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A Note on Multivariate Lyapunov-Type Inequality

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ABSTRACT

We transfer the recent obtained result of univariate Lyapunov-type inequality for third order differential equations to the multivariate setting of a shell via the polar method. Our result is better than the result of Anastassiou [Appl. Math. Letters, 24 (2011), 2167-2171] for third order partial differential equations.

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1. INTRODUCTION AND MAIN RESULT

The Lyapunov inequality and many of its generalizations play a key role in the study of oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations. Up to now, the Lyapunov-type inequalities have been studied extensively such as [1,2,6,9]. However, there are not so many results for partial differential equations or systems except for [3,4] or [5].

Here, we give some notation for constructing the theoretical background given by Anastassiou [3] who was the first interested in the problem of finding on the multivariate Lyapunov-type inequalities in the literature:

Suppose that A be a spherical shell $\subseteq \mathbb{R}^N$ for $N>1$, i.e. $A = B(0,R_3) - \overline{B(0,R_1)}$ for $0<R_1<R_3$, where the ball

$$B(0,R) = \{x \in \mathbb{R}^N : |x| < R\} \tag{1}$$

for $R>0$ and $|\cdot|$ is the Euclidean norm. We also suppose that

$$S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\} \tag{2}$$

is the unit sphere in $\mathbb{R}^N$ with surface area

$$\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \tag{3}$$

i.e.

$$\int_{S^{N-1}} d\omega = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \tag{4}$$

where Γ is the gamma function. It is easy to see that every $x \in \mathbb{R}^N - \{0\}$ has a unique representation of the form $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$ [8, pp. 149-150]. Thus, $\mathbb{R}^N - \{0\}$ may be regarded as the Cartesian product $\overline{A} = [R_1, R_3] \times S^{N-1}$. Therefore, we have

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$$\int_A F(s)ds = \int_{S^{N-1}}\left(\int_{R_1}^{R_3} F(r\omega)r^{N-1}dr\right)d\omega \tag{5}$$

for $F \in C(\overline{A})$. Here, we deal with the partial differential equations involving radial derivatives of functions on $\overline{A}$, using the polar coordinates r, ω. If $f \in C^n(\overline{A})$ for $n \in \mathbb{N}$, then $f(\cdot\omega) \in C^n([R_1, R_3])$ for a fixed $\omega \in S^{N-1}$.

Recently, by using the result of Çakmak [6], Anastassiou [3] obtained the following result.

Theorem A. Suppose that $n \in \mathbb{N}$, $n \geq 2$ and $q \in C(\overline{A})$. If $f \in C^n(\overline{A})$ is a solution of the following partial differential equations

$$\frac{\partial^n f(x)}{\partial r^n} + q(x)f(x) = 0 \ , \ \ \forall x \in \overline{A}, \tag{6}$$

with the boundary value conditions

$$f(\partial B(0,R_1)) = f(\partial B(0,t_2)) = \cdots = f(\partial B(0,t_{n-1})) = f(\partial B(0,R_3)) = 0 \tag{7}$$

where $R_1 = t_1 < t_2 < \cdots < t_{n-1} < t_n = R_3$, and $f(t\omega) \not\equiv 0$, $\forall \omega \in S^{N-1}$, for any $t \in (t_k, t_{k+1})$, $k = 1,2,\ldots,n-1$, then the following inequality

$$\int_A |q(s)|ds > \left(\frac{(n-2)!n^n R_1^{N-1}}{(n-1)^{n-1}(R_3 - R_1)^{n-1}}\right)\left(\frac{2\pi^{N/2}}{\Gamma(N/2)}\right) \tag{8}$$

holds.

In 1907, Lyapunov [7] established the first Lyapunov inequality

$$\int_a^c |q(s)|ds > \frac{4}{c-a}, \tag{9}$$

if

$$x''(t) + q(t)x(t) = 0 \tag{10}$$

has a real solution $x(t)$ satisfying the boundary value conditions

$$x(a) = x(c) = 0 \tag{11}$$

for $x(t) \not\equiv 0$ for $t \in (a,c)$.

Since the appearance of Lyapunov's fundamental paper, various proofs and generalizations or improvements have appeared in the literature.

More recently, Aktaş et al. [1] obtained the following Lyapunov-type inequality for third order differential equations

$$x'''(t) + q(t)x(t) = 0, \tag{12}$$

where $q \in C([a,c])$, with the boundary value conditions

$$x(a) = x(b) = x(c) = 0 \tag{13}$$

for $x(t) \not\equiv 0$ for $t \in (a,b) \cup (b,c)$.

Theorem B. If the equation (12) has a solution $x(t)$ satisfying the boundary value conditions (13), then the following inequality

$$\int_a^c |q(s)|ds > \frac{16}{(c-a)^2} \tag{14}$$

holds.

Now, motivated by the recent results of Anastassiou [3], we transfer the univariate inequality (14) in Theorem B to the multivariate setting of a shell via the polar method.

Theorem 1. Suppose that $q \in C(\overline{A})$. If $f \in C^3(\overline{A})$ is a solution of the following third order partial differential equations

$$\frac{\partial^3 f(x)}{\partial r^3} + q(x)f(x) = 0 \ , \ \ \forall x \in \overline{A}, \tag{15}$$

with the boundary value conditions

$$f(\partial B(0,R_1)) = f(\partial B(0,R_2)) = f(\partial B(0,R_3)) = 0 \tag{16}$$

where $R_1 < R_2 < R_3$, and $f(t\omega) \not\equiv 0$, $\forall \omega \in S^{N-1}$, for any $t \in (R_1, R_2) \cup (R_2, R_3)$, then the following inequality

$$\int_A |q(s)|ds > \left(\frac{16 R_1^{N-1}}{(R_3 - R_1)^2}\right)\left(\frac{2\pi^{N/2}}{\Gamma(N/2)}\right) \tag{17}$$

holds.

Proof. One can rewrite (15) as

$$\frac{\partial^3 f(r\omega)}{\partial r^3} + q(r\omega)f(r\omega) = 0 \ , \ \ \forall (r,\omega) \in [R_1, R_3] \times S^{N-1}, \tag{18}$$

where $q(\cdot\omega) \in C([R_1, R_3])$, $\forall \omega \in S^{N-1}$, such that the boundary value conditions

$$f(R_1\omega) = f(R_2\omega) = f(R_3\omega) = 0 \tag{19}$$

for $\forall \omega \in S^{N-1}$. In addition, $f(r\omega) \not\equiv 0$ holds for any $r \in (R_1, R_2) \cup (R_2, R_3)$ and $\forall \omega \in S^{N-1}$. Thus, from inequality (14), we get

$$\frac{16}{\left(R_3-R_1\right)^2}<\int_{R_1}^{R_3}\left|q(r\omega)\right|dr=$$

$$=\int_{R_1}^{R_3}r^{1-N}r^{N-1}\left|q(r\omega)\right|dr\leq\left(\int_{R_1}^{R_3}r^{N-1}\left|q(r\omega)\right|dr\right)R_1^{1-N} \tag{20}$$

for a fixed $\omega\in S^{N-1}$. Therefore, we have the following inequality

$$\int_{R_1}^{R_3}r^{N-1}\left|q(r\omega)\right|dr>\frac{16R_1^{N-1}}{\left(R_3-R_1\right)^2} \tag{21}$$

for $\forall\omega\in S^{N-1}$ and

$$\int_{S^{N-1}}\left(\int_{R_1}^{R_3}r^{N-1}\left|q(r\omega)\right|dr\right)d\omega>\left(\frac{16R_1^{N-1}}{(R_3-R_1)^2}\right)\left(\frac{2\pi^{N/2}}{\Gamma\left(N/2\right)}\right), \tag{22}$$

which by (5), proves the inequality (17).

Remark 1. It is easy to see that the inequality (17) is better than the inequality (8) with $n=3$ in Theorem A given by Anastassiou [3] in the sense that (8) with $n=3$ follows from (17), but not conversely.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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