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An Efficient Method Based on Lucas Polynomials for Solving High-Order Linear Boundary Value Problems

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ABSTRACT

In this paper, a collocation method based on Lucas polynomials for solving high-order linear differential equations with variable coefficients under the boundary conditions is presented by transforming the problem into a system of linear algebraic equations with Lucas coefficients. The proposed approach is applied to fourth, fifth, sixth and eighth-order two-point boundary values problems occurring in science and engineering, and compared by existing methods. The technique gives better approximations than other methods, and has a lower computational cost. In addition, the error analysis based on residual function is developed for the present method and the improved approximate solution is obtained. Moreover, numerical examples are included to illustrate the practical usefulness and efficiency of the method.

Keywords : Lucas polynomials, boundary value problems, high-order differential equations, collocation method, residual error analysis.

1. INTRODUCTION

In recent years, much attention have been given to solve the high-order boundary value problems (BVPs) which have application in various branches of pure and applied sciences. Several numerical and analytical methods have been developed for solving these problems. But it may not be possible to find the analytical solutions of such problems for all coefficient functions.

A very special form of high-order problems is the fourth-order BVPs. It is well known that these problems arise in the mathematical modelling of viscoelastic and inelastic flows [1-3], deformation of beams [2], plate deflection theory [2,3], stress distribution in a spherical membrane, fluid dynamics and bending of lateral load circular plate [1].

Several numerical and analytical methods including finite difference method, Adomian decomposition method, differential transform method and variational iteration method [2], homotophy perturbation method [2,3], nonpolynomial spline method [1,3], B-spline collocation method [1] have been developed for solving general fourth-order BVPs.

Special fifth-order BVPs arise in the mathematical modelling of viscoelastic flows and other branches of mathematical, physical and engineering sciences [4,5]. Fifth-order BVPs investigated by many authors using finite difference method, quartic spline method [4], spectral Galarkin and collocation method [5] and therein.

Sixth-order BVPs arise in astro-physics [6-9]; the narrow convecting, layers bounded by stable layers which are believed to surround A-type stars can be modelled by sixth-order BVPs. Also, when an infinite horizontal layers of fluid is heated from below and is subject to rotation, instability occurs. When this instability is like ordinary convection, the differential

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equation is sixth-order. The analytic solution of this type equations subject to boundary conditions can not be obtained for arbitrary choices of coefficient functions. For the numerical solutions of the sixth-order BVPs, the following methods have been used in the literature [3,6,7,8]: Spline collocation method, quartic spline method, septic spline method and Sinc-Galerkin method [9].

In the literature, the eighth-order BVPs arise in the mathematical modelling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences [10,11]. For the solution of problems, the following numerical methods have been developed: Spectral Galarkin and collocation, B-spline method, decomposition method, spline collocation method, Chow-Yorke algorithm and homotopy perturbation method [10,11] and references therein.

Since the beginnig of 1994, Taylor, Chebyshev, Legendre, Berstein, Hermite, Laguerre and Bessel matrix methods have been used by Sezer et al. [12-19] to solve linear differential, multi-pantograph, generalized pantograph, Fredholm integral and Fredholm integro-differential-difference equations.

In this study, in the light of the above-mentioned methods and by means of the matrix relations between the Lucas polynomials and their derivatives, we apply a collocation method for solving the high-order linear differential equations with variable coefficients

$$L\left[y(x)\right]=\sum_{k=0}^{m}p_k(x)y^{(k)}(x)=g(x)\ ,\ 0\le a\le x\le b \qquad (1)$$

under the mixed conditions (initial and boundary conditions)

$$\sum_{k=0}^{m-1}\left(a_{jk}y^{(k)}(a)+b_{jk}y^{(k)}(b)\right)=c_j,\qquad j=0,1,...,m-1 \qquad (2)$$

where $y^{(0)}(x)=y(x)$ is an unknown function; $p_k(x)$ and $g(x)$ are the known continuous functions defined on interval $[a,b]$; $y^{(k)}(x)$ represents to the k th-order derivative of $y(x)$; a_{jk},b_{jk} and c_j are real constants.

Also, by improving the Lucas collocation method with the aid of the residual error function [20-24], we gain an improved approximate solution of (1) expressed in the truncated Lucas series form

$$y_{N,M}(x)=y_N(x)+e_{N,M}(x)$$

where

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}a_nL_n(x) \qquad (3)$$

is the Lucas polynomial solution and

$$e_{N,M}(x)=\sum_{n=0}^{M}a_n^*L_n(x),\qquad (M>N)$$

is the Lucas polynomial solution of the error problem based on the residual error function. Here a_n and a_n^* are the unknown coefficients; and $L_n(x),\ n=0,1,...,N$ are the Lucas polynomials defined by

$$L_0(x)=2\,;$$
$$L_n(x)=\sum_{k=0}^{[n/2]}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k},\ (n\ge 1)$$
$$[n/2]=\begin{cases} n/2 & ,\ n\text{ even}\\ (n-1)/2 & ,\ n\text{ odd}\end{cases}$$

[25-27]. The purpose of this study is to develop a Lucas polynomial solution for BVPs by means of the residual error function and to give an efficient and useful error estimation via the error problem. On the other hand, in order to find a solution of the equation (1) with the conditions (2), we can use the collocation points defined by

$$x_i=a+\frac{b-a}{N}i,\quad i=0,1,...,N,\quad 0\le a\le x\le b\,. \qquad (4)$$

2. LUCAS COLLOCATION METHOD

Firstly, we can write the approximate solution $y_N(x)$ given by (3) in the matrix form

$$y_N(x)=\mathbf{L}(x)\mathbf{A} \qquad (5)$$

where

$$\mathbf{L}(x)=\begin{bmatrix}L_0(x) & L_1(x) & L_2(x) & \cdots & L_N(x)\end{bmatrix}$$

and

$$\mathbf{A}=\begin{bmatrix}a_0 & a_1 & a_2 & \cdots & a_N\end{bmatrix}^T.$$

Also, we can express Eq.(5) as

$$y_N(x)=\mathbf{X}(x)\mathbf{D}^T\mathbf{A} \qquad (6)$$

such that

$$\mathbf{X}(x)=\begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix}$$

and if N is odd,

$$\mathbf{D}=\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & \cdots & 0 \\ \frac{4}{2}\binom{2}{2} & 0 & \frac{4}{3}\binom{3}{1} & 0 & \frac{4}{4}\binom{4}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{n-1}{(n-1)/2}\binom{(n-1)/2}{(n-1)/2} & 0 & \frac{n-1}{(n+1)/2}\binom{(n+1)/2}{(n-3)/2} & 0 & \frac{n-1}{(n+3)/2}\binom{(n+3)/2}{(n-5)/2} & & 0 \\ 0 & \frac{n}{(n+1)/2}\binom{(n+1)/2}{(n-1)/2} & 0 & \frac{n}{(n+3)/2}\binom{(n+3)/2}{(n-3)/2} & 0 & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}.$$

If N is even,

$$\mathbf{D}=\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & \cdots & 0 \\ \frac{4}{2}\binom{2}{2} & 0 & \frac{4}{3}\binom{3}{1} & 0 & \frac{4}{4}\binom{4}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{n-1}{n/2}\binom{n/2}{(n-2)/2} & 0 & \frac{n-1}{(n+2)/2}\binom{(n+2)/2}{(n-4/2} & 0 & \cdots & 0 \\ \frac{n}{n/2}\binom{n/2}{n/2} & 0 & \frac{n}{(n+2)/2}\binom{(n+2)/2}{(n-2)/2} & 0 & \frac{n}{(n+4)/2}\binom{(n+4)/2}{(n-4)/2} & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}.$$

It is clearly seen that the relations between the matrix $\mathbf{X}(x)$ and its derivatives $\mathbf{X}'(x)$, $\mathbf{X}''(x)$ and $\mathbf{X}^{(k)}(x)$ are

$$\mathbf{X}'(x)=\mathbf{X}(x)\mathbf{B}\,,\ \mathbf{X}''(x)=\mathbf{X}(x)\mathbf{B}^2 \text{ and } \mathbf{X}^{(k)}(x)=\mathbf{X}(x)\mathbf{B}^k \tag{7}$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then (6) and (7) yield the matrix relations

$$y_N'(x)=\mathbf{X}(x)\mathbf{B}\mathbf{D}^T\mathbf{A}\,,\ \ y_N''(x)=\mathbf{X}(x)\mathbf{B}^2\mathbf{D}^T\mathbf{A} \text{ and } y_N^{(k)}(x)=\mathbf{X}(x)\mathbf{B}^k\mathbf{D}^T\mathbf{A}. \tag{8}$$

By substituting (6) and (8) into Eq.(1), we obtain the matrix equation

$$\sum_{k=0}^{m} p_k(x)\mathbf{X}(x)\mathbf{B}^k\mathbf{D}^T\mathbf{A}=g(x) \tag{9}$$

and by using the collocation points (4) into Eq.(9), the system of matrix equations

$$\sum_{k=0}^{m} p_k(x_i)\mathbf{X}(x_i)\mathbf{B}^k\mathbf{D}^T\mathbf{A} = g(x_i)$$

or the compact form

$$\left\{\sum_{k=0}^{m}\mathbf{P}_k\mathbf{X}\mathbf{B}^k\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G} \qquad (10)$$

where

$$\mathbf{P}_k = \begin{bmatrix} p_k(x_0) & 0 & \cdots & 0 \\ 0 & p_k(x_1) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & p_k(x_N) \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Thus, the fundamental matrix equation (10) corresponding to Eq.(1) can be written in the form

$$\mathbf{WA=G} \text{ or } [\mathbf{W;G}], \quad \mathbf{W} = \sum_{k=0}^{m}\mathbf{P}_k\mathbf{X}\mathbf{B}^k\mathbf{D}^T. \qquad (11)$$

Eq.(11) indicates a system of $(N+1)$ linear algebraic equations with unknown Lucas coefficients a_n $(n = 0,1,...,N)$. Now, by means of Eq.(8), we obtain the matrix forms for the conditions (2) as follows

$$\sum_{k=0}^{m-1}\left[a_{jk}\mathbf{X}(a) + b_{jk}\mathbf{X}(b)\right]\mathbf{B}^k\mathbf{D}^T\mathbf{A} = \left[c_j\right], \quad j = 0,1,...,m-1$$

or briefly,

$$\mathbf{U}_j\mathbf{A} = \left[c_j\right] \text{ or } \left[\mathbf{U}_j; c_j\right], \quad j = 0,1,...,m-1 \qquad (12)$$

where

$$\mathbf{U}_j = \sum_{k=0}^{m-1}\left[a_{jk}\mathbf{X}(a) + b_{jk}\mathbf{X}(b)\right]\mathbf{B}^k\mathbf{D}^T = \left[u_{j0} \quad u_{j1} \quad u_{j2} \quad \cdots \quad u_{jN}\right], \quad j = 0,1,...,(m-1).$$

Consequently, by replacing the row matrices (12) by last rows of the matrix (11), we have

$$\left[\mathbf{W};\mathbf{G}\right] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-m)0} & w_{(N-m)1} & \cdots & w_{(N-m)N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & c_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & c_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & \cdots & u_{(m-1)N} & ; & c_{m-1} \end{bmatrix},$$

which is a linear algebraic system. If rank $\mathbf{W} = \text{rank}\left[\mathbf{W};\mathbf{G}\right] = N+1$, then we can write $\mathbf{A} = (\mathbf{W})^{-1}\mathbf{G}$. Hence, the unknown Lucas coefficients matrix $\mathbf{A} = \left[a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_N\right]^T$ is determined and by substituting the coefficients $a_0, a_1, a_2, ..., a_n$ into Eq.(3), the Lucas polynomial solution of the differential equation is obtained.

3. RESIDUAL CORRECTION AND ERROR ESTIMATION

In this section, we will give an error estimation for the Lucas polynomial solution (3) with the residual error function [20-24]. Furthermore, we will improve the solution (3) by means of the residual error function. Firstly, we can define the residual function of the method as

$$R_N(x) = L\left[y_N(x)\right] - g(x). \tag{13}$$

Here, $y_N(x)$ is the Lucas polynomial solution given by (3) of the problem (1) and (2). Hence, $y_N(x)$ satisfies the problem

$$\begin{cases} L\left[y_N(x)\right] = \sum_{k=0}^{m} p_k(x) y_N^{(k)}(x) = g(x) + R_N(x) \\ \sum_{k=0}^{m-1}\left(a_{jk} y_N^{(k)}(a) + b_{jk} y_N^{(k)}(b)\right) = c_j, \quad j = 0,1,\ldots,m-1. \end{cases}$$

Also, the error function $e_N(x)$ can be defined as

$$e_N(x) = y(x) - y_N(x) \tag{14}$$

where $y(x)$ is the exact solution of the problem (1) and (2). From Eqs.(1), (2), (13) and (14), we gain the error differential equation

$$L\left[e_N(x)\right] = L\left[y(x)\right] - L\left[y_N(x)\right] = -R_N(x)$$

with the homogeneous mixed conditions

$$\sum_{k=0}^{m-1}\left(a_{jk} e_N^{(k)}(a) + b_{jk} e_N^{(k)}(b)\right) = 0, \quad j = 0,1,\ldots,m-1$$

or openly, the error problem

$$\begin{cases} \sum_{k=0}^{m} p_k(x) e_N^{(k)}(x) = -R_N(x) \\ \sum_{k=0}^{m-1}\left(a_{jk} e_N^{(k)}(a) + b_{jk} e_N^{(k)}(b)\right) = 0, \quad j = 0,1,\ldots,m-1. \end{cases} \tag{15}$$

Here, note that the nonhomegeneous mixed conditions

$$\sum_{k=0}^{m-1}\left(a_{jk} y^{(k)}(a) + b_{jk} y^{(k)}(b)\right) = c_j, \quad j = 0,1,\ldots,m-1$$

and

$$\sum_{k=0}^{m-1}\left(a_{jk} y_N^{(k)}(a) + b_{jk} y_N^{(k)}(b)\right) = c_j, \quad j = 0,1,\ldots,m-1$$

are reduced to homogeneous mixed conditions

$$\sum_{k=0}^{m-1}\left(a_{jk} e_N^{(k)}(a) + b_{jk} e_N^{(k)}(b)\right) = 0, \quad j = 0,1,\ldots,m-1.$$

The error problem (15) can be solved by using the prosedure given in Section 2. Thus, we obtain the approximation $e_{N,M}(x)$ to $e_N(x)$ as follows

$$e_{N,M}(x)=\sum_{n=0}^{M}a_n^{*}L_n(x),\qquad M>N\,.$$

Consequently, the corrected Lucas polynomial solution $y_{N,M}(x)=y_N(x)+e_{N,M}(x)$ is obtained by means of the polynomials $y_N(x)$ and $e_{N,M}(x)$. Also, we construct the error function $e_N(x)=y(x)-y_N(x)$, the estimated error function $e_{N,M}(x)$ and the corrected error function $E_{N,M}(x)=e_N(x)-e_{N,M}(x)=y(x)-y_{N,M}(x)$.

4. NUMERICAL EXAMPLES

In this section, the numerical results of fourth, fifth, sixth and eighth-order boundary value problems are given to show the efficiency and applicability of the method. In tables, we calculate the values of the Lucas polynomial solution $y_N(x)$, the corrected Lucas polynomial solution $y_{N,M}(x)=y_N(x)+e_{N,M}(x)$, the absolute error function $|e_N(x)|=|y(x)-y_N(x)|$, the corrected absolute error function $|E_{N,M}(x)|=|y(x)-y_{N,M}(x)|$ and the estimated absolute error function $|e_{N,M}(x)|$. All numerical computations are calculated by using a computer programme written in *Maple*.

Example 1 : Let us consider the boundary value problem given by

$$\begin{aligned}&y^{(4)}(x)+xy(x)=-(8+7x+x^3)e^x\,,\qquad 0\le x\le 1\\&y(0)=0\,,\quad y''(0)=0\\&y(1)=0\,,\quad y''(1)=-4e\end{aligned}\tag{16}$$

which has the exact solution $y(x)=x(1-x)e^x$ [3,28].

The approximate solution $y_5(x)$ by the truncated Lucas series for $N=5$ is given by

$$y_5(x)=\sum_{n=0}^{5}a_nL_n(x).$$

Now, let us compute the coefficients a_n, $(n=0,1,2,3,4,5)$ of the approximate solution. The set of the collocation points given by (4) for $a=0$, $b=1$ and $N=5$ is calculated as

$$\left\{x_0=0,\ x_1=\frac{1}{5},\ x_2=\frac{2}{5},\ x_3=\frac{3}{5},\ x_4=\frac{4}{5},\ x_5=1\right\}.$$

By applying the procedure given by in Section 2, we gain the Lucas polynomial solution for $N=5$ as

$$\begin{aligned}y_5(x)=&-0.147300000126457x^5-0.333333333333333x^4-0.65452121855119x^3\\&+1.13515455201097x.\end{aligned}$$

In order to compute the corrected Lucas polynomial solution, let us consider the error problem

$$\begin{cases}e_5^{(4)}(x)+xe_5(x)=-R_5(x)\\e_5(0)=0,\ e_5''(0)=0,\ e_5(1)=0,\ e_5''(1)=0\end{cases}\tag{17}$$

such that the residual function is

$$R_5(x)=y_5^{(4)}(x)+xy_5(x)+(8+7x+x^3)e^x\,.$$

By solving the error problem (17) for $M=6$ with the method in Section 2, the estimeted Lucas error function approximation $e_{5,6}(x)$ to $e_5(x)$ is obtained as

$$e_{5,6}(x) = -(0.42649685187551e-1)x^6 + (0.25496786604631e-1)x^5 + 0.12825913725566x^3 - 0.11110623867273x.$$

Thus, we can calculate the corrected Lucas polynomial solution

$$y_{5,6}(x) = -0.121803213521826x^5 - 0.333333333333333x^4 - 0.52626208129553x^3 + 1.02404831333824x - (0.42649685187551e-1)x^6.$$

In Table 1, the numerical values of the exact solution, the Lucas polynomial solutions and the corrected Lucas polynomial solutions are compared. In Table 2, the actual absolute errors are compared with the estimated absolute errors. Table 3 shows the numerical results of the corrected absolute error functions for different values of N and M.

Table 1. Comparison of the exact and approximate solutions for $N = 5,7$ and $M = 6,8,9,11$ of the Problem (16)

	Exact Solution	Lucas Polynomial Solution	Corrected Lucas Polynomial Solution	
x_i	$y(x_i)$	$y_5(x_i)$	$y_{5,6}(x_i)$	$y_{5,9}(x_i)$
1/8	0.123938112	0.140530082	0.126892923	0.124016671
1/16	0.062372722	0.070782138	0.063869332	0.062412524
1/32	0.031234420	0.035453283	0.031985128	0.031254386
x_i	$y(x_i)$	$y_7(x_i)$	$y_{7,8}(x_i)$	$y_{7,11}(x_i)$
1/8	0.123938112	0.124258347	0.123951009	0.123910335
1/16	0.062372722	0.062534817	0.062379238	0.062358648
1/32	0.031234420	0.031315714	0.031237686	0.031227359

Table 2. Comparison of the actual and estimated absolute errors for $N = 5,7$ and $M = 6,8,9,11$ of the Problem (16)

	Actual absolute errors	Estimated absolute errors	
x_i	$\lvert e_5(x_i)\rvert = \lvert y(x_i) - y_5(x_i)\rvert$	$\lvert e_{5,6}(x_i)\rvert$	$\lvert e_{5,9}(x_i)\rvert$
1/8	1.6592e-2	1.3637e-2	1.6513e-2
1/16	8.4094e-3	6.9128e-3	8.3696e-3
1/32	4.2187e-3	3.4682e-3	4.1989e-3
x_i	$\lvert e_7(x_i)\rvert = \lvert y(x_i) - y_7(x_i)\rvert$	$\lvert e_{7,8}(x_i)\rvert$	$\lvert e_{7,11}(x_i)\rvert$
1/8	3.2023e-4	3.0734e-4	3.4801e-4
1/16	1.6209e-4	1.5558e-4	1.7617e-4
1/32	8.1294e-5	7.8028e-5	8.8354e-5

Table 3. Numerical results of the corrected absolute error functions for $N = 5,7$ and $M = 6,8,9,11$ of the Problem (16)

	Corrected absolute errors $\lvert E_{N,M}(x)\rvert = \lvert y(x) - y_{N,M}(x)\rvert$			
x_i	$\lvert E_{5,6}(x_i)\rvert$	$\lvert E_{5,9}(x_i)\rvert$	$\lvert E_{7,8}(x_i)\rvert$	$\lvert E_{7,11}(x_i)\rvert$
1/8	2.9548e-3	7.8559e-5	1.2897e-5	2.7777e-5
1/16	1.4966e-3	3.9802e-5	6.5161e-6	1.4074e-5
1/32	7.5071e-4	1.9966e-5	3.2665e-6	7.0602e-6

In Table 4, we compare the actual absolute errors $\lvert e_N(x)\rvert$ obtained by Lucas collocation method for different values of N. In addition, in Table 5, the actual absolute errors of our method and other methods given by [3], [28] are compared for different values of N for the problem (16).

Table 4. Actual absolute errors for different values of N of the Problem (16)

x_i	$\lvert e_5(x_i)\rvert$	$\lvert e_6(x_i)\rvert$	$\lvert e_7(x_i)\rvert$	$\lvert e_8(x_i)\rvert$
1/8	1.66e-2	2.90e-3	3.20e-4	5.88e-5
1/16	8.41e-3	1.47e-3	1.62e-4	2.98e-5
1/32	4.22e-3	7.36e-4	8.13e-5	1.50e-5

Table 5. Comparison of actual absolute errors of different methods of the Problem (16)

x_i	Exact Solution	Quintic Spline Method [3]	Sixth-Order Method [28]	Present Method (for $N=8$)
1/8	0.123938112	2.13e-4	1.74e-3	5.88e-5
1/16	0.062372722	1.32e-6	4.33e-4	2.98e-5
1/32	0.031234420	3.81e-6	1.08e-4	1.50e-5

It is seen from Table 4 and Table 5 that the actual absolute errors in the present method are very close to zero when the value of N is increased and Lucas collocation method gives better approximations than other methods in [3] and [28].

Example 2 : Let us consider the boundary value problem given by

$$y^{(5)}(x)-y(x)=-(15+10x)e^{x},\qquad 0\le x\le 1 \tag{18}$$
$$y(0)=0,\quad y'(0)=1,\quad y''(0)=0$$
$$y(1)=0,\quad y'(1)=-e$$

which has the analytic solution $y(x)=x(1-x)e^{x}$ [4,29,30].

By appliying the technique introduced in Section 2, we obtain the Lucas polynomial solution for $a=0$, $b=1$ and $N=6$ as

$$y_6(x)=-(0.377003259532790e-1)x^6-0.125000000000000x^5-0.35518085059922x^4$$
$$-0.48211882344751x^3+x.$$

and then the error problem

$$\begin{cases} e_6^{(5)}(x)-e_6(x)=-R_6(x) \\ e_6(0)=0,\ e_6{}'(0)=0,\ e_6{}''(0)=0,\ e_6(1)=0,\ e_6{}'(1)=0 \end{cases} \tag{19}$$

where the residual function is

$$R_6(x)=y_6^{(5)}(x)-y_6(x)+(15+10x)e^{x}.$$

By solving the error problem (19) for $M=7$ we find the estimated error approximation $e_{6,7}(x)$ to $e_6(x)$ as

$$e_{6,7}(x)=(0.1e-10)x^2-(0.156648233e-1)x^3+(0.1925419347e-1)x^4+(0.489671273518969e-2)x^6$$
$$-(0.848608291898198e-2)x^7$$

and hence, the corrected Lucas poynomial solution is

$$y_{6,7}(x)=-(0.328036132180893e-1)x^6-0.125x^5-0.33592665712922x^4-0.49778364674751x^3$$
$$+x+(0.1e-10)x^2-(0.848608291898198e-2)x^7.$$

In Table 6, the numerical results of the exact solution, the Lucas polynomial solutions and the corrected Lucas polynomial solutions are compared and, in Table 7, the actual absolute errors are compared with the estimated absolute errors for the different values of N and M. Also, in Table 8, the corrected absolute errors are given.

Table 6. Comparison of the exact solution and the approximate solutions for $N = 6,7$ and $M = 7,8,9$ of the Problem (18)

	Exact Solution	Lucas Polynomial Solution	Corrected Lucas Polynomial Solution	
x_i	$y(x_i)$	$y_6(x_i)$	$y_{6,7}(x_i)$	$y_{6,9}(x_i)$
$1/10$	0.099465383	0.099481075	0.099467340	0.099465355
$1/20$	0.049935377	0.049937476	0.049935638	0.049935373
$1/40$	0.024992056	0.024992327	0.024992090	0.024992056
x_i	$y(x_i)$	$y_7(x_i)$	$y_{7,8}(x_i)$	$y_{7,9}(x_i)$
$1/10$	0.099465383	0.099467427	0.099465586	0.099465417
$1/20$	0.049935377	0.049935650	0.049935404	0.049935381
$1/40$	0.024992056	0.024992091	0.024992059	0.024992057

Table 7. Comparison of the actual and estimated absolute errors for $N = 6,7$ and $M = 7,8,9$ of the Problem (18)

	Actual absolute errors	Estimated absolute errors	
x_i	$\lvert e_6(x_i)\rvert = \lvert y(x_i) - y_6(x_i)\rvert$	$\lvert e_{6,7}(x_i)\rvert$	$\lvert e_{6,9}(x_i)\rvert$
$1/10$	1.5693e-5	1.3735e-5	1.5720e-5
$1/20$	2.0985e-6	1.8377e-6	2.1022e-6
$1/40$	2.7086e-7	2.3724e-7	2.7132e-7
x_i	$\lvert e_7(x_i)\rvert = \lvert y(x_i) - y_7(x_i)\rvert$	$\lvert e_{7,8}(x_i)\rvert$	$\lvert e_{7,9}(x_i)\rvert$
$1/10$	2.0441e-6	1.8411e-6	2.0101e-6
$1/20$	2.7248e-7	2.4553e-7	2.6793e-7
$1/40$	3.5122e-8	3.1687e-8	3.4528e-8

Table 8. Numerical results of the corrected absolute error functions for $N = 6,7$ and $M = 7,8,9$ of the Problem (18)

	Corrected absolute errors $\lvert E_{N,M}(x)\rvert = \lvert y(x) - y_{N,M}(x)\rvert$			
x_i	$\lvert E_{6,7}(x_i)\rvert$	$\lvert E_{6,9}(x_i)\rvert$	$\lvert E_{7,8}(x_i)\rvert$	$\lvert E_{7,9}(x_i)\rvert$
$1/10$	1.9557e-6	2.7597e-8	2.0296e-7	3.3994e-8
$1/20$	2.6084e-7	3.7035e-9	2.6946e-8	4.5441e-9
$1/40$	3.3618e-8	4.6467e-10	3.4352e-9	5.9382e-10

Table 9 shows the actual absolute errors $\lvert e_N(x)\rvert$ obtained by Lucas collacation method for different values of N. These errors are very close to zero when the values of N is increased. In Table 10, the absolute errors of our method are compared with the error values given by Khan [30], Caglar et al. [29] and Siddiqi et al. [4]. It is seen that the collocation method based on Lucas polynomials is very effective and applicability than the other methods for fifth order boundary value problems.

Table 9. Actual absolute errors for $N = 6,7,8,9,10,11$ of the Problem (18)

x_i	$\lvert e_6(x_i)\rvert$	$\lvert e_7(x_i)\rvert$	$\lvert e_8(x_i)\rvert$	$\lvert e_9(x_i)\rvert$	$\lvert e_{10}(x_i)\rvert$	$\lvert e_{11}(x_i)\rvert$
$1/10$	1.5693e-5	2.0441e-6	2.7165e-7	1.0264e-7	8.6681e-8	8.5597e-8
$1/20$	2.0985e-6	2.7248e-7	3.6178e-8	1.3728e-8	1.1608e-8	1.1464e-8
$1/40$	2.7086e-7	3.5122e-8	4.6609e-9	1.7722e-9	1.4993e-9	1.4807e-9

Table 10. Comparison of actual absolute errors of different methods of the Problem (18)

x_i	Exact Solution	Quartic Spline Method [4]	Sixth-Degree B-Spline Method [29]	Finite Difference Method [30]	Present Method (for $N=11$)
$1/10$	0.099481075	3.6000e-3	0.1570	0.4025e-2	8.5597e-8
$1/20$	0.049937476	5.5531e-4	0.0747	0.3911e-2	1.1464e-8
$1/40$	0.024992327	7.6625e-5	0.0208	0.1145e-1	1.4807e-9

Example 3 : We consider the boundary value problem given by

$$y^{(6)}(x) - y(x) = -6e^{x}, \qquad 0 \le x \le 1, \tag{20}$$
$$y(0)=1, \quad y'(0)=0, \quad y''(0)=-1$$
$$y(1)=0, \quad y'(1)=-e, \quad y''(1)=-2e$$

which has the exact solution $y(x) = (1-x)e^{x}$ [31-34].

This problem was solved by using the Adomian decomposition method [31], sixth degree B-spline functions method [32], the variational iteration method [33] and the quartic B-Splines method [34]. Now, we calculate the approximate solutions and the absolute errors of the problem by using the Lucas collocation method for the different values of N (see Table 11) and in Table 12, we compare the obtained results with the results given in [31-34].

Table 11. Actual absolute errors for $N = 6,7,8,9,10,11,12$ of the Problem (20)

x_i	$\lvert e_7(x_i)\rvert$	$\lvert e_8(x_i)\rvert$	$\lvert e_9(x_i)\rvert$	$\lvert e_{10}(x_i)\rvert$	$\lvert e_{11}(x_i)\rvert$	$\lvert e_{12}(x_i)\rvert$
0.1	7.6676e-7	1.0496e-7	1.6514e-8	7.3623e-9	6.6409e-9	6.5959e-9
0.2	4.5773e-6	6.3552e-7	1.0014e-7	4.4204e-8	3.9771e-8	3.9505e-8
0.3	1.1009e-5	1.5548e-6	2.4556e-7	1.0712e-7	9.6057e-8	9.5399e-8
0.4	1.7498e-5	2.5212e-6	3.9966e-7	1.7187e-7	1.5348e-7	1.5236e-7
0.5	2.1074e-5	3.1069e-6	4.9526e-7	2.0944e-7	1.8605e-7	1.8461e-7
0.6	1.9878e-5	3.0070e-6	4.8311e-7	2.0040e-7	1.7684e-7	1.7534e-7
0.7	1.4202e-5	2.2100e-6	3.5884e-7	1.4565e-7	1.2747e-7	1.2629e-7
0.8	6.7004e-6	1.0750e-6	1.7692e-7	7.0120e-8	6.0743e-8	6.0141e-8
0.9	1.2722e-6	2.1085e-7	3.5277e-8	1.3630e-8	1.1661e-8	1.1517e-8

Table 12. Absolute errors of different methods of the Problem (20)

x_i	Exact Solution	Adomian Decomposition Method [31]	Sixth Degree B-Spline Method [32]	Variational Iteration Method [33]	Quartic B-Spline Method [34]	Present Method (for $N=12$)
0.1	0.99465383	4.0933e-4	1.2159e-5	4.0933e-4	4.5092e-6	6.5959e-9
0.2	0.97712221	7.7820e-4	2.7418e-5	7.7820e-4	1.2619e-5	3.9505e-8
0.3	0.94490117	1.0704e-3	2.2053e-6	1.0704e-3	1.9154e-5	9.5399e-8
0.4	0.89509482	1.2578e-3	2.5033e-6	1.2578e-3	2.1632e-5	1.5236e-7
0.5	0.82436064	1.3223e-3	5.4836e-6	1.3223e-3	1.9704e-5	1.8461e-7
0.6	0.72884752	1.2578e-3	1.6212e-5	1.2578e-3	1.4548e-5	1.7534e-7
0.7	0.60412581	1.0704e-3	2.0682e-5	1.0704e-3	8.2238e-6	1.2629e-7
0.8	0.44510819	7.7820e-4	2.2619e-5	7.7820e-4	2.9420e-6	6.0141e-8
0.9	0.24596031	4.0933e-4	1.9460e-5	4.0933e-4	2.3610e-7	1.1517e-8

From Table 11 and Table 12, we can say that the Lucas collocation method is very effective and better than other methods for sixth order boundary value problem (20).

Example 4 : We consider the boundary value problem given by

$$y^{(8)}(x)+xy(x)=-(48+15x+x^3)e^x\,,\qquad 0\le x\le 1 \tag{21}$$
$$y(0)=0\,,\quad y'(0)=1,\quad y''(0)=0\,,\quad y'''(0)=-3$$
$$y(1)=0\,,\quad y'(1)=-e\,,\quad y''(1)=-4e\,,\quad y'''(1)=-9e$$

which has the analytic solution $y(x)=x(1-x)e^x$ [11,35].

In Table 13 and Table 14, the computed results are compared with the results of the methods given by [11] and [35]. The obtained results shows that the collocation method based on Lucas polynomials is very effective and applicability than the other methods for eighth order boundary value problems.

Table 13. Comparison of absolute errors of the Problem (21)

x_i	$\lvert e_9(x_i)\rvert$	$\lvert e_{10}(x_i)\rvert$	$\lvert e_{11}(x_i)\rvert$	$\lvert e_{12}(x_i)\rvert$
0.1	1.5743e-8	2.3989e-9	2.5241e-10	2.9618e-12
0.2	1.6491e-7	2.5487e-8	2.7115e-9	3.6009e-11
0.3	5.1355e-7	8.0638e-8	8.6914e-9	1.3116e-10
0.4	9.1991e-7	1.4701e-7	1.6088e-8	2.7701e-10
0.5	1.1378e-6	1.8536e-7	2.0647e-8	4.0704e-10
0.6	1.0156e-6	1.6895e-7	1.9204e-8	4.3448e-10
0.7	6.2580e-7	1.0647e-7	1.2383e-8	3.2192e-10
0.8	2.2174e-7	3.8633e-8	4.6095e-9	1.3756e-10
0.9	2.3345e-8	4.1706e-9	5.1178e-10	1.7447e-11

Table 14. Comparison of absolute errors of different methods of the Problem (21)

x_i	Exact Solution	Nonic Spline Method [11]	Adomian Decomposition Method [35]	Present Method (for $N=12$)
0.1	0.099465398	5.62e-10	3.73e-9	2.9618e-12
0.2	0.195424606	4.88e-9	6.61e-9	3.6009e-11
0.3	0.283470863	1.37e-8	2.33e-8	1.3116e-10
0.4	0.358038847	2.29e-8	5.17e-8	2.7701e-10
0.5	0.412181455	2.71e-8	9.76e-8	4.0704e-10
0.6	0.437309528	2.38e-8	1.78e-6	4.3448e-10
0.7	0.422888694	1.49e-8	4.12e-6	3.2192e-10
0.8	0.356086770	5.54e-9	1.83e-4	1.3756e-10
0.9	0.221364303			1.7447e-11

5. CONCLUSION

In this paper, we have developed a new method based on Lucas polynomials with the aid of the residual error function for solving high-order linear boundary value problems. When the obtained results are investigated in examples, it can be seen that the improved method is very effective than other methods for boundary value problems. Furthermore, when the exact solution of the problem is not known, the actual absolute error $|e_N(x)|$ can be approximately computed with the aid of the estimated absolute error function. It is seen from the examples in Section 4 that the actual absolute errors $|e_N(x)|$ are close to the estimated absolute errors $|e_{N,M}(x)|$. Moreover, tables in Section 4 show that the errors decrease when the values of N and M increase. In addition, the obtained numerical results and comparisons in examples show that the developed method based on Lucas polynomials is very effective than other methods in the literaure. One of the advanteges of the present method that the approximate solutions are obtained easily by using computer programmes. In this paper, *Maple* is used. Also, It may not be possible to find the analytical solution of the equation (1) for each functions $p_k(x)$ and $g(x)$ in some methods. But our method is practicable for each continuous functions $p_k(x)$ and $g(x)$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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