Gazi University Journal of Science
GU J Sci
28(3):473-482 (2015)

Lyapunov-Type Inequalities for Two Classes of Difference Systems with Dirichlet Boundary Conditions

Mustafa Fahri AKTAŞ[1], Devrim ÇAKMAK[2,♠], Merve ARITÜRK[1]

[1]*Faculty of Sciences, Department of Mathematics, Gazi University, 06500 Teknikokullar, Ankara, TURKEY*
[2]*Faculty of Education, Department of Mathematics Education, Gazi University, 06500 Teknikokullar, Ankara, TURKEY*

Received: 26/12/2014 Accepted: 13/01/2015

ABSTRACT

In this paper, we establish Lyapunov-type inequalities for two classes of difference systems which improve all existing ones in the literature. Applying our inequalities, we obtain a lower bound for the eigenvalues of corresponding systems.

AMS Subject Classification: 34C10, 39A10, 39A12.

Keywords: Difference systems; Lyapunov-type inequalities; Dirichlet boundary conditions.

1. INTRODUCTION

In 1983, Cheng [4] obtained the following inequality

$$\Im(b-a)\sum_{\tau=a}^{b-2} f_1(\tau) \geq 4, \tag{1.1}$$

where $f_1(n) \geq 0$ for all $n \in \mathbb{Z}$ and

$$\Im(z) = \begin{cases} \dfrac{z^2-1}{z}, & \text{if } z-1 \text{ is even} \\ z \quad , & \text{if } z-1 \text{ is odd} \end{cases} \tag{1.2}$$

if the second-order difference equation

$$-\Delta^2 u_1(n) = f_1(n)u_1(n+1) \tag{1.3}$$

♠Corresponding author, e-mail: dcakmak@gazi.edu.tr

has a real solution $u_1(n)$ satisfying Dirichlet boundary conditions

$$u_1(a) = 0 = u_1(b), u_1(n) \not\equiv 0, n \in \mathbb{Z}[a,b], \tag{1.4}$$

$a, b \in \mathbb{Z}$ with $a \le b-2$, and $\mathbb{Z}[a,b] = \{a, a+1, a+2, \ldots, b-1, b\}$, f_1 is a real-valued function defined on $\mathbb{Z}$. The inequality (1.1) is a discrete analogue of the following so-called Lyapunov inequality

$$(b-a)\int_a^b |f_1(s)|\, ds > 4 \tag{1.5}$$

if Hill's equation

$$-u_1''(t) = f_1(t)u_1(t), \tag{1.6}$$

where $f_1 \in C([a,b], \mathbb{R})$, has a real solution $u_1(t)$ such that Dirichlet boundary conditions

$$u_1(a) = 0 = u_1(b), u_1(t) \not\equiv 0, t \in (a,b), \tag{1.7}$$

where $a, b \in \mathbb{R}$ with $a < b$ [7].

In 2012, Zhang and Tang [15] obtained the following Lyapunov-type inequality for the $2k$-th order difference equations

$$-\Delta^{2k}u_1(n) = (-1)^{k-1}f_1(n)u_1(n+1) \tag{1.8}$$

with the boundary conditions

$$\Delta^{2i}u_1(a) = 0 = \Delta^{2i}u_1(b), i = 0,1,\ldots,k-1;\ u_1(n) \not\equiv 0, n \in \mathbb{Z}[a,b], \tag{1.9}$$

where $k \in \mathbb{N}$, $n \in \mathbb{Z}$ and $f_1(n)$ is a real-valued function defined on $\mathbb{Z}$.

Theorem A. *If (1.8) has a solution $u_1(n)$ satisfying the boundary conditions (1.9), then the following inequality*

$$\sum_{\tau=a}^{b-1}[|f_1(\tau)|(\tau-a+1)(b-\tau-1)] \ge \frac{2^{3(k-1)}}{(b-a)^{2k-3}} \tag{1.10}$$

holds.

It is easy to see that the inequality (1.10) is rewritten as

$$\sum_{\tau=a}^{b-2}[|f_1(\tau)|(\tau-a+1)(b-\tau-1)] \ge \frac{2^{3(k-1)}}{(b-a)^{2k-3}}. \tag{1.11}$$

Now, throughout the paper for the sake of brevity, we denote

$$\zeta_i(n) = \sum_{\tau=a}^{n} r_i^{1/(1-p_i)}(\tau) \quad \text{and} \quad \eta_i(n) = \sum_{\tau=n+1}^{b-1} r_i^{1/(1-p_i)}(\tau) \tag{1.12}$$

for $i = 1,2,\ldots,m$.

In 2012, Zhang and Tang [14] obtained Lyapunov-type inequalities for the following systems

$$\begin{cases} -\Delta\left(r_1(n)|\Delta u_1(n)|^{p_1-2}\,\Delta u_1(n)\right) = f_1(n)|u_1(n+1)|^{\alpha_1-2}u_1(n+1)|u_2(n+1)|^{\alpha_2} \\ -\Delta\left(r_2(n)|\Delta u_2(n)|^{p_2-2}\Delta u_2(n)\right) = f_2(n)|u_1(n+1)|^{\beta_1}|u_2(n+1)|^{\beta_2-2}u_2(n+1) \end{cases} \tag{1.13}$$

and

$$\begin{cases} -\Delta\left(r_1(n)|\Delta u_1(n)|^{p_1-2}\Delta u_1(n)\right) = f_1(n)|u_1(n+1)|^{\alpha_1-2}u_1(n+1)|u_2(n+1)|^{\alpha_2}\cdots|u_m(n+1)|^{\alpha_m} \\ -\Delta\left(r_2(n)|\Delta u_2(n)|^{p_2-2}\Delta u_2(n)\right) = f_2(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2-2}u_2(n+1)\cdots|u_m(n+1)|^{\alpha_m} \\ \cdots \\ -\Delta\left(r_m(n)|\Delta u_m(n)|^{p_m-2}\Delta u_m(n)\right) = f_m(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2}\cdots|u_m(n+1)|^{\alpha_m-2}u_m(n+1) \end{cases}. \tag{1.14}$$

For the sake of convenience, we give the following hypotheses:

(H_1) $r_i(n)$ and $f_i(n)$ are real-valued functions and $r_i(n) > 0$, $\forall n \in \mathbb{Z}$ and $i = 1,2,\ldots,m$,

(H_2) $1 < p_1, p_2, \alpha_1, \beta_2 < \infty$, $\alpha_2, \beta_1 \ge 0$ satisfy $\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1$ and $\frac{\beta_1}{p_1} + \frac{\beta_2}{p_2} = 1$,

(H_3) $1 < p_k < \infty$, and $\alpha_k \geq 0$ for $k = 1,2,\dots,m$ satisfy $\sum_{i=1}^{m} \frac{\alpha_i}{p_i} = 1$.

Theorem B. *Let $a, b \in \mathbb{Z}$ with $a \leq b-2$. Suppose that hypotheses (H_1) with $i = 1,2$ and (H_2) are satisfied. If the system (1.13) has a solution $(u_1(n), u_2(n))$ satisfying Dirichlet boundary conditions*

$$u_i(a) = 0 = u_i(b),\ u_i(n) \not\equiv 0, n \in \mathbb{Z}[a,b], i = 1,2, \tag{1.15}$$

then the following inequality

$$\left(\sum_{\tau=a}^{b-2} \frac{(\zeta_1(\tau)\eta_1(\tau))^{p_1-1}}{\zeta_1^{p_1-1}(\tau)+\eta_1^{p_1-1}(\tau)} f_1^+(\tau)\right)^{\frac{\alpha_1\beta_1}{p_1^2}} \left(\sum_{\tau=a}^{b-2} \frac{(\zeta_1(\tau)\eta_1(\tau))^{p_1-1}}{\zeta_1^{p_1-1}(\tau)+\eta_1^{p_1-1}(\tau)} f_2^+(\tau)\right)^{\frac{\beta_1\alpha_2}{p_1p_2}} \times$$

$$\left(\sum_{\tau=a}^{b-2} \frac{(\zeta_2(\tau)\eta_2(\tau))^{p_2-1}}{\zeta_2^{p_2-1}(\tau)+\eta_2^{p_2-1}(\tau)} f_1^+(\tau)\right)^{\frac{\beta_1\alpha_2}{p_1p_2}} \left(\sum_{\tau=a}^{b-2} \frac{(\zeta_2(\tau)\eta_2(\tau))^{p_2-1}}{\zeta_2^{p_2-1}(\tau)+\eta_2^{p_2-1}(\tau)} f_2^+(\tau)\right)^{\frac{\alpha_2\beta_2}{p_2^2}} \geq 1 \tag{1.16}$$

holds, where $f_i^+(n) = max\{0, f_i(n)\}, i = 1,2.$

Theorem C. *Let $a, b \in \mathbb{Z}$ with $a \leq b-2$. Suppose that hypotheses (H_1) and (H_3) are satisfied. If the system (1.14) has a solution $(u_1(n), u_2(n), \dots, u_m(n))$ satisfying Dirichlet boundary conditions*

$$u_i(a) = 0 = u_i(b),\ u_i(n) \not\equiv 0, n \in \mathbb{Z}[a,b], i = 1,2,\dots,m, \tag{1.17}$$

then the following inequality

$$\prod_{k=1}^{m}\prod_{i=1}^{m}\left(\sum_{\tau=a}^{b-2} \frac{\left(\zeta_k(\tau)\eta_k(\tau)\right)^{p_k-1}}{\zeta_k^{p_k-1}(\tau)+\eta_k^{p_k-1}(\tau)} f_i^+(\tau)\right)^{\frac{\alpha_k\alpha_i}{p_kp_i}} \geq 1 \tag{1.18}$$

holds, where $f_i^+(n) = max\{0, f_i(n)\}, i = 1,2,\dots,m.$

Remark 1.1. *It is clear that the system (1.13) with (1.4), (H_2), and the condition α_2=0 or β_1=0, or the system (1.14) with (1.4) and (H_3) for $m = 1$ reduces to the following problem*

$$-\Delta\left(r_1(n)|\Delta u_1(n)|^{p_1-2}\Delta u_1(n)\right) = f_1(n)|u_1(n+1)|^{p_1-2}u_1(n+1) \tag{1.19}$$

$$u_1(a) = 0 = u_1(b). \tag{1.20}$$

Moreover, when $\alpha_i = p_i$ for $i = 1,2,\dots,m$, and for $k \neq i, \alpha_k = 0$ for $k = 1,2,\dots,m$, we obtain a single equation similar to the equation (1.19) from the system (1.14).

Aktaş et al. [1], Aktaş [2], Çakmak and Tiryaki [5, 6], Tang and He [9], and Tiryaki et al. [11] established Lyapunov-type inequalities for the continuous cases of systems (1.13) and/or (1.14) and their special cases. For some of the most recent works on Lyapunov-type inequalities, the reader is referred to [4, 6, 8-10, 12]. Motivated by the above-mentioned papers, we establish Lyapunov-type inequalities for systems (1.13) and (1.14) which are better than that of Zhang and Tang [14].

2. MAIN RESULTS

One of the main results of this paper for the system (1.13) is as follows.

Theorem 2.1. *Let $a, b \in \mathbb{Z}$ with $a \leq b-2$. Suppose that hypotheses (H_1) with $i = 1,2$ and (H_2) are satisfied. If the system (1.13) has a solution $(u_1(n), u_2(n))$ satisfying Dirichlet boundary conditions (1.15), then the following inequality*

$$\left(\sum_{\tau=a}^{b-2} f_1^+(\tau)\left(\frac{(\zeta_1(\tau)\eta_1(\tau))^{p_1-1}}{\zeta_1^{p_1-1}(\tau)+\eta_1^{p_1-1}(\tau)}\right)^{\frac{\alpha_1}{p_1}}\left(\frac{(\zeta_2(\tau)\eta_2(\tau))^{p_2-1}}{\zeta_2^{p_2-1}(\tau)+\eta_2^{p_2-1}(\tau)}\right)^{\frac{\alpha_2}{p_2}}\right)^{\frac{\beta_1}{p_1}} \times$$

$$\left(\sum_{\tau=a}^{b-2} f_2^+(\tau)\left(\frac{(\zeta_1(\tau)\eta_1(\tau))^{p_1-1}}{\zeta_1^{p_1-1}(\tau)+\eta_1^{p_1-1}(\tau)}\right)^{\frac{\beta_1}{p_1}}\left(\frac{(\zeta_2(\tau)\eta_2(\tau))^{p_2-1}}{\zeta_2^{p_2-1}(\tau)+\eta_2^{p_2-1}(\tau)}\right)^{\frac{\beta_2}{p_2}}\right)^{\frac{\alpha_2}{p_2}} \geq 1 \tag{2.1}$$

holds, where $f_i^+(n) = max\{0, f_i(n)\}$ for $i = 1,2$.

Proof. Let $u_i(a) = 0 = u_i(b)$ and $u_i(n) \not\equiv 0,\ n \in \mathbb{Z}[a,b], i = 1,2$ hold. Multiplying the first equation of system (1.13) by $u_1(n+1)$ and the second equation of system (1.13) by $u_2(n+1)$, summing from a to $b-2$ and taking into account that $u_i(a) = 0 = u_i(b)$ for $i = 1,2$, we get

$$\sum_{\tau=a}^{b-1} r_1(\tau)|\Delta u_1(\tau)|^{p_1} = \sum_{\tau=a}^{b-2} f_1(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2} \le \sum_{\tau=a}^{b-2} f_1{}^{+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2} \tag{2.2}$$

and

$$\sum_{\tau=a}^{b-1} r_2(\tau)|\Delta u_2(\tau)|^{p_2} = \sum_{\tau=a}^{b-2} f_2(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2} \le \sum_{\tau=a}^{b-2} f_2^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}. \tag{2.3}$$

It follows from (1.12), (1.15), and Hölder's inequality that

$$|u_i(n+1)|^{p_i} = \left|\sum_{\tau=a}^{n} \Delta u_i(\tau)\right|^{p_i} \le \left(\sum_{\tau=a}^{n} |\Delta u_i(\tau)|\right)^{p_i} \le$$

$$\left(\sum_{\tau=a}^{n} r_i{}^{1/(1-p_i)}(\tau)\right)^{p_i-1} \sum_{\tau=a}^{n} r_i(\tau)|\Delta u_i(\tau)|^{p_i} = \zeta_i{}^{p_i-1}(n) \sum_{\tau=a}^{n} r_i(\tau)|\Delta u_i(\tau)|^{p_i} \tag{2.4}$$

and

$$|u_i(n+1)|^{p_i} = \left|\sum_{\tau=n+1}^{b-1} \Delta u_i(\tau)\right|^{p_i} \le \left(\sum_{\tau=n+1}^{b-1} |\Delta u_i(\tau)|\right)^{p_i} \le$$

$$\left(\sum_{\tau=n+1}^{b-1} r_i{}^{1/(1-p_i)}(\tau)\right)^{p_i-1} \sum_{\tau=n+1}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i} = \eta_i{}^{p_i-1}(n) \sum_{\tau=n+1}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i} \tag{2.5}$$

for $i = 1,2$ and $a \le n \le b-1$. Adding (2.4) and (2.5), we have

$$|u_i(n+1)|^{p_i} \le \frac{(\zeta_i(n)\eta_i(n))^{p_i-1}}{\zeta_i{}^{p_i-1}(n) + \eta_i{}^{p_i-1}(n)} \sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i} \tag{2.6}$$

for $i = 1,2$ and $a \le n \le b-1$. If we take the $\frac{\alpha_1}{p_1}$-th and $\frac{\beta_1}{p_1}$-th powers of both sides of the inequality (2.6) with $i = 1$, we have

$$|u_1(n+1)|^{\alpha_1} \le \left(\frac{\big(\zeta_1(n)\eta_1(n)\big)^{p_1-1}}{\zeta_1{}^{p_1-1}(n) + \eta_1{}^{p_1-1}(n)}\right)^{\frac{\alpha_1}{p_1}} \left(\sum_{\tau=a}^{b-1} r_1(\tau)|\Delta u_1(\tau)|^{p_1}\right)^{\frac{\alpha_1}{p_1}} \tag{2.7}$$

and

$$|u_1(n+1)|^{\beta_1} \le \left(\frac{\big(\zeta_1(n)\eta_1(n)\big)^{p_1-1}}{\zeta_1{}^{p_1-1}(n) + \eta_1{}^{p_1-1}(n)}\right)^{\frac{\beta_1}{p_1}} \left(\sum_{\tau=a}^{b-1} r_1(\tau)|\Delta u_1(\tau)|^{p_1}\right)^{\frac{\beta_1}{p_1}}, \tag{2.8}$$

respectively. Thus, from (2.2), we have

$$|u_1(n+1)|^{\alpha_1} \le \left(\frac{\big(\zeta_1(n)\eta_1(n)\big)^{p_1-1}}{\zeta_1{}^{p_1-1}(n) + \eta_1{}^{p_1-1}(n)}\right)^{\frac{\alpha_1}{p_1}} \left(\sum_{\tau=a}^{b-2} f_1{}^{+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2}\right)^{\frac{\alpha_1}{p_1}} \tag{2.9}$$

and

$$|u_1(n+1)|^{\beta_1} \le \left(\frac{\left(\zeta_1(n)\eta_1(n)\right)^{p_1-1}}{\zeta_1{}^{p_1-1}(n)+\eta_1{}^{p_1-1}(n)}\right)^{\frac{\beta_1}{p_1}} \left(\sum_{\tau=a}^{b-2} f_1{}^{+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2}\right)^{\frac{\beta_1}{p_1}}. \tag{2.10}$$

Multiplying both sides of (2.9) by $f_1{}^{+}(n)|u_2(n+1)|^{\alpha_2}$, summing from a to $b-2$, we have

$$\left(\sum_{\tau=a}^{b-2} f_1{}^{+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2}\right)^{1-\frac{\alpha_1}{p_1}} \le \sum_{\tau=a}^{b-2} f_1{}^{+}(\tau)|u_2(\tau+1)|^{\alpha_2}\left(\frac{\left(\zeta_1(\tau)\eta_1(\tau)\right)^{p_1-1}}{\zeta_1{}^{p_1-1}(\tau)+\eta_1{}^{p_1-1}(\tau)}\right)^{\frac{\alpha_1}{p_1}}. \tag{2.11}$$

Similarly, if we take the $\frac{\alpha_2}{p_2}$- th and $\frac{\beta_2}{p_2}$-th powers of both sides of the inequality (2.6) with $i=2$, we have

$$|u_2(n+1)|^{\alpha_2} \le \left(\frac{\left(\zeta_2(n)\eta_2(n)\right)^{p_2-1}}{\zeta_2{}^{p_2-1}(n)+\eta_2{}^{p_2-1}(n)}\right)^{\frac{\alpha_2}{p_2}} \left(\sum_{\tau=a}^{b-2} f_2^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}\right)^{\frac{\alpha_2}{p_2}} \tag{2.12}$$

and

$$|u_2(n+1)|^{\beta_2} \le \left(\frac{\left(\zeta_2(n)\eta_2(n)\right)^{p_2-1}}{\zeta_2{}^{p_2-1}(n)+\eta_2{}^{p_2-1}(n)}\right)^{\frac{\beta_2}{p_2}} \left(\sum_{\tau=a}^{b-2} f_2^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}\right)^{\frac{\beta_2}{p_2}}, \tag{2.13}$$

respectively. Multiplying both sides of (2.13) by $f_2{}^{+}(n)|u_1(n+1)|^{\beta_1}$, summing from a to $b-2$, we have

$$\left(\sum_{\tau=a}^{b-2} f_2{}^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}\right)^{1-\frac{\beta_2}{p_2}} \le \sum_{\tau=a}^{b-2} f_2^{+}(\tau)|u_1(\tau+1)|^{\beta_1}\left(\frac{\left(\zeta_2(\tau)\eta_2(\tau)\right)^{p_2-1}}{\zeta_2{}^{p_2-1}(\tau)+\eta_2{}^{p_2-1}(\tau)}\right)^{\frac{\beta_2}{p_2}}. \tag{2.14}$$

By using (2.12) in (2.11) and (2.10) in (2.14), we have

$$\left(\sum_{\tau=a}^{b-2} f_1{}^{+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2}\right)^{1-\frac{\alpha_1}{p_1}} \le M_1\left(\sum_{\tau=a}^{b-2} f_2^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}\right)^{\frac{\alpha_2}{p_2}} \tag{2.15}$$

and

$$\left(\sum_{\tau=a}^{b-2} f_2{}^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}\right)^{1-\frac{\beta_2}{p_2}} \le M_2\left(\sum_{\tau=a}^{b-2} f_1^{+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2}\right)^{\frac{\beta_1}{p_1}}, \tag{2.16}$$

where

$$M_1 = \sum_{\tau=a}^{b-2} f_1^{+}(\tau)\left(\frac{\left(\zeta_1(\tau)\eta_1(\tau)\right)^{p_1-1}}{\zeta_1{}^{p_1-1}(\tau)+\eta_1{}^{p_1-1}(\tau)}\right)^{\frac{\alpha_1}{p_1}}\left(\frac{\left(\zeta_2(\tau)\eta_2(\tau)\right)^{p_2-1}}{\zeta_2{}^{p_2-1}(\tau)+\eta_2{}^{p_2-1}(\tau)}\right)^{\frac{\alpha_2}{p_2}} \tag{2.17}$$

and

$$M_2 = \sum_{\tau=a}^{b-2} f_2^{+}(\tau)\left(\frac{\left(\zeta_1(\tau)\eta_1(\tau)\right)^{p_1-1}}{\zeta_1{}^{p_1-1}(\tau)+\eta_1{}^{p_1-1}(\tau)}\right)^{\frac{\beta_1}{p_1}}\left(\frac{\left(\zeta_2(\tau)\eta_2(\tau)\right)^{p_2-1}}{\zeta_2{}^{p_2-1}(\tau)+\eta_2{}^{p_2-1}(\tau)}\right)^{\frac{\beta_2}{p_2}}, \tag{2.18}$$

respectively. If we take e_1-th and e_2-th powers of both sides of inequalities (2.15) and (2.16), and multiplying the resulting inequalities, we obtain

$$\left(\sum_{\tau=a}^{b-2} f_1{}^{+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2}\right)^{\left(1-\frac{\alpha_1}{p_1}\right)e_1}\left(\sum_{\tau=a}^{b-2} f_2{}^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}\right)^{\left(1-\frac{\beta_2}{p_2}\right)e_2} \le$$

$$M_1{}^{e_1}\left(\sum_{\tau=a}^{b-2} f_2^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}\right)^{\frac{\alpha_2}{p_2}e_1} M_2{}^{e_2}\left(\sum_{\tau=a}^{b-2} f_1^{+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2}\right)^{\frac{\beta_1}{p_1}e_2}. \tag{2.19}$$

Next, we prove that

$$0 < \sum_{\tau=a}^{b-2} f_1^{\,+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2}. \tag{2.20}$$

If (2.20) is not true, then

$$\sum_{\tau=a}^{b-2} f_1^{\,+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2} = 0. \tag{2.21}$$

From (2.2) and (2.21), we have

$$0 \le \sum_{\tau=a}^{b-1} r_1(\tau)|\Delta u_1(\tau)|^{p_1} = \sum_{\tau=a}^{b-2} f_1(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2} \le$$

$$\sum_{\tau=a}^{b-2} f_1^{\,+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2} = 0. \tag{2.22}$$

It follows from (H_1) with $i = 1$ that

$$\Delta u_1(n) \equiv 0 \tag{2.23}$$

for $a \le n \le b-1$. Combining (2.6) for $i = 1$ with (2.23), we obtain that $u_1(n) \equiv 0$ for $a \le n \le b$, which contradicts (1.15) with $i = 1$. Therefore, (2.20) holds. Similarly, we have

$$0 < \sum_{\tau=a}^{b-2} f_2^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2}. \tag{2.24}$$

Now, we choose e_1 and e_2 such that

$$0 < \textstyle\sum_{\tau=a}^{b-2} f_1^{\,+}(\tau)|u_1(\tau+1)|^{\alpha_1}|u_2(\tau+1)|^{\alpha_2} \quad \text{and} \quad 0 < \sum_{\tau=a}^{b-2} f_2^{+}(\tau)|u_1(\tau+1)|^{\beta_1}|u_2(\tau+1)|^{\beta_2} \tag{2.25}$$

cancel out in the inequality (2.19), i.e. solve the homogeneous linear system

$$\begin{cases} \left(1-\dfrac{\alpha_1}{p_1}\right)e_1 - \dfrac{\beta_1}{p_1}e_2 = 0 \\ \dfrac{\alpha_2}{p_2}e_1 - \left(1-\dfrac{\beta_2}{p_2}\right)e_2 = 0 \end{cases}. \tag{2.26}$$

We observe that by hypotheses $\frac{\alpha_1}{p_1}+\frac{\alpha_2}{p_2} = 1$ and $\frac{\beta_1}{p_1}+\frac{\beta_2}{p_2} = 1$, this system admits a nontrivial solution, indeed all equations are equivalent to $\left(1-\frac{\alpha_1}{p_1}\right)e_1 = \frac{\beta_1}{p_1}e_2$ and $\frac{\alpha_2}{p_2}e_1 = \left(1-\frac{\beta_2}{p_2}\right)e_2$. Hence, we may take $e_1 = \frac{\beta_1}{p_1}$ and $e_2 = \frac{\alpha_2}{p_2}$, and we get the inequality (2.1) which completes the proof. ■

The following result gives the new Lyapunov-type inequality for the system (1.14).

Theorem 2.2. *Let $a, b \in \mathbb{Z}$ with $a \le b-2$. Suppose that hypotheses (H_1) and (H_3) are satisfied. If the system (1.14) has a solution $(u_1(n), u_2(n), \dots, u_m(n))$ satisfying Dirichlet boundary conditions (1.17), then the following inequality*

$$\prod_{i=1}^{m}\left[\sum_{\tau=a}^{b-2} f_i^{+}(\tau)\prod_{k=1}^{m}\left(\frac{\left(\zeta_k(\tau)\eta_k(\tau)\right)^{p_k-1}}{\zeta_k^{\,p_k-1}(\tau)+\eta_k^{\,p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}}\right]^{\frac{\alpha_i}{p_i}} \ge 1 \tag{2.27}$$

holds, where $f_i^{+}(n) = max\{0, f_i(n)\}$ for $i = 1,2,\dots,m$.

Proof. Let $u_i(a) = 0 = u_i(b)$ and $u_i(n) \not\equiv 0, n \in \mathbb{Z}[a,b], i = 1,2,\dots,m$ hold. Multiplying the i-th equation of system (1.14) by $u_i(n+1)$ and summing from a to $b-2$ and taking into account that $u_i(a) = 0 = u_i(b)$ for $i = 1,2,\dots m$, we get

$$\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i} = \sum_{\tau=a}^{b-2}\left[f_i(\tau)\prod_{k=1}^{m}|u_k(\tau+1)|^{\alpha_k}\right] \le \sum_{\tau=a}^{b-2}\left[f_i^{+}(\tau)\prod_{k=1}^{m}|u_k(\tau+1)|^{\alpha_k}\right] \tag{2.28}$$

for $i = 1,2,\dots,m$. By using $u_i(a) = 0$, (1.12) and Hölder's inequality, we get

$$|u_i(n+1)|^{p_i} = \left|\sum_{\tau=a}^{n}\Delta u_i(\tau)\right|^{p_i} \le$$

$$\left(\sum_{\tau=a}^{n} r_i^{1/(1-p_i)}(\tau)\right)^{p_i-1}\left(\sum_{\tau=a}^{n} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)=\zeta_i^{p_i-1}(n)\sum_{\tau=a}^{n} r_i(\tau)|\Delta u_i(\tau)|^{p_i} \tag{2.29}$$

for $i=1,2,\dots,m$ and $a\le n\le b-1$. Similarly, by using $u_i(b)=0$, (1.12) and Hölder's inequality, we get

$$|u_i(n+1)|^{p_i}\le \eta_i^{p_i-1}(n)\sum_{\tau=n+1}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i} \tag{2.30}$$

for $i=1,2,\dots,m$ and $a\le n\le b-1$. Adding (2.29) and (2.30), we have

$$|u_i(n+1)|^{p_i}\le \frac{\left(\zeta_i(n)\eta_i(n)\right)^{p_i-1}}{\zeta_i^{p_i-1}(n)+\eta_i^{p_i-1}(n)}\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i} \tag{2.31}$$

for $i=1,2,\dots,m$ and $a\le n\le b-1$. If we take the $\frac{\alpha_i}{p_i}$-th power of both sides of the inequality (2.31), we obtain

$$|u_i(n+1)|^{\alpha_i}\le \left(\frac{\left(\zeta_i(n)\eta_i(n)\right)^{p_i-1}}{\zeta_i^{p_i-1}(n)+\eta_i^{p_i-1}(n)}\right)^{\frac{\alpha_i}{p_i}}\left(\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{\frac{\alpha_i}{p_i}}. \tag{2.32}$$

Multiplying both sides of (2.31) by $f_i^+(n)\prod_{\substack{k=1\\k\neq i}}^{m}|u_k(n+1)|^{\alpha_k}$, summing from a to $b-2$, we have

$$\sum_{\tau=a}^{b-2} f_i^+(\tau)\prod_{k=1}^{m}|u_k(\tau+1)|^{\alpha_k}\le \sum_{\tau=a}^{b-2}\left(\frac{(\zeta_i(\tau)\eta_i(\tau))^{p_i-1}}{\zeta_i^{p_i-1}(\tau)+\eta_i^{p_i-1}(\tau)}\right)^{\frac{\alpha_i}{p_i}}\times$$
$$\left(\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{\frac{\alpha_i}{p_i}} f_i^+(\tau)\prod_{\substack{k=1\\k\neq i}}^{m}|u_k(\tau+1)|^{\alpha_k} \tag{2.33}$$

for $i=1,2,\dots,m$. By using (2.28) in (2.33), we have

$$\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\le \sum_{\tau=a}^{b-2}\left(\frac{(\zeta_i(\tau)\eta_i(\tau))^{p_i-1}}{\zeta_i^{p_i-1}(\tau)+\eta_i^{p_i-1}(\tau)}\right)^{\frac{\alpha_i}{p_i}}\left(\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{\frac{\alpha_i}{p_i}} f_i^+(\tau)\prod_{\substack{k=1\\k\neq i}}^{m}|u_k(\tau+1)|^{\alpha_k} \tag{2.34}$$

for $i=1,2,\dots,m$. Therefore, by using (2.31) in (2.34), we have

$$\left(\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{1-\frac{\alpha_i}{p_i}}\le \prod_{\substack{k=1\\k\neq i}}^{m}\left(\sum_{\tau=a}^{b-1} r_k(\tau)|\Delta u_k(\tau)|^{p_k}\right)^{\frac{\alpha_k}{p_k}}\sum_{\tau=a}^{b-2} f_i^+(\tau)\prod_{k=1}^{m}\left(\frac{\left(\zeta_k(\tau)\eta_k(\tau)\right)^{p_k-1}}{\zeta_k^{p_k-1}(\tau)+\eta_k^{p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}} \tag{2.35}$$

for $i=1,2,\dots,m$. If we take the e_i-th power of both side of the inequalities (2.35) for $i=1,2,\dots,m$, and multiplying the resulting inequalities, we obtain

$$\prod_{i=1}^{m}\left(\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{\left(1-\frac{\alpha_i}{p_i}\right)e_i}\le$$
$$\prod_{i=1}^{m}\left[\prod_{\substack{k=1\\k\neq i}}^{m}\left(\sum_{\tau=a}^{b-1} r_k(\tau)|\Delta u_k(\tau)|^{p_k}\right)^{\frac{\alpha_k}{p_k}}\sum_{\tau=a}^{b-2} f_i^+(\tau)\prod_{k=1}^{m}\left(\frac{\left(\zeta_k(\tau)\eta_k(\tau)\right)^{p_k-1}}{\zeta_k^{p_k-1}(\tau)+\eta_k^{p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}}\right]^{e_i} \tag{2.36}$$

and hence

$$\prod_{i=1}^{m}\left(\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{\left(1-\frac{\alpha_i}{p_i}\right)e_i}\le \prod_{k=1}^{m}\left(\sum_{\tau=a}^{b-1} r_k(\tau)|\Delta u_k(\tau)|^{p_k}\right)^{\frac{\alpha_k}{p_k}\sum_{\substack{i=1\\i\neq k}}^{m} e_i}\times$$
$$\prod_{i=1}^{m}\left(\sum_{\tau=a}^{b-2} f_i^+(\tau)\prod_{k=1}^{m}\left(\frac{\left(\zeta_k(\tau)\eta_k(\tau)\right)^{p_k-1}}{\zeta_k^{p_k-1}(\tau)+\eta_k^{p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}}\right)^{e_i}. \tag{2.37}$$

It is easy to see that by using similar technique to the proof of Theorem 2.1, we obtain the following inequality

$$0<\sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i} \tag{2.38}$$

for $i = 1,2,\dots,m$. Now, we choose e_i such that $0 < \sum_{\tau=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}$ for $i = 1,2,\dots,m$ cancel out in the inequality (2.37), i.e. solve the homogeneous linear system

$$\begin{cases} (p_1-\alpha_1)e_1 - \alpha_1 e_2 - \alpha_1 e_3 - \cdots - \alpha_1 e_m = 0 \\ -\alpha_2 e_1 + (p_2-\alpha_2)e_2 - \alpha_2 e_3 - \cdots - \alpha_2 e_m = 0 \\ \cdots \\ -\alpha_m e_1 - \alpha_m e_2 - \alpha_m e_3 - \cdots + (p_m-\alpha_m)e_m = 0 \end{cases}. \tag{2.39}$$

We observe that by hypothesis $\sum_{i=1}^{m}\frac{\alpha_i}{p_i} = 1$, this system admits a nontrivial solution, indeed all equations are equivalent to

$$\frac{\alpha_i}{p_i}\left(\sum_{\substack{k=1\\k\neq i}}^{m} e_k\right) = e_i\left(\sum_{\substack{k=1\\k\neq i}}^{m}\frac{\alpha_k}{p_k}\right)$$

for $i = 1,2,\dots,m$. Hence, we may take $e_i = \frac{\alpha_i}{p_i}$ for $i = 1,2,\dots,m$, and we get the inequality (2.27) which completes the proof.■

Remark 2.1. *It is easy to see that if we use generalized Hölder's inequality to the inequalities (2.1) and (2.27), then they reduce to the inequalities (1.16) and (1.18) obtained by Zhang and Tang* [14], *respectively. Thus, they are sharper than (1.16) and (1.18). Moreover, if we take $r_1(n) = 1$ and $p_1 = 2$ in the problem (1.19)-(1.20), then Theorems 2.1, 2.2, B, and C are equivalent. In this case, from the inequalities (1.16), (1.18), (2.1), and (2.27), we get*

$$\sum_{\tau=a}^{b-2} f_1^+(\tau)(\tau-a+1)(b-\tau-1) \geq b-a. \tag{2.40}$$

If we also take $m = 2$ in the system (1.14), and $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$ in the system (1.13), then Theorems 2.1 and 2.2 are equivalent.

Remark 2.2. *Note that since $f_1^+(n) \leq |f_1(n)|$, the inequality (2.40) is better than the inequality (1.11) with $k = 1$. Moreover, by using*

$$M(n) = (n-a+1)(b-n-1) \leq \max_{a\leq n\leq b-1} M(n) = M\left(\frac{a+b}{2}-1\right) = \left(\frac{b-a}{2}\right)^2$$

in the inequality (2.40), we get

$$\sum_{\tau=a}^{b-2} f_1^+(\tau) \geq \frac{4}{b-a}. \tag{2.41}$$

Therefore, if we take $f_1(n) \geq 0$, then when $b-a-1$ is odd, (2.41) is the same as (1.1). However, (2.41) is worse than (1.1) when $b-a-1$ is even.

Now, we apply our Lyapunov-type inequalities to obtain a lower bound for the first eigencurve in the generalized spectra. Let $a, b \in \mathbb{Z}$ with $a \leq b-2$. We consider here the following difference system

$$\begin{cases} -\Delta\left(|\Delta u_1(n)|^{p_1-2}\Delta u_1(n)\right) = \lambda_1\alpha_1 q(n)|u_1(n+1)|^{\alpha_1-2}u_1(n+1)|u_2(n+1)|^{\alpha_2}\cdots|u_m(n+1)|^{\alpha_m} \\ -\Delta\left(|\Delta u_2(n)|^{p_2-2}\Delta u_2(n)\right) = \lambda_2\alpha_2 q(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2-2}u_2(n+1)\cdots|u_m(n+1)|^{\alpha_m} \\ \cdots \\ -\Delta\left(|\Delta u_m(n)|^{p_m-2}\Delta u_m(n)\right) = \lambda_m\alpha_m q(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2}\cdots|u_m(n+1)|^{\alpha_m-2}u_m(n+1) \end{cases}, \tag{2.42}$$

where $q(n) > 0, \lambda_i \in \mathbb{R}$, p_i and α_i are the same as those in the hypothesis (H_3), and u_i satisfies Dirichlet boundary conditions

$$u_i(a) = 0 = u_i(b),\ u_i(n) \not\equiv 0, n \in \mathbb{Z}[a+1, b-1], i = 1,2,\dots,m. \tag{2.43}$$

We define the generalized spectrum S of a nonlinear difference system as the set of vector $(\lambda_1,\lambda_2,\dots,\lambda_m) \in \mathbb{R}^m$ such that the eigenvalue problem (2.42)-(2.43) admits a nontrivial solution.

Boundary problem (2.42)-(2.43) is a generalization of the following p_1-Laplacian difference equation

$$-\Delta\left(|\Delta u_1(n)|^{p_1-2}\Delta u_1(n)\right) = \lambda_1 p_1 q(n)|u_1(n+1)|^{p_1-2}u_1(n+1) \tag{2.44}$$

with Dirichlet boundary conditions

$$u_1(a) = 0 = u_1(b),\ u_1(n) \not\equiv 0, n \in \mathbb{Z}[a+1, b-1], \tag{2.45}$$

where $p_1 > 1, \lambda_1 \in \mathbb{R}$, and $q(n) > 0$. When $p_1 = 2$, Atkinson [3, Theorems 4.3.1 and 4.3.5] investigated the existence of eigenvalues for (2.44)-(2.45), see also [13].

Let $f_i(n) = \lambda_i\alpha_i q(n)$ for $i = 1,2,\dots,m$. Then we can apply Theorem 2.2 to boundary problem (2.42)-(2.43) and obtain a lower bound for the m-th component of any generalized eigenvalue $(\lambda_1,\lambda_2,\dots,\lambda_m)$ of the system (2.42).

Theorem 2.3. *Let $a, b \in \mathbb{Z}$ with $a \leq b-2$. Assume that $1 < p_i < \infty, \alpha_i > 0$ satisfy $\sum_{i=1}^{m} \frac{\alpha_i}{p_i} = 1$, and $q(n) > 0$ for all $n \in \mathbb{Z}$. Then there exists a function $h(\lambda_1, \lambda_2, \dots, \lambda_{m-1})$ such that $|\lambda_m| \geq h(\lambda_1, \lambda_2, \dots, \lambda_{m-1})$ for every generalized eigenvalue $(\lambda_1, \lambda_2, \dots, \lambda_m)$ of problem (2.42)-(2.43), where $h(\lambda_1, \lambda_2, \dots, \lambda_{m-1})$ is given by*

$$h(\lambda_1, \lambda_2, \dots, \lambda_{m-1}) = \frac{1}{\alpha_m}\left[\prod_{i=1}^{m-1}(|\lambda_i|\alpha_i)^{\frac{\alpha_i}{p_i}} \sum_{\tau=a}^{b-2} q(\tau) \prod_{k=1}^{m}\left(\frac{\left(\zeta_k(\tau)\eta_k(\tau)\right)^{p_k-1}}{\zeta_k{}^{p_k-1}(\tau)+\eta_k{}^{p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}}\right]^{\frac{-p_m}{\alpha_m}}. \quad (2.46)$$

Proof. For the eigenvalue $(\lambda_1, \lambda_2, \dots, \lambda_m)$, (2.42)-(2.43) has a nontrivial solution $\left(u_1(n), u_2(n), \dots, u_m(n)\right)$. That is the system (1.14) with $f_i(n) = \lambda_i \alpha_i q(n)$ has a solution $\left(u_1(n), u_2(n), \dots, u_m(n)\right)$ satisfying (1.17), it follows from (2.27) that $f_i(n) = \lambda_i \alpha_i q(n)$, for all $n \in \mathbb{Z}, i = 1,2, \dots, m$, and that

$$1 \leq \prod_{i=1}^{m}\left[\sum_{\tau=a}^{b-2} f_i^{+}(\tau) \prod_{k=1}^{m}\left(\frac{(\zeta_k(\tau)\eta_k(\tau))^{p_k-1}}{\zeta_k{}^{p_k-1}(\tau)+\eta_k{}^{p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}}\right]^{\frac{\alpha_i}{p_i}} \leq$$

$$\prod_{i=1}^{m}(|\lambda_i|\alpha_i)^{\frac{\alpha_i}{p_i}} \prod_{i=1}^{m}\left[\sum_{\tau=a}^{b-2} q(\tau) \prod_{k=1}^{m}\left(\frac{(\zeta_k(\tau)\eta_k(\tau))^{p_k-1}}{\zeta_k{}^{p_k-1}(\tau)+\eta_k{}^{p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}}\right]^{\frac{\alpha_i}{p_i}} =$$

$$\prod_{i=1}^{m}(|\lambda_i|\alpha_i)^{\frac{\alpha_i}{p_i}} \sum_{\tau=a}^{b-2} q(\tau) \prod_{k=1}^{m}\left(\frac{(\zeta_k(\tau)\eta_k(\tau))^{p_k-1}}{\zeta_k{}^{p_k-1}(\tau)+\eta_k{}^{p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}}.$$

Hence, we have

$$|\lambda_m| \geq \frac{1}{\alpha_m}\left[\prod_{i=1}^{m-1}(|\lambda_i|\alpha_i)^{\frac{\alpha_i}{p_i}} \sum_{\tau=a}^{b-2} q(\tau) \prod_{k=1}^{m}\left(\frac{\left(\zeta_k(\tau)\eta_k(\tau)\right)^{p_k-1}}{\zeta_k{}^{p_k-1}(\tau)+\eta_k{}^{p_k-1}(\tau)}\right)^{\frac{\alpha_k}{p_k}}\right]^{\frac{-p_m}{\alpha_m}}. \quad (2.47)$$

This completes the proof.■

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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