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Tripled Fixed Point Results in Generalized Metric Spaces Under Nonlinear Type Contractions Depended on Another Function

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ABSTRACT

In 2006, Mustafa and Sims [18-19] introduced an improved version of the generalized metric space structure which they called G-metric spaces and in 2011; Berinde and Borcut [11] introduced the concept of triple fixed point. The intent of this paper is to establish some tripled fixed point theorems for mappings having mixed monotone property under nonlinear type contractions depended on another function in the framework of a G-metric space X enclosed with partial order. The presented results generalize, improve and extend corresponding results of H. Aydi et al. [13] (Tripled Fixed Point Results in Generalized Metric Spaces" Journal of Applied Mathematics Volume 2012, Article ID 314279, 10 pages, doi:10.1155/2012/314279). Moreover, some examples are provided to illustrate the usability of the obtained results.

Keywords and Phrases: Tripled fixed point; nonlinear contractions; partially ordered sets; G-metric spaces; mixed monotone; ICS mapping.

Mathematics Subject Classification (2010): 47H10, 54H25;

1. INTRODUCTION

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and is a very active field of research at present. Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tool for proving the existence and eventually the uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities).

Basic topological properties of an ordered set like convergence were introduced by Wolk [1]. In 1981, Monjardet [2] considered metrics on partially ordered sets. Ran and Reurings [3] proved and analog of Banach Contraction mapping principle in partially ordered metric spaces. Bhashkar and Lakshmikantham in [4] introduced the concept of coupled fixed point of a mapping $F\colon X \times X \to X$ and investigated the existence and uniqueness of a coupled fixed point theorem in partially ordered complete metric spaces. Lakshmikantham and Ciric in [5] defined mixed g-monotone property and coupled coincidence point in partially ordered metric spaces. Following this trend, Berinde and Borcut [11] introduced

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the concept of triple fixed point and established some triple fixed point theorems in partially ordered metric spaces. A tripled fixed point is a generalization of the well-known concept of coupled fixed point. The study of tripled fixed point is a very interesting research area in fixed point theory.

The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhage [14-17]. Recently, Mustafa and Sims [18-19] have shown that most of the results concerning Dhage's D-metric spaces are invalid. In [18-19], they introduced an improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces, one can refer to the papers [24-30, 33-35]. Hassen et al. [13] established some tripled fixed point results in G-Metric Spaces.

The purpose of this paper is three fold which can be described as follows.

1. We give some example which shows the weakness of Theorem 16 and Theorem 17 (Theorem 2.1 and 2.4 of Hassen et al. [13]).

2. We establish some tripled fixed point theorems for continuous mappings having mixed monotone property under nonlinear type contractions depended on another function $T: X \to X$ (T is an ICS) in the frame work of a G-metric space X enclosed with partial order. Also these theorems, are still valid for F, not necessarily continuous, assuming $(X, G, \leq)$ is regular. We prove the uniqueness of tripled fixed point for such mappings in this setup. Our results are extensions of the main results of Hassen et al. [13].

3. We present some examples to illustrate the effectiveness of our results. Also, we give a simple example which shows that if T is not an ICS mapping then the conclusion of main results fail.

2. DEFINITIONS AND PREMILINARIES

Throughout this paper, we denote $\mathbb{R}^+$ the set of all positive real numbers and $\mathbb{N}$ the set of all natural numbers. The triple $(X, G, \leq)$ is called a partially ordered G-metric space if $(X, \leq)$ is a partially ordered set and (X, G) is a G-metric space. Further, if (X, G) is complete metric space, and then the triple $(X, G, \leq)$ is called a partially ordered complete G-metric space. We assume that $X \neq \emptyset$ and

$$X^k = \underbrace{X \times X \times \dots \dots \dots \times X}_{k-\text{times}} \qquad (1)$$

In what follows, we collect some relevant definitions, fundamental results, examples for our further use.

Definition 1 (2006, Mustafa and Sims [19]) Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$, if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $x \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (Symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z), \forall\ x, y, z, a \in X$, (Rectangle inequality).

Then the function G is called generalized metric or more specially, G-metric on X, and the pair (X, G) is called a G-metric space. Every G-metric on X will define a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \ \forall\ x, y \in X \qquad (2)$$

Example 2 (Hassen et al. [13]) Let (X, d) be a metric space. The function $G : X \times X \times X \to \mathbb{R}^+$ defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\} \qquad (3)$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x) \qquad (4)$$

for all $x, y, z \in X$, is a G-metric on X. ∎

Definition 3 (see [19]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x_m, x_n, x) = 0$, and one say that the sequence $\{x_n\}$ is G-convergent to x. Thus, that if $x_n \to x$ in a G-metric space (X, G), then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_m, x_n, x) < \varepsilon$, for all $n, m \geq N$.

Proposition 4 (see [19]) Let (X, G) be a G-metric space. The following are equivalent:

(1). $\{x_n\}$ is G-convergent to x,
(2). $G(x_n, x_n, x) \to 0$ as $n \to +\infty$,
(3). $G(x_n, x, x) \to 0$ as $n \to +\infty$,
(4). $G(x_m, x_n, x) \to 0$ as $m, n \to +\infty$. ∎

Definition 5 (see [19]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 6 (see [19]) Let (X, G) is a G-metric space. The following are equivalent:

(1). The sequence $\{x_n\}$ is G-Cauchy.
(2). For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq N$. ∎

Definition 7 (see [19]) Let (X, G) and (X', G') are two G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 8 (see [19]) Let (X, G) and (X', G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, we have $\{f(x_n)\}$ is G'-convergent to $f(x)$. ∎

Proposition 9 (see [19]) Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables. ■

Definition 10 (see [19]) A G-metric space (X, G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence is G-convergent in (X, G).

Definition 11 (see [19]) A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Now, we give an example of a non-symmetric G-metric.

Example 12 Let $X = \{0,1,2\}$ with G-metric defined by

$$G(0,0,0) = G(1,1,1) = G(2,2,2) = 0,$$

$$\begin{aligned} &G(0,0,1) = G(0,1,0) = G(1,0,0) \\ &= G(0,0,2) = G(0,2,0) = G(2,0,0) \\ &= G(0,2,2) = G(2,0,2) = G(2,2,0) = 1, \end{aligned}$$

$$\begin{aligned} &G(0,1,1) = G(1,0,1) = G(1,1,0) \\ &= G(1,1,2) = G(1,2,1) = G(2,1,1) \\ &= G(1,2,2) = G(2,1,2) = G(2,2,1) = 2, \end{aligned}$$

$$\begin{aligned} &G(0,1,2) = G(0,2,1) = G(1,0,2) \\ &= G(1,2,0) = G(2,0,1) = G(2,1,0) = 2. \end{aligned}$$

is a non-symmetric G-metric on X because $G(0,0,1) \neq G(0,1,1)$. ■

Definition 13 (Berinde and Borcut [11]) Let $(X, \leq)$ be a partially ordered set and $F: X^3 \to X$. The mapping F is said to have the mixed monotone property if, for any $x, y, z \in X$,

$$\begin{aligned} &x_1, x_2 \in X,\ x_1 \leq x_2 \implies F(x_1, y, z) \leq F(x_2, y, z), \\ &y_1, y_2 \in X,\ y_1 \leq y_2 \implies F(x, y_1, z) \geq F(x, y_2, z), \\ &z_1, z_2 \in X,\ z_1 \leq z_2 \implies F(x, y, z_1) \leq F(x, y, z_2). \end{aligned} \tag{5}$$

Definition 14 (see [11]) Let $F: X^3 \to X$. An element (x, y, z) is called a tripled fixed point of F if

$$F(x, y, z) = x,\ F(y, x, y) = y,\ F(z, y, x) = z. \tag{6}$$

Definition 15 (see [13]) Let (X, G) be a G-metric space. A mapping $F: X^3 \to X$ is said to be continuous if for any three G-convergent sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converging to $x, y,$ and z, respectively, $\{F(x_n, y_n, z_n)\}$ is G-convergent to $F(x, y, z)$.

Theorem 16 (Hassen et al. [13]) Let $(X, \leq)$ be partially ordered set and suppose there is a G-metric G on X such that (X, G) is a G-complete. Suppose also $F: X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $\phi \in \Phi$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, one has

$$\begin{aligned} &G\big(F(x, y, z), F(a, b, c), F(u, v, w)\big) \\ &\quad \leq \varphi(\max\{G(x, a, u), G(y, b, v), G(z, c, w)\}) \end{aligned} \tag{7}$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0),\ y_0 \geq F(y_0, x_0, y_0),$$

$$z_0 \leq F(z_0, y_0, x_0) \tag{8}$$

Then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x, y, z) = x,\ F(y, x, y) = y, F(z, y, x) = z. \quad ■$$

Theorem 17 (see [13]) Let $(X, \leq)$ be partially ordered set and suppose there is a G-metric G on X such that (X, G) is a G-complete. Suppose also $F: X^3 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $\phi \in \Phi$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, one has (7). If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0),\ y_0 \geq F(y_0, x_0, y_0),$$

$$z_0 \leq F(z_0, y_0, x_0)$$

Assume also that X has the following properties:

a) if a non-decreasing sequence $x_n \to x$ in X, then $x_n \leq x$, $\forall n$.
b) if a non-increasing sequence $y_n \to y$ in X, then $y_n \geq y$, $\forall n$.

Then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x, y, z) = x,\ F(y, x, y) = y,\ F(z, y, x) = z. \quad ■$$

3. MAIN RESULTS

We start this section with some examples which shows the weakness of Theorem 16 and 17 (Theorem 2.1 and 2.4 in [13]).

Example 18(a) Take $X = \left[\frac{1}{2}, 64\right]$ endowed with the complete G-metric:

$$G(x, y, z) = |x - y| + |y - z| + |z - x|,\ \forall x, y, z \in X$$

and $F: X^3 \to X$ defined by

$$F(x, y, z) = 8\left(\frac{\sqrt{xz}}{y}\right)^{\frac{1}{6}},\ \forall x, y, z \in X.$$

The mapping F is continuous and has the mixed monotone property. Also, there exist $x_0 = 1 = z_0$ and $y_0 = 64$ such that

$$\begin{aligned} F(x_0, y_0, z_0) &= F(1, 64, 1) = 8\left(\frac{1}{64}\right)^{\frac{1}{6}} = 4 > 1 = x_0, \\ F(y_0, x_0, y_0) &= F(64, 1, 64) = 8(64)^{\frac{1}{6}} = 16 < 64 = y_0, \\ F(z_0, y_0, x_0) &= F(1, 64, 1) = 8\left(\frac{1}{64}\right)^{\frac{1}{6}} = 4 > 1 = z_0. \end{aligned}$$

and then, the condition (8) holds. Taking $x = w = a = c = z = 1$, $y = b = v = 64, u = \frac{1}{2}$,

$$G\big(F(x, y, z), F(a, b, c), F(u, v, w)\big)$$

$$= G\left(8\left(\frac{\sqrt{xz}}{y}\right)^{\frac{1}{6}}, 8\left(\frac{\sqrt{ac}}{b}\right)^{\frac{1}{6}}, 8\left(\frac{\sqrt{uw}}{v}\right)^{\frac{1}{6}}\right)$$
$$= G\left(8\left(\frac{1}{64}\right)^{\frac{1}{6}}, 8\left(\frac{1}{64}\right)^{\frac{1}{6}}, 8\left(\frac{\sqrt{\frac{1}{2}}}{64}\right)^{\frac{1}{6}}\right)$$
$$\approx G(4,4,3.77549) = 11.77549$$

and

$$\max\{G(x,a,u), G(y,b,v), G(z,c,w)\}$$
$$= \max\left\{G\left(1,1,\tfrac{1}{2}\right), G(64,64,64), G(1,1,1)\right\}$$
$$= \max\left\{\left|1-\tfrac{1}{2}\right| + \left|\tfrac{1}{2}-1\right|, 0, 0\right\} = 1.$$

It is clear that here is no $\phi \in \Phi$ (no $k \in [0,1)$) for which the inequality (2.1) (inequalities (2.14) and (2.16)) of Theorem 2.1 (Corollary 2.2 and 2.3) holds of [13]. Notice, however, that (8,8,8) is the unique tripled fixed point of F. ∎

Example 18(b) Take $X = [1,64]$ endowed with the complete G-metric:

$$G(x,y,z) = |x-y| + |y-z| + |z-x|, \forall x,y,z \in X$$

and $F: X^3 \to X$ defined by

$$F(x,y,z) = 8\sqrt[3]{\frac{x}{y}}, \ \forall x,y,z \in X.$$

The mapping F is continuous and has the mixed monotone property. Moreover, taking $x_0 = y_0 = z_0 = 8$, the condition (8) holds. Taking $u = 1, a = x = 2,\ v = 2, y = b = z = c = w = 1$,

$$G\big(F(x,y,z), F(a,b,c), F(u,v,w)\big)$$
$$= 8\left(\left|(2)^{\frac{1}{3}} - \left(\frac{1}{2}\right)^{\frac{1}{3}}\right| + \left|\left(\frac{1}{2}\right)^{\frac{1}{3}} - (2)^{\frac{1}{3}}\right|\right) \approx 7.458$$

and

$$\max\{G(x,a,u), G(y,b,v), G(z,c,w)\} = 2.$$

It is clear that here is no $\phi \in \Phi$ (no $k \in [0,1)$) for which the inequality (2.1) (inequalities (2.14) and (2.16)) of Theorem 2.1 (Corollary 2.2 and 2.3) holds of [13]. Notice, however, that (8,8,8) is the unique tripled fixed point of F. ∎

Now, motivated by the work in [31-32], we give the following definition.

Definition 19 Let (X,G) be a G-metric space. A mapping $T: X \to X$ is said to be an ICS mapping if T is injective, continuous, and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then, $\{x_n\}$ is also convergent.

Before starting to introduce our main results, let us consider the set of functions.

$$\Phi = \{\phi: \mathbb{R}^+ \to \mathbb{R}^+ \mid \phi \text{ is nondecreasing}, \phi(t) < t \text{ and } \lim_{r\to t^+}\phi(r) < t, \forall\ t > 0\} \quad (9)$$

Note that $\phi(t) < t$ and $\lim_{r\to t^+}\phi(r) < t$ imply $\lim_{k\to+\infty}\phi^k(t) = 0$ for each $t > 0$, where ϕ^k denotes the k-times repeated composition of ϕ with itself.

Our first main result is given by the following:

Theorem 20 Let $(X,\le)$ be partially ordered set and suppose there is a G-metric G on X such that (X,G) is a G-complete. Suppose also that $T: X \to X$ is an ICS mapping and $F: X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $\phi \in \Phi$ such that for $x,y,z,a,b,c,u,v,w \in X$ with $x \ge a \ge u$, $y \le b \le v$, and $z \ge c \ge w$, one has

$$G\big(TF(x,y,z), TF(a,b,c), TF(u,v,w)\big) \le \phi(\max\{G(Tx,Ta,Tu), G(Ty,Tb,Tv), G(Tz,Tc,Tw)\}) \quad (10)$$

If there exist $x_0, y_0, z_0 \in X$ be as in (8). Then F has a tripled fixed point in X.

Proof: Suppose $x_0, y_0, z_0 \in X$are such that

$$x_0 \le F(x_0,y_0,z_0),\ y_0 \ge F(y_0,x_0,y_0),$$
$$z_0 \le F(z_0,y_0,x_0)$$

Define

$$x_1 = F(x_0,y_0,z_0),\ y_1 = F(y_0,x_0,y_0),$$
$$z_1 = F(z_0,y_0,x_0).$$

Then $x_0 \le x_1, y_1 \le y_0$, and $z_0 \le z_1$. Again, define

$$x_2 = F(x_1,y_1,z_1),\ y_2 = F(y_1,x_1,y_1),$$
$$z_2 = F(z_1,y_1,x_1).$$

Since F has the mixed monotone property on X, we have

$$x_0 \le x_1 \le x_2,\ y_2 \le y_1 \le y_0,\ z_0 \le z_1 \le z_2.$$

Repeating this process, we can construct the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in X such that

$$x_{n+1} = F(x_n,y_n,z_n),\ \ y_{n+1} = F(y_n,x_n,y_n),$$
$$z_{n+1} = F(z_n,y_n,x_n), \forall\ n \ge 0. \quad (11)$$

We claim that;

$$x_n \le x_{n+1},\quad y_{n+1} \le y_n,\quad z_n \le z_{n+1}. \quad (12)$$

Indeed, we will use the mathematical induction to prove (12). The inequalities in (12) hold for $n = 1, 2$ because, we have $x_0 \le x_1 \le x_2$, $y_2 \le y_1 \le y_0$, and $z_0 \le z_1 \le z_2$. Now, suppose that the inequalities in (12) hold for $n = m$. In that case,

$$x_m \le x_{m+1},\quad y_{m+1} \le y_m,\quad z_m \le z_{m+1}. \quad (13)$$

If we consider (11) and mixed monotone property of F together with (13), we have

$$\begin{aligned} x_{m+1} &= F(x_m, y_m, z_m) \\ &\leq F(x_{m+1}, y_m, z_m) \\ &\leq F(x_{m+1}, y_{m+1}, z_m) \\ &\leq F(x_{m+1}, y_{m+1}, z_{m+1}) = x_{m+2}, \end{aligned}$$

$$\begin{aligned} y_{m+1} &= F(y_m, x_m, y_m) \\ &\geq F(y_{m+1}, x_m, y_m) \\ &\geq F(y_{m+1}, x_{m+1}, y_m) \\ &\geq F(y_{m+1}, x_{m+1}, y_{m+1}) = y_{m+2}, \end{aligned} \tag{14}$$

$$\begin{aligned} z_{m+1} &= F(z_m, y_m, x_m) \\ &\leq F(z_{m+1}, y_m, x_m) \\ &\leq F(z_{m+1}, y_{m+1}, x_m) \\ &\leq F(z_{m+1}, y_{m+1}, x_{m+1}) = z_{m+2}. \end{aligned}$$

Thus, (12) is satisfied for all $n \geq 1$. If for some positive integer n, we have $(x_{n+1}, y_{n+1}, z_{n+1}) = (x_n, y_n, z_n)$, then $x_n = F(x_n, y_n, z_n)$, $y_n = F(y_n, x_n, y_n)$, and $z_n = F(z_n, y_n, x_n)$, that is, (x_n, y_n, z_n) is a tripled fixed point of F. Thus, we will assume that $(x_{n+1}, y_{n+1}, z_{n+1}) \neq (x_n, y_n, z_n)$, $\forall\ n \in \mathbb{N}$; that is, we assume that either $x_{n+1} \neq x_n$ or $y_{n+1} \neq y_n$ or $z_{n+1} \neq z_n$. Since T is injective, for any $n \in \mathbb{N}$,

$$\begin{aligned} 0 < max\{&G(Tx_{n+1}, Tx_n, Tx_n), \\ &G(Ty_{n+1}, Ty_n, Ty_n), \\ &G(Tz_{n+1}, Tz_n, Tz_n)\} \end{aligned} \tag{15}$$

Due to (10) and (11), for any $n \in \mathbb{N}$, we have

$$\begin{aligned} G(Tx_{n+1}, Tx_n, Tx_n) &\coloneqq G(TF(x_n, y_n, z_n), \\ &\qquad TF(x_{n-1}, y_{n-1}, z_{n-1}), \\ &\qquad TF(x_{n-1}, y_{n-1}, z_{n-1})) \\ &\leq \phi(max\{G(Tx_n, Tx_{n-1}, Tx_{n-1}), \\ &\qquad G(Ty_n, Ty_{n-1}, Ty_{n-1}), \\ &\qquad G(Tz_n, Tz_{n-1}, Tz_{n-1}))\} \end{aligned}$$

$$\begin{aligned} G(Ty_{n+1}, Ty_n, Ty_n) &\coloneqq G(TF(y_n, x_n, y_n), \\ &\qquad TF(y_{n-1}, x_{n-1}, y_{n-1}) \\ &\qquad TF(y_{n-1}, x_{n-1}, y_{n-1})) \\ &\leq \phi(max\{G(Ty_n, Ty_{n-1}, Ty_{n-1}), \\ &\qquad G(Tx_n, Tx_{n-1}, Tx_{n-1})\}\} \\ &\leq \phi(max\{G(Tx_n, Tx_{n-1}, Tx_{n-1}), \\ &\qquad G(Ty_n, Ty_{n-1}, Ty_{n-1}), \\ &\qquad G(Tz_n, Tz_{n-1}, Tz_{n-1})\}) \end{aligned}$$

$$\begin{aligned} G(Tz_{n+1}, Tz_n, Tz_n) &\coloneqq G(TF(z_n, y_n, x_n), \\ &\qquad TF(z_{n-1}, y_{n-1}, x_{n-1}), \\ &\qquad TF(z_{n-1}, y_{n-1}, x_{n-1})) \\ &\leq \phi(max\{G(Tz_n, Tz_{n-1}, Tz_{n-1}), \\ &\qquad G(Ty_n, Ty_{n-1}, Ty_{n-1}), \\ &\qquad G(Tx_n, Tx_{n-1}, Tx_{n-1})\}) \end{aligned} \tag{16}$$

Having in mind that $\phi(t) < t$ for all $t > 0$, sofrom (16), we obtain that

$$\begin{aligned} 0 < max\{&G(Tx_{n+1}, Tx_n, Tx_n), \\ &G(Ty_{n+1}, Ty_n, Ty_n), \\ &G(Tz_{n+1}, Tz_n, Tz_n)\} \\ \leq \phi(max\{&G(Tx_n, Tx_{n-1}, Tx_{n-1}), \\ &G(Ty_n, Ty_{n-1}, Ty_{n-1}), \\ &G(Tz_n, Tz_{n-1}, Tz_{n-1})\}) \\ < max\{&G(Tx_n, Tx_{n-1}, Tx_{n-1}), \\ &G(Ty_n, Ty_{n-1}, Ty_{n-1}) \\ &G(Tz_n, Tz_{n-1}, Tz_{n-1})\} \end{aligned} \tag{17}$$

Set

$$\begin{aligned} \Delta_n \coloneqq max\{G(Tx_{n+1}, Tx_n, Tx_n), G(Ty_{n+1}, Ty_n, Ty_n), \\ G(Tz_{n+1}, Tz_n, Tz_n)\} \end{aligned} \tag{18}$$

It follows that $\Delta_n < \Delta_{n-1}$. Thus, $\{\Delta_n\}$ is a non-increasing sequence of positive real numbers. Hence, there exists $\gamma \geq 0$ such that

$$lim_{n\to+\infty} \Delta_n = \gamma \tag{19}$$

We shall claim that $\gamma = 0$. On the contrary, suppose that $\gamma > 0$. Letting $n \to +\infty$ in (17), we obtain that

$$\begin{aligned} 0 < \gamma &= lim_{n\to+\infty} \Delta_n \\ &= lim_{n\to+\infty} max\{G(Tx_{n+1}, Tx_n, Tx_n), \\ &\qquad G(Ty_{n+1}, Ty_n, Ty_n), \\ &\qquad G(Tz_{n+1}, Tz_n, Tz_n)\} \\ &\leq lim_{n\to+\infty} \phi(max\{G(Tx_n, Tx_{n-1}, Tx_{n-1}), \\ &\qquad G(Ty_n, Ty_{n-1}, Ty_{n-1}), \\ &\qquad G(Tz_n, Tz_{n-1}, Tz_{n-1})\}) \\ &= lim_{n\to+\infty} \phi(\Delta_{n-1}) \\ &= lim_{\Delta_{n-1}\to\gamma^+} \phi(\Delta_{n-1}) < \gamma^+ \end{aligned} \tag{20}$$

which is a contradiction. Therefore, we conclude that $\gamma = 0$, that is,

$$lim_{n\to+\infty} \Delta_n = 0. \tag{21}$$

In what follows, we shall show that $\{Tx_n\}$, $\{Ty_n\}$, and $\{Tz_n\}$ are G-Cauchy sequences. Suppose, to the contrary, that at least of $\{Tx_n\}$ or $\{Ty_n\}$ or $\{Tz_n\}$ is not a G-Cauchy sequence, consequently,

$$lim_{n,m\to+\infty} G(Tx_n, Tx_m, Tx_m) \neq 0,$$

or $$lim_{n,m\to+\infty} G(Ty_n, Ty_m, Ty_m) \neq 0,$$

or $$lim_{n,m\to+\infty} G(Tz_n, Tz_m, Tz_m) \neq 0. \tag{22}$$

This means there exists an $\varepsilon > 0$ for which we can find subsequences $\{Tx_{n_k}\}$, $\{Tx_{m_k}\}$ of $\{Tx_n\}$, $\{Ty_{n_k}\}$, $\{Ty_{m_k}\}$ of $\{Ty_n\}$and $\{Tz_{n_k}\}$, $\{Tz_{m_k}\}$ of $\{Tz_n\}$with $n_k > m_k > k$ such that

$$\begin{aligned} max\{&G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), \\ &G(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}), \\ &G(Tz_{n_k}, Tz_{m_k}, Tz_{m_k})\} \geq \varepsilon. \end{aligned} \tag{23}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k \geq k$ satisfying (23). Then

$$max\{G(Tx_{n_k-1}, Tx_{m_k}, Tx_{m_k}), G(Ty_{n_k-1}, Ty_{m_k}, Ty_{m_k}), G(Tz_{n_k-1}, Tz_{m_k}, Tz_{m_k})\} < \varepsilon. \quad (24)$$

By rectangle inequality and (24), we have

$$\begin{aligned} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k}) &\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{n_k}) \\ &\leq \varepsilon + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{n_k}) \end{aligned} \quad (25)$$

Letting $k \to +\infty$ in (25) and using (21), we obtain

$$\begin{aligned} &lim_{k\to+\infty} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k}) \\ &\leq lim_{k\to+\infty} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k-1}) \leq \varepsilon. \end{aligned} \quad (26)$$

Similarly,

$$\begin{aligned} &lim_{k\to+\infty} G(Ty_{m_k}, Ty_{m_k}, Ty_{n_k}) \\ &\leq lim_{k\to+\infty} G(Ty_{m_k}, Ty_{m_k}, Ty_{n_k-1}) \leq \varepsilon, \end{aligned}$$

$$\begin{aligned} &lim_{k\to+\infty} G(Tz_{m_k}, Tz_{m_k}, Tz_{n_k}) \\ &\leq lim_{k\to+\infty} G(Tz_{m_k}, Tz_{m_k}, Tz_{n_k-1}) \leq \varepsilon. \end{aligned} \quad (27)$$

Again, by (24), we have

$$\begin{aligned} \varepsilon &\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k}) \\ &\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1}) + G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k}) \\ &\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1}) + G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{n_k}) \\ &\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1}) + G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{m_k}) + G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{n_k}) \\ &\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1}) + G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{m_k}) + \varepsilon + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{n_k}) \end{aligned} \quad (28)$$

Letting $k \to +\infty$ in (28) and using (21), we get

$$\begin{aligned} \varepsilon &\leq lim_{k\to+\infty} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k}) \\ &\leq lim_{k\to+\infty} G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}) \leq \varepsilon. \end{aligned} \quad (29)$$

Similarly, we have

$$\begin{aligned} \varepsilon &\leq lim_{k\to+\infty} G(Ty_{m_k}, Ty_{m_k}, Ty_{n_k}) \\ &\leq lim_{k\to+\infty} G(Ty_{m_k-1}, Ty_{m_k-1}, Ty_{n_k-1}) \leq \varepsilon, \end{aligned}$$

$$\begin{aligned} \varepsilon &\leq lim_{k\to+\infty} G(Tz_{m_k}, Tz_{m_k}, Tz_{n_k}) \\ &\leq lim_{k\to+\infty} G(Tz_{m_k-1}, Tz_{m_k-1}, Tz_{n_k-1}) \leq \varepsilon. \end{aligned} \quad (30)$$

Set

$$r_k := max\{G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), G(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}), G(Tz_{n_k}, Tz_{m_k}, Tz_{m_k})\}$$

Using (23) and (29)-(30), we have

$$\varepsilon = lim_{k\to+\infty} r_k \quad (31)$$

Now, using inequality (10), we obtain

$$\begin{aligned} G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) &:= G(TF(x_{n_k-1}, y_{n_k-1}, z_{n_k-1}), TF(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}), TF(x_{m_k-1}, y_{m_k-1}, z_{m_k-1})) \\ &\leq \phi(max\{G(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}), G(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}), G(Tz_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1})\}) \end{aligned}$$

$$\begin{aligned} G(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}) &:= G(TF(y_{n_k-1}, x_{n_k-1}, y_{n_k-1}), TF(y_{m_k-1}, x_{m_k-1}, y_{m_k-1}), TF(y_{m_k-1}, x_{m_k-1}, y_{m_k-1})) \\ &\leq \phi(max\{G(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}), G(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1})\} \\ &\leq \phi(max\{G(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}), G(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}), G(Tz_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1})\}) \end{aligned}$$

$$\begin{aligned} G(Tz_{n_k}, Tz_{m_k}, Tz_{m_k}) &:= G(TF(z_{n_k-1}, x_{n_k-1}, y_{n_k-1}), TF(z_{m_k-1}, x_{m_k-1}, y_{m_k-1}), TF(z_{m_k-1}, x_{m_k-1}, y_{m_k-1})) \\ &\leq \phi(max\{G(Tz_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1}), G(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}), G(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1})\}) \end{aligned} \quad (32)$$

From (32), we deduce that

$$\begin{aligned} &max\{G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), G(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}), G(Tz_{n_k}, Tz_{m_k}, Tz_{m_k})\} \\ &\leq \phi(max\{G(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}), G(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}), G(Tz_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1})\}) \end{aligned}$$

That is,

$$r_k \leq \phi(r_{k-1}) \quad (33)$$

Letting $k \to +\infty$ in (33) and having in mind (31), we get that

$$\begin{aligned} 0 < \varepsilon = lim_{k\to+\infty} r_k &\leq lim_{k\to+\infty} \phi(r_{k-1}) \\ &= lim_{r_{k-1}\to\varepsilon^+} \phi(r_{k-1}) < \varepsilon^+, \end{aligned}$$

which is a contradiction. Thus, $\{Tx_n\}, \{Ty_n\}$, and $\{Tz_n\}$ are G-Cauchy sequences in (X, G). Since (X, G) is a G-complete, $\{Tx_n\},\{Ty_n\}$ and $\{Tz_n\}$ are convergent sequences.

Since T is an ICS mapping, there exist $x, y, z \in X$ such that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to $x, y,$ and z , respectively; that is,

$$lim_{n\to+\infty} x_n = x ,\quad lim_{n\to+\infty} y_n = y,\quad lim_{n\to+\infty} y_n = y. \tag{34}$$

Finally, we show that $(x, y, z) \in X^3$ is a tripled fixed point of F. Since F is continuous and $(x_n, y_n, z_n) \to (x, y, z)$, we have

$$x_{n+1} = F(x_n, y_n, z_n) \to F(x, y, z),$$
$$y_{n+1} = F(y_n, x_n, y_n) \to F(y, x, y),$$
$$z_{n+1} = F(z_n, x_n, y_n) \to F(z, x, y).$$

By the uniqueness of limit, we get that $x = F(x, y, z)$, $y = F(y, x, y)$, and $z = F(z, x, y)$. So (x, y, z) is a tripled fixed point of F. This completes the proof. ■

Corollary 21 Let $(X, \leq)$ be partially ordered set and suppose there is a G-metric G on X such that (X, G) is a G-complete. Suppose also that $T: X \to X$ is an ICS mapping and $F: X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $\phi \in \Phi$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, one has

$$G\big(TF(x, y, z), TF(a, b, c), TF(u, v, w)\big) \leq \phi\left(\frac{G(Tx,Ta,Tu)+G(Ty,Tb,Tv)+G(Tz,Tc,Tw)}{3}\right) \tag{35}$$

If there exist $x_0, y_0, z_0 \in X$ be as in (8). Then F has a tripled fixed point in X.

Proof: It suffices to remark that

$$\frac{G(Tx,Ta,Tu)+G(Ty,Tb,Tv)+G(Tz,Tc,Tw)}{3} \leq max\{TF(x, y, z), TF(a, b, c), TF(u, v, w)\}$$

Then, we apply Theorem 20 because that ϕ is non-decreasing. ■

For each $k \in [0,1)$, setting $\phi(t) = kt$ in Theorem 20, we obtain the following Corollary.

Corollary 22 Let $(X, \leq)$ be partially ordered set and suppose there is a G-metric G on X such that (X, G) is a G-complete. Suppose also that $T: X \to X$ is an ICS mapping and $F: X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0,1)$ such that for $x, y, z, a, b, c, u, v, w \in X$ with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, one has

$$G\big(TF(x, y, z), TF(a, b, c), TF(u, v, w)\big) \leq k\, max\{G(Tx, Ta, Tu), G(Ty, Tb, Tv), G(Tz, Tc, Tw)\} \tag{36}$$

If there exist $x_0, y_0, z_0 \in X$ be as in (8). Then F has a tripled fixed point in X. ■

Corollary 23 Let $(X, \leq)$ be partially ordered set and suppose there is a G-metric G on X such that (X, G) is a G-complete. Suppose also that $T: X \to X$ is an ICS mapping and $F: X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0,1)$ such that for $x, y, z, a, b, c, u, v, w \in X$ with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, one has

$$G\big(TF(x, y, z), TF(a, b, c), TF(u, v, w)\big) \leq \frac{k}{3}\big(G(Tx, Ta, Tu) + G(Ty, Tb, Tv) + G(Tz, Tc, Tw)\big) \tag{37}$$

If there exist $x_0, y_0, z_0 \in X$ be as in (8). Then F has a tripled fixed point in X.

Proof: Note that

$$G(Tx, Ta, Tu) + G(Ty, Tb, Tv) + G(Tz, Tc, Tw) \leq 3\, max\{G(Tx, Ta, Tu), G(Ty, Tb, Tv), G(Tz, Tc, Tw)\} \tag{38}$$

Then, the proof follows from Corollary 22. ■

Remark 24 Taking $T = I_{G_X}$, the identity on X, in Theorem 20; we get main result (Theorem 2.1) of Hassen et al. [13]. Therefore, Corollary 22 and 23 are generalization of Corollary 2.2 and 2.3 of Hassen et al. [13], respectively. ■

In the following theorem, we omit the continuity hypothesis of F. We need the following definition.

Definition 25 Let (X, ≤) be a partially ordered set and (X, G) be a G-metric. We say that $(X, G, \leq)$ is regular if the following conditions hold in X:

c) if a non-decreasing sequence $x_n \to x$ in X, then $x_n \leq x,\ \forall\, n$.
d) if a non-increasing sequence $y_n \to y$ in X, then $y_n \geq y,\ \forall\, n$.

Theorem 26 Let $(X, \leq)$ be partially ordered set and suppose there is a G-metric G on X such that (X, G) is a G-complete. Suppose also that $T: X \to X$ is an ICS mapping and $F: X^3 \to X$ be a mapping having the mixed monotone property on X . Assume that there exists $\phi \in \Phi$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$, and $z \geq c \geq w$, one has (10). If there exist $x_0, y_0, z_0 \in X$ be as in (8). Assume also $(X, G, \leq)$ is regular. Then F has a tripled fixed point in X.

Proof: Following proof of Theorem 20 step by step, we can construct three sequences $\{x_n\}$,$\{y_n\}$, and $\{z_n\}$ in X such that $x_{n+1} = F(x_n, y_n, z_n)$, $y_{n+1} = F(y_n, x_n, y_n)$, and $z_{n+1} = F(z_n, y_n, x_n)$ with $x_n \leq x_{n+1}$, $y_{n+1} \leq y_n$, and $z_n \leq z_{n+1}$. Then, $\{Tx_n\}, \{Ty_n\}$, and $\{Tz_n\}$ are G-Cauchy sequences in (X, G) . Since (X, G) is a G-complete, $\{Tx_n\},\{Ty_n\}$ and $\{Tz_n\}$ are convergent sequences. Since T is an ICS mapping, there exist

$x, y, z \in X$ such that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to x, y, and z, respectively. Since T is continuous, we have

$$lim_{n\to+\infty} Tx_n = Tx\,,\quad lim_{n\to+\infty} Ty_n = Ty,\quad lim_{n\to+\infty} Tz_n = Tz. \tag{39}$$

We remain to show that F has a tripled fixed point (x, y, z) in X. To this aim, suppose that assumption "$(X, G, \leq)$ is regular" holds. Since $\{x_n\}$ and $\{z_n\}$ are non-decreasing with $x_n \to x$ and $z_n \to z$ and also $\{y_n\}$ is non-increasing with $y_n \to y$. We have $x_n \leq x$, $y_n \geq y$ and $z_n \leq z$, $\forall\, n$. If for some $n \geq 0$, $(z_n, y_n, x_n) = (x, y, z)$; that is, $x_n = x$, $y_n = y$ and $z_n = z$, then $x = x_n \leq x_{n+1} \leq x = x_n$, $y = y_n \geq y_{n+1} \geq y = y_n$, and $z = z_n \leq z_{n+1} \leq z = z_n$. This means that $x_n = x_{n+1} = F(x_n, y_n, z_n)$, $y_n = y_{n+1} = F(y_n, x_n, y_n)$ and $z_n = z_{n+1} = F(z_n, y_n, x_n)$; that is, $(x_n, y_n, z_n) \in X^3$ is a tripled fixed point of F. Now, assume that, $\forall\, n \geq 0$, $(z_n, y_n, x_n) \neq (x, y, z)$. Thus, $\forall\, n \geq 0$,

$$max\{G(Tx, Tx, Tx_n), G(Tx, Tx, Tx_n), G(Tx, Tx, Tx_n)\} > 0. \tag{40}$$

From (10), we have

$$\begin{aligned} G(TF(x,y,z), TF(x,y,z), Tx_{n+1}) &\coloneqq G\big(TF(x,y,z), TF(x,y,z), TF(x_n,y_n,z_n)\big) \\ &\leq \phi(max\{G(Tx,Tx,Tx_n), G(Ty,Ty,Ty_n), G(Tz,Tz,Tz_n)\}) \end{aligned}$$

$$\begin{aligned} G\big(Ty_{n+1}, TF(y,x,y), TF(y,x,y)\big) &\coloneqq G\big(TF(y_n,x_n,y_n), TF(y,x,y), TF(y,x,y)\big) \\ &\leq \phi(max\{G(Ty_n,Ty,Ty), G(Tx_n,Tx,Tx)\}) \\ &\leq \phi(max\{G(Ty_n,Ty,Ty), G(Tx_n,Tx,Tx), G(Tz,Tz,Tz_n)\}) \end{aligned}$$

$$\begin{aligned} G(TF(z,y,x), TF(z,y,x), Tz_{n+1}) &\coloneqq G\big(TF(z,y,x), TF(z,y,x), TF(z_n,y_n,x_n)\big) \\ &\leq \phi(max\{G(Tz,Tz,Tz_n), G(Ty,Ty,Ty_n), G(Tx,Tx,Tx_n)\}) \end{aligned} \tag{41}$$

Letting $n \to +\infty$ in (41), using (40) in the fact that $\phi(t) < t$ for all $t \in (0, +\infty)$ and (39), the right-hand of all inequalities in (41) tends to 0, so we get that

$$G(TF(x,y,z), TF(x,y,z), Tx) = 0,$$
$$G\big(Ty, TF(y,x,y), TF(y,x,y)\big) = 0,$$
$$G(TF(z,y,x), TF(z,y,x), Tz) = 0.$$

This means that $TF(x, y, z) = Tx$, $TF(y, x, y) = Ty$, and$TF(z, y, x) = Tz$. Since T is injective, it follows that

$$F(x,y,z) = x,\ \ F(y,x,y) = y, F(z,y,x) = z.$$

Thus, we proved that F has a tripled fixed point in X. This completes the proof. ∎

Remark 27 Results similar to Corollary 21, 22 and 23 omitting the continuity hypotheses of F and involving hypotheses $(X, G, \leq)$ is regular corresponding to Theorem 26 can also be derived. Due to repetition, the details are avoided. ∎

4. UNIQUENESS OF A TRIPLE FIXED POINT

Now, we shall prove the uniqueness of a triple fixed point. For a product $X^3 = X \times X \times X$ of a partial ordered set $(X, \leq)$, we define a partial ordering in the following way: for all $(x, y, z), (u, v, w) \in X^3$,

$$(x,y,z) \leq (u,v,w) \Leftrightarrow x \leq u,\ y \geq v,\ z \leq w. \tag{42}$$

We say that (x, y, z) and (u, v, w) are comparable if

$$(x,y,z) \leq (u,v,w)$$
Or
$$(u,v,w) \leq (x,y,z). \tag{43}$$

Also, we say that (x, y, z) is equal to (u, v, w) if and only if $x = u$, $y = v$ and $z = w$.

Theorem 28 In addition to hypotheses of Theorem 28, suppose that for all (x, y, z), $(u, v, w) \in X^3$, there exists $(a, b, c) \in X^3$ such that

$$\big(F(a,b,c), F(b,a,b), F(c,b,a)\big)$$

is comparable to

$$(F(x,y,z), F(y,x,y), F(z,y,x),)$$

and $\big(F(u,v,w), F(v,u,v), F(w,v,u)\big)$.

Then F has a unique triple fixed point (x, y, z).

Proof: Due to Theorem 20, the set of tripled fixed points of F is not empty. Suppose (x, y, z) and (u, v, w), are triple fixed points of the mapping $F : X^3 \to X$ such that$(x, y, z) \neq (u, v, w)$; that is,

$$\begin{aligned} F(x,y,z) = x,&\ \ F(u,v,w) = u, \\ F(y,x,y) = y,&\ \ F(v,u,v) = v, \\ F(z,y,x) = z,&\ \ F(w,v,u) = w. \end{aligned} \tag{44}$$

We shall show that (x, y, z) and (u, v, w) are equal. By assumption, there exists $(a, b, c) \in X^3$ such that

$$\big(F(a,b,c), F(b,a,b), F(c,b,a)\big)$$

is comparable to

$$\big(F(x,y,z), F(y,x,y), F(z,y,x)\big)$$

and

$$\big(F(u,v,w), F(v,u,v), F(w,v,u)\big).$$

Put $a_0 = a, b_0 = b, c_0 = c$ and choose $a_1, b_1, c_1 \in X$ such that

$$a_1 = F(a_0,b_0,c_0), b_1 = F(b_0,a_0,b_0), c_1 = F(c_0,b_0,a_0).$$

Thus, we can define three sequences $\{a_n\}$, $\{b_n\}$,and $\{c_n\}$ as

$$\begin{aligned} a_n &= F(a_{n-1}, b_{n-1}, c_{n-1}), \\ b_n &= F(b_{n-1}, a_{n-1}, b_{n-1}), \\ c_n &= F(c_{n-1}, b_{n-1}, a_{n-1}). \end{aligned} \tag{45}$$

for any $n \geq 1$. Further set $x_0 = x,\ y_0 = y,\ z_0 = z$ and $u_0 = u,\ v_0 = v,\ w_0 = w$. and on the same way define the sequences $\{x_n\},\{y_n\}$, and $\{z_n\}$ and $\{u_n\},\{v_n\}$, and $\{w_n\}$. Then, it is easy that

$$\begin{aligned} x_n &= F(x,y,z), & u_n &= F(u,v,w), \\ y_n &= F(y,x,y), & v_n &= F(v,u,w), \\ z_n &= F(z,y,x), & w_n &= F(w,v,u). \end{aligned} \quad (46)$$

Since

$$\big(F(x,y,z),F(y,x,y),F(z,y,x)\big) = (x_1,y_1,z_1) = (x,y,z)$$

is comparable to

$$\big(F(a,b,c),F(b,a,b),F(c,b,a)\big) = (a_1,b_1,c_1),\ \forall\, n.$$

That is, $x \leq a_1, y \geq b_1, z \leq c_1, \forall\, n$. Recursively, we get that:

$$x \leq a_n, \qquad y \geq b_n, \qquad z \leq c_n, \qquad \forall\, n.$$

By (10), (44) and (45), we get;

$$\begin{aligned} &G(Tx,Tx,Ta_{n+1}) \\ &\quad = G\big(TF(x,y,z),TF(x,y,z),TF(a_n,b_n,c_n)\big) \\ &\quad \leq \phi(max\{G(Tx,Tx,Ta_n),G(Ty,Ty,Tb_n), \\ &\qquad\qquad G(Tz,Tz,Tc_n)\}) \end{aligned} \quad (46)$$

Similarly, we have;

$$\begin{aligned} &G(Ty,Ty,Tb_{n+1}) \\ &\quad = G\big(TF(y,x,y),TF(y,x,y),TF(b_n,a_n,b_n)\big) \\ &\quad \leq \phi(max\{G(Ty,Ty,Tb_n),G(Tx,Tx,Ta_n)\}) \\ &\quad \leq \phi(max\{G(Tx,Tx,Ta_n),G(Ty,Ty,Tb_n), \\ &\qquad\qquad G(Tz,Tz,Tc_n)\}) \end{aligned} \quad (47)$$

$$\begin{aligned} &G(Tz,Tz,Tc_{n+1}) \\ &\quad = G\big(TF(z,y,x),TF(z,y,x),TF(c_n,b_n,a_n)\big) \\ &\quad \leq \phi(max\{G(Tx,Tx,Ta_n),G(Ty,Ty,Tb_n), \\ &\qquad\qquad G(Tz,Tz,Tc_n)\}) \end{aligned}$$

It follows from (46) and (47) that

$$\begin{aligned} &max\{G(Tx,Tx,Ta_{n+1}),G(Ty,Ty,Tb_{n+1}), \\ &\qquad G(Tz,Tz,Tc_{n+1})\} \\ &\leq \phi(max\{G(Tx,Tx,Ta_n),G(Ty,Ty,Tb_n), \\ &\qquad G(Tz,Tz,Tc_n)\}) \end{aligned} \quad (48)$$

Therefore, for each $n \geq 1$,

$$\begin{aligned} &max\{G(Tx,Tx,Ta_n),G(Ty,Ty,Tb_n),G(Tz,Tz,Tc_n)\} \\ &\leq \phi^n(max\{G(Tx,Tx,Ta_0),G(Ty,Ty,Tb_0), \\ &\qquad G(Tz,Tz,Tc_0)\}) \end{aligned} \quad (49)$$

It is known that $\phi(t) < t$ and $lim_{r \to t^+} \phi(r) < t$ imply $lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$. Thus, from (49), we have

$$\begin{aligned} &lim_{n\to\infty} max\{G(Tx,Tx,Ta_n),G(Ty,Ty,Tb_n), \\ &\qquad G(Tz,Tz,Tc_n)\} = 0. \end{aligned} \quad (50)$$

This yields that

$$\begin{aligned} &lim_{n\to\infty} G(Tx,Tx,Ta_n) = 0, \\ &lim_{n\to\infty} G(Ty,Ty,Tb_n) = 0, \\ &lim_{n\to\infty} G(Ty,Ty,Tc_n) = 0, \end{aligned} \quad (51)$$

Analogously, we show that

$$\begin{aligned} &lim_{n\to\infty} G(Tu,Tu,Ta_n) = 0, \\ &lim_{n\to\infty} G(Tv,Tv,Tb_n) = 0, \\ &lim_{n\to\infty} G(Tw,Tw,Tc_n) = 0. \end{aligned} \quad (52)$$

Combining (51) to (52) yields that (Tx,Ty,Tz) and (Tu,Tv,Tw) are equal. The fact T is injective gives us $(x,y,z) = (u,v,w)$. ■

Similarly, we can prove the following statement:

Theorem 29 In addition to hypotheses of Theorem 26, suppose that for all $(x,y,z),\ (u,v,w) \in X^3$, there exists $(a,b,c) \in X^3$ such that

$$\big(F(a,b,c),F(b,a,b),F(c,b,a)\big)$$

is comparable to

$$\big(F(x,y,z),F(y,x,y),F(z,y,x)\big)$$

and

$$\big(F(u,v,w),F(v,u,v),F(w,v,u)\big).$$

Then F has a unique triple fixed point (x,y,z). ■

5. EXAMPLES

In this section, we state some examples showing that our results are effective.

Example 30 As in Example 18 (a), define $T: X \to X$ be defined by $Tx = ln(x) + 1, \forall\ x \in X$. Obviously, T is an ICS mapping. Define $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\phi(t) = {}^t/_2, \forall\, t > 0$, then $\phi \in \Phi$. Taking $x,y,z,a,b,c,u,v,w \in X$, with $x \geq a \geq u$,$y \leq b \leq v$, and $z \geq c \geq w$, we have

$$\begin{aligned} &G\big(TF(x,y,z),TF(a,b,c),TF(u,v,w)\big) \\ &= G\left(T\left(8\left(\frac{\sqrt{xz}}{y}\right)^{\frac{1}{6}}\right),T\left(8\left(\frac{\sqrt{ac}}{b}\right)^{\frac{1}{6}}\right),T\left(8\left(\frac{\sqrt{uw}}{v}\right)^{\frac{1}{6}}\right)\right) \\ &= G\left(ln\left(8\left(\frac{\sqrt{xz}}{y}\right)^{\frac{1}{6}}\right)+1, ln\left(8\left(\frac{\sqrt{ac}}{b}\right)^{\frac{1}{6}}\right)+1,\right. \\ &\qquad\qquad \left. ln\left(8\left(\frac{\sqrt{uw}}{v}\right)^{\frac{1}{6}}\right)+1\right) \\ &= \left|ln\left(8\left(\frac{\sqrt{xz}}{y}\right)^{\frac{1}{6}}\right) - ln\left(8\left(\frac{\sqrt{ac}}{b}\right)^{\frac{1}{6}}\right)\right| \\ &\quad + \left|ln\left(8\left(\frac{\sqrt{ac}}{b}\right)^{\frac{1}{6}}\right) - ln\left(8\left(\frac{\sqrt{uw}}{v}\right)^{\frac{1}{6}}\right)\right| \\ &\quad + \left|ln\left(8\left(\frac{\sqrt{uw}}{v}\right)^{\frac{1}{6}}\right) - ln\left(8\left(\frac{\sqrt{xz}}{y}\right)^{\frac{1}{6}}\right)\right| \\ &\leq \frac{1}{12}\left\{\begin{matrix} |ln(x) - ln(a)| + |ln(a) - ln(u)| \\ +|ln(u) - ln(x)| \end{matrix}\right\} \end{aligned}$$

$$+\frac{1}{6}\left\{\begin{matrix}|ln(y)-ln(b)|+|ln(b)-ln(v)|\\ +|ln(v)-ln(y)|\end{matrix}\right\}$$
$$+\frac{1}{12}\left\{\begin{matrix}|ln(z)-ln(c)|+|ln(c)-ln(w)|\\ +|ln(w)-ln(z)|\end{matrix}\right\}$$

$$=\frac{1}{12}G(ln(x)+1,ln(a)+1,ln(u)+1)$$
$$+\frac{1}{6}G(ln(y)+1,ln(b)+1,ln(v)+1)$$
$$+\frac{1}{12}G(ln(z)+1,ln(c)+1,ln(w)+1)$$

$$=\frac{1}{12}G(Tx,Ta,Tu)+\frac{1}{6}G(Tz,Tc,Tw)$$
$$+\frac{1}{12}G(Tw,Tb,Tv)$$

$$\leq\frac{1}{6}\begin{pmatrix}G(Tx,Ta,Tu)+G(Tz,Tc,Tw)\\ +G(Tw,Tb,Tv)\end{pmatrix}$$

$$\leq\frac{1}{2}max\{G(Tx,Ta,Tu),G(Tz,Tc,Tw),G(Tw,Tb,Tv)\}$$

$$=\phi(max\{G(Tx,Ta,Tu),G(Ty,Tb,Tv),G(Tz,Tc,Tw)\})$$

This is the contractive condition (10). Evidently, for every $(x,y,z),(u.v.w)\in X^3$, there always exists a point $(a,b,c)\in X^3$ that is comparable to $(x,y,z)$and $(u.v.w)$. So Theorem 20 can be applied to this example to conclude that F has a unique triple fixed point (8,8,8), since also the hypotheses of Theorem 28 hold. ■

Example 31 As in Example 18 (b), define $T:X\to X$ by $Tx=ln(x)+1,\ \forall\, x\in X$. Obviously, T is an ICS mapping. Set $\phi(t)=\frac{2t}{3}\in\Phi$. Taking $x,y,z,a,b,c,u,v,$ $w\in X$, with $x\geq a\geq u$,$y\leq b\leq v$, and $z\geq c\geq w$,

$$G\big(TF(x,y,z),TF(a,b,c),TF(u,v,w)\big)$$
$$=G\left(T\left(8\sqrt[3]{\frac{x}{y}}\right),T\left(8\sqrt[3]{\frac{a}{b}}\right),T\left(8\sqrt[3]{\frac{u}{v}}\right)\right)$$
$$=G\left(ln\left(8\sqrt[3]{\frac{x}{y}}\right)+1,ln\left(8\sqrt[3]{\frac{a}{b}}\right)+1,ln\left(8\sqrt[3]{\frac{u}{v}}\right)+1\right)$$
$$=\frac{1}{3}\left\{\left|ln\left(\frac{x}{y}\right)-ln\left(\frac{a}{b}\right)\right|+\left|ln\,ln\left(\frac{a}{b}\right)-ln\left(\frac{u}{v}\right)\right|\right.$$
$$\left.+\left|ln\left(\frac{u}{v}\right)-ln\left(\frac{x}{y}\right)\right|\right\}$$

$$=\frac{1}{3}\{|ln(x)-ln(y)-ln(a)+ln(b)|$$
$$+|ln(a)-ln(b)-ln(u)+ln(v)|$$
$$+|ln(u)-ln(v)-ln(x)+ln(y)|\}$$

$$\leq\frac{1}{3}\{|ln(x)-ln(a)|+|ln(y)-ln(b)|$$
$$+|ln(a)-ln(u)|+|ln(b)-ln(v)|$$
$$+|ln(u)-ln(x)|+|ln(v)-ln(y)|\}$$

$$=\frac{1}{3}\{|ln(x)-ln(a)|+|ln(a)-ln(u)|$$
$$+|ln(u)-ln(x)|+|ln(y)-ln(b)|$$
$$+|ln(b)-ln(v)|+|ln(v)-ln(y)|\}$$

$$=\frac{G(Tx,Ta,Tu)+G(Ty,Tb,Tv)}{3}$$
$$\leq\frac{2}{3}max\{G(Tx,Ta,Tu),G(Ty,Tb,Tv)\}$$

$$\leq\frac{2}{3}max\{G(Tx,Ta,Tu),G(Ty,Tb,Tv)G(Tz,Tc,Tw)\}$$

$$=\phi(max\{G(Tx,Ta,Tu),G(Ty,Tb,Tv),G(Tz,Tc,Tw)\})$$

This is the contractive condition (10). Evidently, for every $(x,y,z),(u.v.w)\in X^3$, there always exists a point $(a,b,c)\in X^3$ that is comparable to (x,y,z) and $(u.v.w)$. So Theorem 20 can be applied to this example to conclude that F has a unique triple fixed point (8,8,8), since also the hypotheses of Theorem 28 hold. ■

Now, we give a simple example which shows that if T is not an ICS mapping then the conclusion of Theorem 20 and 26 fail.

Example 32 Let $X=\mathbb{R}$ and define $G: X\times X\times X\to\mathbb{R}^+$ by $\forall\, x,y,z\in X$.

$$G(x,y,z)=max\{|x-y|,|x-z|,|y-z|\}$$

Let $\leq$ be usual order. Then, (X,G) is a G-complete G-metric space. Let $F:X^3\to X$ be given by

$$F(x,y,z)=2x-y+1,\ \forall\, x,y,z\in X.$$

It is clear that F is continuous and has the mixed monotone property. Moreover, taking $x_0=z_0=1$ and $y_0=0$, we have

$$F(x_0,y_0,z_0)=F(1,0,1)=3>1=x_0,$$
$$F(y_0,x_0,y_0)=F(0,1,0)=-1<0=y_0,$$
$$F(z_0,y_0,x_0)=F(1,0,1)=3>1=z_0,$$

it is condition (8). Let $T:X\to X$ be define by $Tx=1,\forall\, x\in X$. Then T is not an ICS mapping. It is obvious that the condition (10) holds for any $\phi\in\Phi$. However, F has no tripled fixed point. ■

Example 33 As in Example 2.5 of [13], let $T:X\to X$ be given by $Tx=x,\ \forall\, x\in X$. Obviously, T is an ICS mapping. The mapping $F:X^3\to X$ has unique tripled fixed point (0,0,0). ■

Example 34 As in Example 2.6 of [13], let $T:X\to X$ be given by $Tx=x,\ \forall\, x\in X$. Obviously, T is an ICS mapping. The mapping $F:X^3\to X$ has unique tripled fixed point $\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$. ■

5. CONCLUSION

In this paper, we established some tripled fixed point theorems for mappings having mixed monotone property under nonlinear type contractions depended on another function $T:X\to X$ (where T is an ICS mapping) in the framework of a G-metric space X enclosed with partial order. Our results are generalized, improved and extended some well-known results in the literature. These results are extensions of results in [13] to the case triple fixed points depending on another function. Inequality (10) does not reduce to any metric inequality with the metric d_G, [this metric is given by (2)]. Hence our theorems do not reduce to fixed point problems in the corresponding metric space (X,d_G). Also, in all Theorem 20 (Theorem 26) is genuinely different to Theorem 2.1 (Theorem 3.4) of Hassen et al. [13]. If mapping $T:X\to X$

is not an ICS mapping then the conclusion of main results (Theorems 20 and 26) fail (see Example 31). Also, presented examples are showing that our results are real generalization of known ones in triple fixed point theory. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications. ■

COMPETING INTERESTS

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTION

Both authors contributed equally and significantly to writing this paper. Both authors read and approved the final manuscript.

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