$, (U_n) becomes the Fibonacci sequence (F_n), the Lucas sequence (L_n), the Jacobsthal sequence (J_n) and the Jacobsthal-Lucas sequence (j_n), respectively. In 1970 Lind [7] obtained a formula for the determinant of $F = circ(F_r, F_{r+1}, \dots, F_{r+n-1})$ $(r \ge 1)$. In 2005 Solak [9] investigated matrix norms of $F = circ(F_1, F_2, \dots, F_n)$ and $L = circ(L_1, L_2, \dots, L_n)$. In 2011 Shen, Cen and Hao [8] showed that

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$$\det(F) = (1-F_{n+1})^{n-1} + F_n^{n-2}\sum_{k=1}^{n-1} F_k\left(\frac{1-F_{n+1}}{F_n}\right)^{k-1}$$

and

$$\det(L) = (1-L_{n+1})^{n-1} + \left(L_n - 2)^{n-2}\sum_{k=1}^{n-1}(L_{k+2} - 3L_{k+1}\right)\left(\frac{1-L_{n+1}}{L_n-2}\right)^{k-1}.$$

Recently, Bozkurt and Tam [3] have obtained determinant formulae for

$$J = circ(J_1, J_2, \dots, J_n) \text{ and } \mathbb{J} = circ(j_1, j_2, \dots, j_n)$$

using the same method. Then Bozkurt [2] has given a generalization of these determinant formulae as

$$\det(U) = (a^2 - bU_n)(a - U_{n+1})^{n-2} + \sum_{k=2}^{n-1}(aU_{k+1} - bU_k)(a - U_{n+1})^{k-2}(qU_n - b + qa)^{n-k}, \qquad (2)$$

where $\{U_k\}$ is the sequence in (1).

In all of the above-mentioned papers authors calculated determinants of circulant matrices associated to a sequence defined by a second order recurrence relation by using the same method. In this paper we generalize determinantal results of these papers for certain sequences defined by a recurrence relation of order $m \geq 1$.

2. THE MAIN RESULT

Let $c_1, c_2, \dots, c_m$ be real numbers and $c_m \neq 0$. Consider the sequence $\{a_k\}$ defined by the mth order linear homogenous recurrence relation

$$a_k = c_1 a_{k-1} + c_2 a_{k-1} + \cdots + c_m a_{k-m} \qquad (k \geq m+1) \qquad (3)$$

with initial conditions

$$a_1, a_2, \dots, a_m, \qquad (4)$$

which are given real numbers. Let $n > m$ and $A = circ(a_1, a_2, \dots, a_n)$. Let A_{ij} be the ij −entry of A. It is clear that $A_{ij} = a_{j-i+1}$ if $j \geq i$ and $a_{n+j-i+1}$ otherwise. On the other hand, for simplicity, we write $A_{ij} = a_{(j-i+1)}$ in both case. Our main goal is to reduce the order n of the determinant of A and to calculate it in a simpler way. In order to perform this, first we define an $n \times n$ matrix $P = (P_{ij})$, where

$$P_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i + j = n + 2, \\ -c_m & \text{if } i = m + 1 \text{ and } j = 1, \\ -c_t & \text{if } i + j - t = n + 2 \text{ and } i \geq m + 1 \text{ and } 1 \leq t \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then the ij −entry of the product of P and A is

$$(PA)_{ij} = \begin{cases} A_{1j} & \text{if } i = 1, \\ A_{n-i+2,j} & \text{if } 2 \leq i \leq m, \\ \alpha_t & \text{if } i + j = n + t + 1 \text{ and } 1 \leq t \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\alpha_t = A_{n-m+1,n-m+t} - c_1 A_{n-m+2,n-m+t} - \cdots - c_{m-1} A_{n,n-m+t} - c_m A_{1,n-m+t}. \qquad (5)$$

Now, we define a sequence $\{b_s^{(r)}\}$ for every $r = 1,2, \dots, m-1$ by the recurrence relation

$$b_s^{(r)} = -\frac{\alpha_2}{\alpha_1} b_{s-1}^{(r)} - \frac{\alpha_3}{\alpha_1} b_{s-2}^{(r)} - \cdots - \frac{\alpha_m}{\alpha_1} b_{s-m+1}^{(r)} \qquad (s \geq m) \qquad (6)$$

with initial conditions

$$b_i^{(r)} = \delta_{i,r}, \tag{7}$$

the Kronecker delta, for $i = 1,2,\dots,m-1$. We form another $n \times n$ matrix $Q = (Q_{ij})$ such that

$$Q_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i + j = n + 2, \\ b_{n-i+1}^{(j-1)} & \text{if } 2 \le i \le n-m+1 \text{ and } 2 \le j \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$(PAQ)_{ij} = \begin{cases} A_{1,1} & \text{if } i = j = 1, \\ A_{n-i+2,1} & \text{if } 2 \le i \le m \text{ and } j = 1, \\ \sum_{k=2}^{n} A_{1k}\, b_{n-k+1}^{(j-1)} & \text{if } i = 1 \text{ and } 2 \le j \le m, \\ \sum_{k=2}^{n} A_{n-i+2,k}\, b_{n-k+1}^{(j-1)} & \text{if } 2 \le i,j \le m, \\ \alpha_k & \text{if } i,j > m \text{ and } 1 \le k \le m \text{ and } i - j = k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $A_{ij} = a_{j-i+1}$ if $j \ge i$ and $a_{n+j-i+1}$ otherwise and that we write $A_{i,j} = a_{(j-i+1)}$ for simplicity. Also, it is clear that $\det P = \det Q = (-1)^{\frac{n(n+1)}{2}-1}$ and $\alpha_1 = a_1 - a_{n+1}$. Finally, we get the following lemma.

Lemma 1. *Let $\{a_k\}$ be the sequence defined by the recurrence relation in (3) with initial conditions in (4), $n > m$ and $A = circ(a_1, a_2, \dots, a_n)$. Then*

$$\det A = (a_1 - a_{n+1})^{n-m} \times \sum_{k_1=2}^{n} \cdots \sum_{k_{m-1}=2}^{n} \begin{vmatrix} a_1 & a_{(k_1)} & \cdots & a_{(k_{m-1})} \\ a_2 & a_{(k_1+1)} & \cdots & a_{(k_{m-1}+1)} \\ \vdots & \vdots & & \vdots \\ a_m & a_{(k_1+m-1)} & \cdots & a_{(k_{m-1}+m-1)} \end{vmatrix} \prod_{i=1}^{m-1} b_{n-k_i+1}^{(i)}, \tag{8}$$

where sequences $\{b_s^{(r)}\}$ are defined by the recurrence relation in (6) with initial conditions in (7).

Indeed, the determinant formula for $A = circ(a_1, a_2, \dots, a_n)$ in Lemma 1 is not effective but we obtain it by generalizing the common method of papers [8,3,2] for the sequence $\{a_k\}$ defined by a recurrence relation of order $m \ge 1$. To illustrate our goal we consider the well-known tribonacci sequence. The tribonacci sequence $\{a_k\}$ is defined by the recurrence relation

$$a_k = a_{k-1} + a_{k-2} + a_{k-3} \qquad (k \ge 4)$$

with initial conditions $a_1 = 1,\ a_2 = 1,\ a_3 = 2$. For convenience, we take $a_0 = 0$.

Corollary 1. *Let $\{a_k\}$ be the tribonacci sequence, $n > 3$ and $A = circ(a_1, a_2, \dots, a_n)$. Then*

$$\begin{aligned} det\,(A) = (1 - a_{n+1})^{n-3} \Big(& \sum_{i=2}^{n-3} \sum_{j=i+1}^{n-2} (a_{i-2}a_{j-1} - a_{i-1}a_{j-2}) \Big(\frac{\alpha_3}{\alpha_1}\Big)^{n-j-1} b_{j-i+2}^{(1)} \\ & + \sum_{i=2}^{n-2} \big((a_{i-2} + a_{i-1}) + a_{n-1}(a_{i+2} - 2a_{i+1}) + a_n(2a_i - a_{i+2}) \big) b_{n-i+1}^{(1)} \\ & + \sum_{i=2}^{n-2} \big(-a_{i-1} + a_n(a_{i+2} - 2a_{i+1}) \big) \frac{\alpha_1}{\alpha_3} b_{n-i+2}^{(1)} + (2a_n^2 - 2a_n - a_{n-1} + 1) \Big). \end{aligned}$$

Proof. Let $\{a_k\}$ in Lemma 1 be the tribonacci sequence. Then clearly $m = 3$, $a_1 = a_2 = 1$, $a_3 = 2$, $\alpha_1 = 1 - a_{n+1}$ and by Lemma 1, we have

$$\det(A) = (1 - a_{n+1})^{n-3} \sum_{i=2}^{n} \sum_{j=2}^{n} \begin{vmatrix} 1 & a_{(i)} & a_{(j)} \\ 1 & a_{(i+1)} & a_{(j+1)} \\ 2 & a_{(i+2)} & a_{(j+2)} \end{vmatrix} b_{n-i+1}^{(1)} b_{n-j+1}^{(2)}.$$

We denote the 3×3 determinant in the summation by $\Delta((i),(j))$. It is clear that $\Delta((i),(i)) = 0$ and $\Delta((j),(i)) = -\Delta((i),(j))$. Also, we have $\Delta((i),(j)) = \Delta(i,j)$ if $1 \leq i,j \leq n-3$. Thus

$$\det(A) = (1-a_{n+1})^{n-3} \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} \Delta((i),(j))(b_{n-i+1}^{(1)} b_{n-j+1}^{(2)} - b_{n-j+1}^{(1)} b_{n-i+1}^{(2)}).$$

Now, sequences $\{b_k^{(1)}\}$ and $\{b_k^{(2)}\}$ are generated by the recurrence relation in (6) with different initial conditions, all of which are given in (7). The characteristic equation of the recurrence relation in (6) is $\alpha_1 r^2 + \alpha_2 r + \alpha_3 = 0$, where $\alpha_1 = 1 - a_{n+1}$, $\alpha_2 = -a_n - a_{n-1}$ and $\alpha_3 = -a_n$. Since $\alpha_2^2 - 4\alpha_1\alpha_3 < (-a_n + a_{n-1})(3a_n + a_{n-1}) < 0$ for all $n \geq 1$, the characteristic equation has two distinct complex roots, say λ and μ. Finally, Binet's formula for sequences $b_k^{(1)}$ and $b_k^{(2)}$ are $b_k^{(1)} = \frac{\lambda\mu}{\mu-\lambda}(\lambda^{k-2} - \mu^{k-2})$ and $b_k^{(2)} = \frac{1}{\lambda-\mu}(\lambda^{k-1} - \mu^{k-1})$, respectively. Using Binet's formulae we have the identity

$$b_k^{(1)} b_t^{(2)} - b_t^{(1)} b_k^{(2)} = (\frac{\alpha_3}{\alpha_1})^{t-2} b_{k-t+2}^{(1)},$$

where $k \geq t$. Thus, we have

$$\det(A) = (1-a_{n+1})^{n-3} (\sum_{i=2}^{n-3} \sum_{j=i+1}^{n-2} \Delta(i,j)(\frac{\alpha_3}{\alpha_1})^{n-j-1} b_{j-i+2}^{(1)} + \sum_{i=2}^{n-2} \Delta(i,(n-1)) b_{n-i+1}^{(1)} + \sum_{i=2}^{n-2} \Delta(i,(n)) \frac{\alpha_1}{\alpha_3} b_{n-i+2}^{(1)} + \Delta((n-1),(n)) \frac{\alpha_1}{\alpha_3} b_3^{(1)}).$$

The proof follows from equalities

$$\Delta(i,j) = a_{i-2}a_{j-1} - a_{i-1}a_{j-2},$$
$$\Delta(i,(n-1)) = (2a_n - 1)a_i + (1 - 2a_{n-1})a_{i+1} + (a_{n-1} - a_n)a_{i+2},$$
$$\Delta(i,(n)) = a_i + (1 - 2a_n)a_{i+1} + a_{n-1}a_{i+2},$$
$$\Delta((n-1),(n)) \frac{\alpha_1}{\alpha_3} b_3^{(1)} = 2a_n^2 - 2a_n - a_{n-1} + 1.$$

□

We cannot state that the determinant formula in Corollary 1 is elegant but it reduces an $n \times n$ determinant to a double sum.

Corollary 2. ([2], Theorem 1) *Let $\{U_k\}$ be the sequence defined by the recurrence relation given in (1) with initial conditions $U_1 = a, U_2 = b$, $n > 3$ and $A = circ(U_1, U_2, \ldots, U_n)$. Then*

$$\det(U) = (a^2 - bU_n)(a - U_{n+1})^{n-2} + \sum_{k=2}^{n-1} (aU_{k+1} - bU_k)(a - U_{n+1})^{k-2}(qU_n - bpa)^{n-k}.$$

Proof. Let $\{a_k\}$ in Lemma 1 be the sequence $\{U_k\}$ given in (1) with initial conditions $U_1 = a$ and $U_2 = b$. Then $\alpha_1 = a - U_{n+1}$, $\alpha_2 = b - pU_1 - qU_n$ and hence $b_i^{(1)} = (-\alpha_2/\alpha_1)^{i-1}$. Thus, by Lemma 1, we have

$$\det(A) = (a - U_{n+1})^{n-2} \sum_{k=2}^{n} \begin{vmatrix} a & U_{(k)} \\ b & U_{(k+1)} \end{vmatrix} b_{n-k+1}^{(1)}$$
$$= (a - U_{n+1})^{n-2}[(a^2 - U_n b) + \sum_{k=2}^{n-1} \begin{vmatrix} a & U_k \\ b & U_{k+1} \end{vmatrix} b_{n-k+1}^{(1)}]$$
$$= (a - U_{n+1})^{n-2}[(a^2 - bU_n) + \sum_{k=2}^{n-1} (aU_{k+1} - bU_k)(-\frac{qU_n - b + pa}{a - U_{n+1}})^{n-k}].$$

A simple calculation completes the proof. □

Renaming terms of sequence $\{U_k\}$ as $\{W_{k-1}\}$ we obtain the same formula in Theorem 1 of Bozkurt's paper [2]. Also, by choosing convenient values for p, q, a and b in Corollary 1 we can obtain all determinant formulae in [3,8]. Taking $(p,q,a,b) = (1,1,1,1)$, $(1,1,1,3)$, $(1,2,1,1)$ and $(1,2,1,3)$, we have Theorems 2.1 and 3.1 of [8] and Theorems 2.1 and 2.2 of [3], respectively. Also, by Lemma 1, we can easily evaluate the determinant of $A = circ(a, a^2, a^3, \ldots, a^n)$, where a is a nonzero real number, as $\det(A) = a^n(1 - a^n)^{n-1}$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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