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Fixed Point and Common Fixed Point Results for Contraction Mappings in G_b - Cone Metric Spaces

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ABSTRACT

The intent of this paper is to introduce the concept of G_b-cone metric and we describe some basic properties of such metric. Further, we establish some fixed point theorems for self-mappings satisfying the contractive type conditions and a common fixed theorem for two weakly compatible self-mappings satisfying the contractive condition in G_b-cone metric spaces without the assumption of normality. Moreover, some examples are provided to illustrate the usability of the obtained results.

Keywords: G_b -cone metric spaces; normal cones; non-normal cones; contraction; fixed points.

1. INTRODUCTION

Metric spaces play significant in mathematics and applied sciences. So, some authors havetried to give generalizations of metric spaces in several ways. The main revolution in the existence theory of many linear and non-linear operators happened after the Banach contraction principle. After this principle many researchers put their efforts into studying the existence and solutions for nonlinear equations (algebraic, differential and integral), a system of linear (nonlinear) equations and convergence of many computational methods [23]. Banach contraction gave us many important theories like variational inequalities, optimization theory and many computational theories [23-24]. Due to wide spreading importance of Banach contraction, many authors generalized it inseveral directions [25, 27-28].

The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhage [1-4]. Mustafa and Sims [5-6] have shown that most of the results concerning Dhage's D-metric spaces are invalid. In [5-6], they introduced an improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces, one can refer to the papers [11-17]. Beg, Abbas and Nazir [22] introduced G-cone metric space and established some fixed point theorems. Recently, Asadollah Aghajani, Mujahid Abbas and Jamal Rezaei Roshan [21] introduced G_b-metric space and established common

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fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces.

In this paper, we introduce the concept of G_b-cone metric and we describe some basic properties of such metric. Further, we establish some fixed point theorems for self-mappings satisfying the contractive type conditions and a common fixed theorem for two weakly compatible self-mappings satisfying the contractive condition in G_b-cone metric spaces without the assumption of normality. Moreover, some examples are provided to illustrate the usability of the obtained results.

2. PRELIMINARIES

Throughout this paper $\mathbb{R}$ and $\mathbb{R}_+$ will represents the set of real numbers and nonnegative real numbers, respectively.

The following definition is required in the sequel which can be found in [21].

Definition 2.1 Let X be a non-empty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$satisfies:

(GB1). $G(x, y, z) = 0$ if $x = y = z$,
(GB2). $0 < G(x, x, y)$, $\forall\, x, y \in X$, with $x \neq y$,
(GB3). $G(x, x, y) \leq G(x, y, z), \forall\, x, y, z \in X$ with $y \neq z$,
(GB4). $G(x, y, z) = G\{p(x, z, y)\}$ (Symmetry),
(GB5). $G(x, y, z) \leq s\big(G(x, a, a) + G(a, y, z)\big)$, $\forall$ x, y, z, a $\in$ X (Rectangle inequality).

Then G is called a generalized b-metric, or, more specially, G_b-metric on X, and the pair (X, G) is called a G_b-metric space. If $s = 1$, then G is called a generalized metric on X, and the pair (X, G) is called a G-metric space (see [6]).

Definition 2.2 Let $\mathbb{E}$ be a real Banach space, a subset of P of $\mathbb{E}$ is called a cone if and only if:

a) $P$is closed, non empty and $P \neq \{\theta\}$.
b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$, more generally, if $a, b, c \in \mathbb{R}$, $a, b, c \geq 0, x, y, z \in$ P $\Rightarrow ax + by + cz \in P$
c) $x \in P$ and $-x \in P \Rightarrow x = \theta, e.i. P \cap (-P) = \{\theta\}$

Given a cone $P \subset \mathbb{E}$, we define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. A cone $P \subset \mathbb{E}$ is called normal if there is a number for all $K > 0$ such that for all $x, y \in \mathbb{E}, \theta \preccurlyeq x \preccurlyeq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in intP$, ($intP$ denotes the interior of P), while $x \prec y$ means that $x \preccurlyeq y$ but $x \neq y$.

Now, we introduce the concept of G_b-cone metric space.

Definition 2.3 Let X be a nonempty set and $\mathbb{E}$ be a real Banach space equipped with the partial ordering $\preccurlyeq$ with respect to the cone P. A vector-valued function $G : X \times X \times X \to \mathbb{E}$ is said to be a generalized cone b-metric function on X with the constant $s \geq 1$ if the following conditions are satisfied:

(GBC1). $G(x, y, z) = \theta$ if $x = y = z$,
(GBC2). $\theta \prec G(x, x, y)$,whenever $x \neq y, \forall\, x, y \in X$,
(GBC3). $G(x, x, y) \preccurlyeq G(x, y, z)$, whenever $y \neq z, \forall\, x, y, z \in X$,
(GBC4). $G(x, y, z) = G\{p(x, z, y)\}$ (Symmetry),
(GBC5). $G(x, y, z) \preccurlyeq s\big(G(x, a, a) + G(a, y, z)\big)$, $\forall\ x, y, z, a \in X$, (Rectangle inequality).

Then the pair (X, G) is called a generalized G_b-cone metric space or, more specifically, a G_b-cone metric space. Obverse that if $s = 1$ the ordinary rectangle inequality in a generalized cone metric space is satisfied; however, it does not hold true when $s > 1$. Thus the class of G_b-cone metric spaces are effectively larger than that of ordinary G-cone metric spaces. That is, every G-cone metric space is a G_b-cone metric space, but the converse need not be true. Therefore, it is obvious that G_b-cone metric spaces generalize G_b-metric spaces and G-cone metric spaces.

We can present a number of examples, as follows, which show that introducing a G_b-cone metric space instead of a G-cone metric space is meaningful since there exist a G_b-cone metric spaces which are G-cone metric space.

Example 2.4 Let $X = \mathbb{R}$ and $\mathbb{E} = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{E}: x, y \geq 0\} \subset \mathbb{E}$. Define $G: X \times X \times X \to \mathbb{E}$ by

$$G(x, y, z) = max\{(|x - y|^p, \alpha|x - y|^p), (|y - z|^p, \alpha|y - z|^p), (|z - x|^p, \alpha|z - x|^p)\}$$

$\forall$ x, y, z $\in$ X, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, G) is a G_b-cone metric on X, but not a G-cone metric. In fact, we only need to prove (GBC5) in Definition 2.3 as follows: let $x, y, z, w \in X$. Set $u = x - w$, $v = w - y$, so $u + v = x - y$. From the inequality

$$\begin{aligned}(a + b)^p &\leq (2\, max\{a, b\})^p \\ &\leq 2^p(a^p + b^p),\ \forall a, b \geq 0,\end{aligned}$$

we have

$$\begin{aligned}|x - y|^p &= |u + v|^p \\ &\leq (|u| + |v|)^p \\ &\leq 2^p(|u|^p + |v|^p) \\ &= 2^p(|x - w|^p + |w - y|^p)\end{aligned}$$

Similarly,

$$|y - z|^p \leq 2^p(|w - w|^p + |y - z|^p)$$
$$|x - z|^p \leq 2^p(|x - w|^p + |w - z|^p)$$

Hence

$$\begin{aligned}&max\{(|x - y|^p, \alpha|x - y|^p), (|y - z|^p, \alpha|y - z|^p), (|z - x|^p, \alpha|z - x|^p)\} \\ &\leq 2^p\, max\{(|x - w|^p, \alpha|x - w|^p), (|w - w|^p, \alpha|w - w|^p), , |x - w|^p, \alpha|x - w|^p\}\end{aligned}$$

$$+2^p\, max\{(|w-y|^p, \alpha|w-y|^p)$$
$$,(|y-z|^p, \alpha|y-z|^p)$$
$$,|w-z|^p, \alpha|w-z|^p\}$$

which is implies that

$$G(x,y,z) \preccurlyeq s\big(G(x,,w,w)+G(w,y,z)\big)$$

with $s = 2^p > 1$.Taking account of the inequality

$$(a+b)^p > (a^p + b^p), \forall\, a, b > 0,$$

we arrive at

$$|x-y|^p = |x-w+w-y|^p$$
$$= (x-w+w-y)^p$$
$$> (x-w)^p + (w-y)^p$$
$$= |x-w|^p + |w-y|^p$$

for all $x > w > y$. Similarly,

$$|y-z|^p \geq |w-w|^p + |y-z|^p$$
$$|x-z|^p \geq |x-w|^p + |w-z|^p.$$

Thus, rectangle inequality in Definition G- cone metric space (see [22]) is not satisfied, *i.e.*, (X, G) is not a G-cone metric space.

Example 2.5 Let $X = [0, +\infty)$ and $\mathbb{E} = C_R^1[0,1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider

$$P = \{x \in \mathbb{E}: x(t) \geq 0 \text{ on } [0,1]\}.$$

Define $G: X \times X \times X \to \mathbb{E}$ by

$$G(x,y,z) = \max\{|x-y|^p, |y-z|^p, |z-x|^p\}\,\varphi,$$

$\forall\, x, y, z \in X$, with $p \geq 1$, where $\varphi: [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$ .Then (X, G) is a complete G_b-cone metric space with the coefficient $s = 2^{p-1}$.

If we define $G: X \times X \times X \to \mathbb{E}$ by

$$G(x,y,z) = (|x-y|^2 + |y-z|^2 + |z-x|^2)\varphi,$$

$\forall\, x, y, z \in X$.Then (X, G) is not a G_b-cone metric space. However,

$$G(x,y,z) = \max\{|x-y|^2, |y-z|^2, |z-x|^2\}\varphi$$

is a G_b-cone metric space on X with $s = 2$.

Example 2.6 Let $X = [0,1]$ and $\mathbb{E} = C_R^1[0,1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider

$$P = \{x \in \mathbb{E}: x(t) \geq 0 \text{ on } [0,1]\}.$$

Define $G: X \times X \times X \to \mathbb{E}$ by

$$G(x,y,z) = (|x-y| + |y-z| + |z-x|)^2\varphi,$$

$\forall\, x, y, z \in X$, where $\varphi: [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. Then (X, G) is a complete G_b-cone metric space with the coefficient $s = 2$.

Definition 2.7 A G_b-cone metric space (X, G) is said to symmetric if $\forall\, x, y \in X, G(x,y,y) = G(y,x,x)$.

Let (X, G) be a G_b-cone metric space, define $d_{G_b}: X \times X \to \mathbb{E}$ by

$$d_{G_b}(x,y) = G(x,y,y) + G(y,x,x).$$

Then $(X, d_{G_b})$is a cone b-metric space. It can be noted that

$$G(x,y,y) \preccurlyeq \tfrac{2s}{2s+1} d_{G_b}(x,y).$$

Obverse that if $s = 1$, that is, G be a G-cone metric on X, then

$$G(x,y,y) \preccurlyeq \tfrac{2}{3} d_G(x,y)$$

If X is a symmetric G_b-cone metric space, then

$$d_{G_b}(x,y) = 2G(x,y,y).$$

Definition 2.8 Let (X, G) be a G_b-cone metric space. A sequence (x_n) in X is said to be:

1) a G_b-cone Cauchy sequence if, for every $c \in \mathbb{E}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for all $n, m, l > N, G(x_n, x_m, x_l) \ll c$.
2) a G_b-cone convergent sequence if, for every $c \in \mathbb{E}$ with $\theta \ll c$, there is $N \in \mathbb{N}$ such that for all $m, n > N, G(x_n, x_m, x) \ll c$ for some fixed x in X. Here x is called the G_b-limit of (x_n) and is denoted by $G_b - \lim_{n \to +\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.

Definition 2.9 A G_b-cone metric space X is said to be a G_b-complete cone metric space, if every G_b-cone Cauchy sequence in X is G_b-cone convergent in$X$.

Using above definitions, we prove the following some lemmas and propositions.

Lemma 2.10 Let (X, G) be a G_b-cone metric space. Then for $c \in \mathbb{E}$ with $c \gg \theta$, there is $\delta > 0$such that $\|x\| < \delta$ implies $c - x \in intP$.

Proof Since $c \gg \theta$, then $\in intP$. Hence find $\delta > 0$ such that $N_\delta(c) = \{x \in \mathbb{E} : \|x - c\| < \delta\} \subset intP$. Since $\|c - c\| = \theta < \delta$, then $c \in N_\delta(c)$ and so $N_\delta(c) \neq \emptyset$. Now if $\|x\| < \delta$, then

$$\|x\| = \|x - c + c\|$$
$$= |-1|\|x - c + c\|$$
$$= \|-x + c - c\|$$
$$= \|(c - x) - c\| < \delta.$$

Then $c - x \in N_\delta(c)$ and since $N_\delta(c) \subset P$. So $c - x \in intP$.

Lemma 2.11 Let (X, G) be a G_b-cone metric space, P be a normal cone with normal constant K. Let (x_n) be a sequence in X. Then (x_n) is G_b-cone convergent to x if and only if $G(x_m, x_n, x) \to \theta$ as $m, n \to +\infty$.

Proof: Suppose that (x_n) is G_b-cone convergent to x. For every real $\varepsilon > 0$, choose $c \in \mathbb{E}$ with $c \gg \theta$ and $K\|c\| < \varepsilon$. Then there is $N \in \mathbb{N}$, for all $n, m \in \mathbb{N}, G(x_m, x_n, x) \ll c$. Since P is a normal cone with normal constant K, when $n, m \in \mathbb{N}$, $\|G(x_m, x_n, x)\| \leq K\|c\| < \varepsilon$. Therefore $G(x_m, x_n, x) \to \theta$, as $m, n \to +\infty$. Conversely, suppose that means $G(x_m, x_n, x) \to \theta$, as $m, n \to +\infty$. From Lemma 2.10, for every $c \in \mathbb{E}$ with $c \gg \theta$, there is $\delta >$

0 such that $\|x\| < \delta$ implies $-x \in intP$. For this δ there is $N \in \mathbb{N}$, such that $\|G(x_m, x_n, x)\| < \delta$ for all $n, m, l \geq N$. So $c - G(x_m, x_n, x) \in intP$. This means $G(x_m, x_n, x) \ll c$. Therefore (x_n) is G-cone convergent to x.

Proposition 2.12 Let (X, G) be a G_b-cone metric space, P be a normal cone with normal constant K, then the following are equivalent:

(1). (x_n) is G_b-cone convergent to x,
(2). $G(x_n, x_n, x) \to \theta$, as $n \to +\infty$,
(3). $G(x_n, x, x) \to \theta$, as $n \to +\infty$,
(4). $G(x_m, x_n, x) \to \theta$, as $m, n \to +\infty$.

Proof $(1) \Rightarrow (2)$: Suppose that the sequence (x_n) is G_b-cone convergent to x. From Lemma 2.11, $G(x_m, x_n, x) \to \theta$, as $m, n \to +\infty$. If we choose $m = n$, then $G(x_n, x_n, x) \to \theta$, as $n \to +\infty$. $(2) \Rightarrow (3)$: Suppose that $G(x_n, x_n, x) \to \theta$, as $n \to +\infty$. From $(GBC5)$,

$$G(x, x_n, x) \preccurlyeq s[G(x, x_n, x_n) + G(x_n, x_n, x)].$$

From $(GBC4)$, we have $G(x, x_n, x_n) = G(x_n, x_n, x)$ and since $G(x_n, x_n, x) \to \theta$, as $n \to +\infty$, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $\|G(x_n, x_n, x)\| < \frac{\varepsilon}{2sK}$. Since P be a normal cone with normal constant K,

$$\begin{aligned}\|G(x, x_n, x)\| &\preccurlyeq sK\|G(x, x_n, x_n) + G(x_n, x_n, x)\| \\ &\preccurlyeq sK\|G(x, x_n, x_n)\| + sK\|G(x_n, x_n, x)\| \\ &\prec sK\left(\frac{\varepsilon}{2sK} + \frac{\varepsilon}{2sK}\right) = \varepsilon.\end{aligned}$$

From $(GBC4)$, $G(x, x_n, x) = G(x_n, x, x)$, so for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $\|G(x_n, x, x)\| < \varepsilon$. This means $G(x_n, x, x) \to \theta$, as $n \to +\infty$. $(3) \Rightarrow (4)$: Suppose that $G(x_n, x, x) \to \theta$, as $n \to +\infty$. From $(GBC5)$,

$$G(x_m, x_n, x) \preccurlyeq s[G(x_m, x, x) + G(x, x_n, x)].$$

From $(GBC4)$, $G(x, x_n, x) = G(x_n, x, x) \to \theta$, as $n \to +\infty$ and $G(x_m, x, x) \to \theta$, as $m \to +\infty$. So for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n, m > N$, $\|G(x_m, x, x)\| < \frac{\varepsilon}{2sK}$ and $\|G(x, x_n, x)\| < \frac{\varepsilon}{2sK}$. Since P be a normal cone with normal constant K,

$$\begin{aligned}\|G(x_m, x_n, x)\| &\preccurlyeq sK\|G(x_m, x, x) + G(x, x_n, x)\| \\ &= sK\|G(x_m, x, x)\| + sK\|G(x, x_n, x)\| \\ &\prec sK\left(\frac{\varepsilon}{2sK} + \frac{\varepsilon}{2sK}\right) = \varepsilon.\end{aligned}$$

So for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n, m > N$, $\|G(x_m, x_n, x)\| < \varepsilon$. This means $G(x_m, x_n, x) \to \theta$, as $n, m \to +\infty$. $(4) \Rightarrow (1)$: Suppose that $G(x_m, x_n, x) \to \theta$, as $n, m \to +\infty$. From Lemma 2.11, the sequence (x_n) is G_b-cone convergent to x.

Lemma 2.13 Let (X, G) be a complete G_b-cone metric space with the coefficient $s \geq 1$, P be a normal cone with normal constant K. Let (x_n) be a sequence in X. If (x_n) G_b-cone converges to x and also (x_n) G_b-cone converges to y, then $x = y$. That is the limit of (x_n) is unique.

Proof From $x_n \to x$ and $x_n \to y$ as $n \to +\infty$, For every real $\varepsilon > 0$, choose $c \in \mathbb{E}$ with $c \gg \theta$ and $K\|c\| < \varepsilon$. Then there is $N \in \mathbb{N}$ such that for all $n, m > N$,

$$G(x_m, x_m, x) \ll \frac{c}{3s^2}, G(x_m, x_n, y) \ll \frac{c}{3s^2},$$

and $G(x_n, x_n, y) \ll \frac{c}{3s}$.

From $(GBC5)$ and $(GBC4)$, we have

$$\begin{aligned}G(x, x_n, y) &\preccurlyeq s[G(x, x_m, x_m) + G(x_m, x_n, y)] \\ &\ll s\left(\frac{c}{3s^2} + \frac{c}{3s^2}\right) = \frac{2c}{3s}.\end{aligned}$$

Again, from $(GBC5)$and $(GBC4)$, we have

$$\begin{aligned}G(x, x, y) &\preccurlyeq s[G(x, x_n, x_n) + G(x_n, x, y)] \\ &\ll s\left(\frac{c}{3s} + \frac{2c}{3s}\right) = c.\end{aligned}$$

Since P be a normal cone with normal constant K, $\|G(x, x, y)\| \preccurlyeq K\|c\| < \varepsilon$. Since ε is arbitrary, $G(x, x, y) = \theta$, therefore $x = y$.

Proposition 2.14 Let (X, G) is a G_b-cone metric space, P be a normal cone with normal constant K. Then sequence (x_n) is G_b-cone Cauchy if and only if $G(x_n, x_m, x_l) \to \theta$, as $n, m, l \to +\infty$.

Proof Suppose that $(x_n)$is G_b-cone Cauchy. For every real $\varepsilon > 0$, choose $c \in \mathbb{E}$ with $c \gg \theta$ and $K\|c\| < \varepsilon$.Then there is $N \in \mathbb{N}$ such that for all $n, m, l > N$, $G(x_n, x_m, x_l) \ll c$. Since P be a normal cone with normal constant K, when $n, m, l > N$, $\|G(x_n, x_m, x_l)\| \preccurlyeq K\|c\| < \varepsilon$.Therefore $G(x_n, x_m, x_l) \to \theta$, as $n, m, l \to +\infty$. Conversely, Suppose that $G(x_n, x_m, x_l) \to \theta$, as $n, m, l \to +\infty$. From Lemma 2.10, for every $c \in \mathbb{E}$ with $c \gg \theta$, there is $\delta > 0$ such that $\|x\| < \delta$ implies $-x \in intP$. For this δ there is $N \in \mathbb{N}$ such that for all $n, m, l \geq N$, $\|G(x_n, x_m, x_l)\| < \delta$.So $c - G(x_n, x_m, x_l) \in intP$. This means $G(x_n, x_m, x_l) \ll c$. Therefore (x_n) is G_b-cone Cauchy.

Lemma 2.15 Let (X, G) be a G_b-cone metric space, (x_n) be a sequence in X. If (x_n) is a G_b-cone convergent to x in X, then (x_n) is a G_b-cone Cauchy sequence in X.

Proof Suppose that (x_n) is a G_b-cone convergent to x in X. For any $c \in \mathbb{E}$ with $c \gg \theta$, there is $N \in \mathbb{N}$ such that for all $n, m, l > N$, $G(x, x_n, x_m) \ll \frac{c}{2s}$ and $G(x, x_m, x_l) \ll \frac{c}{2s}$. From $(GBC5)$, we have

$$G(x_n, x_m, x_l) \preccurlyeq s[G(x_n, x, x) + G(x, x_m, x_l)].$$

Also from $(GBC3)$ and $(GBC4)$, we have

$$\begin{aligned}G(x_n, x, x) &= G(x, x, x_n) \\ &\preccurlyeq G(x, x_m, x_n) \\ &= G(x, x_n, x_m).\end{aligned}$$

Thus

$$G(x_n, x, x) \preccurlyeq G(x, x_n, x_m) \ll \frac{c}{2s}$$

and then

$$G(x_n, x_m, x_l) \ll s\left(\frac{c}{2s} + \frac{c}{2s}\right) = c.$$

Therefore, (x_n) is a G_b-cone Cauchy sequence in X.

Proposition 2.16 Let (X,G) be a G_b-cone metric space, then the following are equivalent:

(1). (x_n) is G_b-cone Cauchy in X.
(2). For every $c \in \mathbb{E}$ with $c \gg \theta$, there is $N \in \mathbb{N}$ such that for all $n,m > N$, $G(x_n,x_m,x_m) \ll c$.

Proof $(1) \Rightarrow (2)$: Suppose that (x_n) is a G_b-cone Cauchy in X, for every $c \in \mathbb{E}$ with $c \gg \theta$,there is $N \in \mathbb{N}$ such that for all $n,m,l > N$, $G(x_n,x_m,x_l) \ll c$. If we choose $m = l$ then, for every $c \in \mathbb{E}$ with $c \gg \theta$,there is $N \in \mathbb{N}$ such that for all $n,m > N$, $G(x_n,x_m,x_m) \ll c$. $(2) \Rightarrow (3)$: Suppose that for every $c \in \mathbb{E}$ with $c \gg \theta$, there is $N \in \mathbb{N}$ such that for all $n,m > N$, $G(x_n,x_m,x_m) \ll c$. From (GBC5) and (GBC4), we get

$$G(x_n,x_m,x_l) \preccurlyeq s[G(x_n,x_m,x_m) + G(x_m,x_m,x_l)].$$

For this arbitrary for every $c \in \mathbb{E}$ with $c \gg \theta$, there is $N \in \mathbb{N}$ such that for all $n,m,l > N$, $G(x_n,x_m,x_m) \ll \frac{c}{2s}$ and $G(x_m,x_m,x_l) \ll \frac{c}{2s}$. Hence using (GBC4), we have

$$G(x_n,x_m,x_l) \ll s\left(\frac{c}{2s} + \frac{c}{2s}\right) = c.$$

This means (x_n) is G_b-cone Cauchy in X.

Proposition 2.17 Let (X,G) is a G_b-cone metric space, P be a normal cone with normal constant K. Let (x_n) and (y_n) be two sequences in X and suppose that $x_n \to x, y_n \to y$ as $n \to +\infty$.Then $G(x_n,x_n,y_n) \to s^2G(x,x,y)$ as $n \to +\infty$.

Proof For every real $\varepsilon > 0$, choose $c \in \mathbb{E}$ with $c \gg \theta$ and $K\|c\| < \varepsilon$. Then there is $N \in \mathbb{N}$ such that for all $n > N$,

$$G(x_n,x_n,x) \ll \frac{c}{2s} \text{ and } G(y_n,y,y) \ll \frac{c}{2s^2}.$$

From (GBC5) and (GBC4), we have

$$\begin{aligned} G(x_n,x_n,y_n) &\preccurlyeq s[G(y_n,x,x) + G(x_n,x_n,x)]. \\ &\preccurlyeq s^2[G(x,x,y) + G(y_n,y,y)] \\ &\quad + sG(x_n,x_n,x) \end{aligned}$$

Thus

$$\begin{aligned} &G(x_n,x_n,y_n) - s^2G(x,x,y) \\ &\preccurlyeq s^2G(y_n,y,y) + sG(x_n,x_n,x) \\ &\ll \frac{c}{2} + \frac{c}{2} = c \end{aligned}$$

Since P be a normal cone with normal constant K, therefore

$$\begin{aligned} &\|G(x_n,x_n,y_n) - s^2G(x,x,y)\| \\ &\preccurlyeq K\|c\| < \varepsilon. \end{aligned}$$

This means $G(x_n,x_n,y_n) \to s^2G(x,x,y)$ as $n \to +\infty$.

Proposition 2.18 Let (X,G) be a G_b-cone metric space, P be a normal cone with normal constant K. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables

Lemma 2.19 [26] For the case of non normal cones, we have the following properties**.**

(PT1). If $u \preccurlyeq v$ and $v \ll w$, then $u \ll w$.
(PT2). If $u \ll v$ and $v \preccurlyeq w$, then $u \ll w$.
(PT3). If $u \ll v$ and $v \ll w$, then $u \ll w$.
(PT4). If $\theta \preccurlyeq u \ll c$ for each $c \in intP$, then $u = \theta$.
(PT5). If $a \preccurlyeq b + c$ for each $c \in intP$, then $a \preccurlyeq b$.
(PT6). If $\mathbb{E}$ be a real Banach space with a cone P, and if $a \preccurlyeq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
(PT7). If $c \in intP$, $a_n \in \mathbb{E}$ and $a_n \to \theta$, then there exists an n_0 such that, for all $n > n_0$, we have $a_n \ll c$.

Definition 2.20 Let (X,G) be a G_b-cone metric space with the coefficient $s \geq 1$. A mapping $T : X \to X$ is called Lipschitzian if there exists $k \in \mathbb{R}$ such that

$$G(Tx,Ty,Tz) \preceq k\, G(x,y,z)$$

for all $x,y,z \in X$. The smallest constant k which satisfies the above inequality is called theLipschitz constant of T, denoted $Lip(T)$. In particular T is a contraction if $Lip(T) \in \left[0,\frac{1}{s}\right)$.

3. MAIN RESULT

In this section, we will present some fixed point and common fixed point theorems for contractive mappings in the setting of G_b-cone metric spaces. Furthermore, we will give examples to support our mainresults. Throughout this section, we not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is, $intP \neq \emptyset$.

We begin with a simple but a useful lemma.

Lemma 3.1 Let (x_n) be a sequence in a G_b-cone metric space (X,G) with the coefficient $s \geq 1$ relative to a solid cone P suchthat

(3.1) $$G(x_n,x_{n+1},x_{n+1}) \preceq \lambda G(x_{n-1},x_n,x_n)$$

where $\lambda \in \left[0,\frac{1}{s}\right)$and $n = 1,2,\ldots$. Then (x_n) is a Cauchy sequence in (X,G).

Proof Let $m > n \geq 1$. It follows that

(3.2) $$\begin{aligned} &G(x_n,x_m,x_m) \\ &\preceq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_m,x_m)] \\ &\preceq sG(x_n,x_{n+1},x_{n+1}) + s^2[G(x_{n+1},x_{n+2},x_{n+2}) \\ &+G(x_{n+2},x_m,x_m)] \\ &\preceq sG(x_n,x_{n+1},x_{n+1}) + s^2G(x_{n+1},x_{n+2},x_{n+2}) \\ &+s^3G(x_{n+2},x_{n+3},x_{n+3}) + \cdots \ldots \ldots \ldots \ldots \\ &+s^{m-n}G(x_{m-1},x_m,x_m) \end{aligned}$$

Now, (3.1) and $s\lambda < 1$ imply that

(3.3) $$\begin{aligned} G(x_n,x_m,x_m) &\preceq sG(x_n,x_{n+1},x_{n+1}) \\ &+s^2G(x_{n+1},x_{n+2},x_{n+2}) \\ &+s^3G(x_{n+2},x_{n+3},x_{n+3}) + \cdots \end{aligned}$$

$$+s^{m-n}G(x_{m-1},x_m,x_m)$$
$$\leq s\lambda^n G(x_0,x_1,x_1)$$
$$+s^2\lambda^{n+1}G(x_0,x_1,x_1)$$
$$+s^3\lambda^{n+2}G(x_0,x_1,x_1)+\cdots$$
$$+s^{m-n}\lambda^{n-1}G(x_0,x_1,x_1)$$
$$= s\lambda^n(1+s\lambda+(s\lambda)^2$$
$$+\cdots+(s\lambda)^{m-n-1})G(x_0,x_1,x_1)$$
$$\leq \frac{s\lambda^n}{1-s\lambda}G(x_0,x_1,x_1)\to\theta as\ n\to\infty.$$

According to Lemma 2.19 (PT7), and for any $c\in\mathbb{E}$ with $c\gg\theta$, there exists $N_0\in\mathbb{N}$ such that for any $n>N_0$, $\frac{s\lambda^n}{1-s\lambda}G(x_0,x_1,x_1)\ll c$. Furthermore, from (3.3) and for any $m>n>N_0$, Lemma 2.19 (PT1) shows that

(3.4) $$G(x_n,x_m,x_m)\ll c.$$

Hence, by Proposition 2.16, (x_n) is a Cauchy sequence in X.

4. FIXED POINT THEOREM

Now, our first main results as follows.

Theorem 3.2 Let (X,G) be a complete G_b-cone metric space with the coefficient $s\geq 1$ relative to a solid cone P. Supposethe mapping $T:X\to X$ satisfies the contractive condition

(3.5) $$G(Tx,Ty,Tz)\leq\lambda G(x,y,z),\forall\, x,y,z\in X,$$

where $\lambda\in\left[0,\frac{1}{s}\right)$is a constant.Then T has a unique fixed point in X. Furthermore, theiterative sequence $(T^n x)$ converges to the fixed point.

Proof Choose $x_0\in X$. We construct the iterative sequence (x_n), where $x_n=Tx_{n-1}, n\geq 1$, i.e. $x_{n+1}=Tx_n=T^{n+1}x_0$. From (3.5), we have

(3.6) $$G(x_n,x_{n+1},x_{n+1})=G(Tx_{n-1},Tx_n,Tx_n)$$
$$\leq\lambda G(x_{n-1},x_n,x_n)$$

So, by Lemma 3.1,(x_n) is Cauchy sequences in (G,X). Since (X,G) is a complete G_b-cone metric space, for any $c\in\mathbb{E}$ with $c\gg\theta$, there exists $x^*\in X$ such that $G(x_n,x_n,x^*)\ll\frac{c}{2s\lambda}$ and $G(x_{n+1},x^*,x^*)\ll\frac{c}{2s}$ for all $n>n_0$. Hence

(3.7) $$G(x^*,x^*,Tx^*)$$
$$\leq s[G(x^*,x^*,Tx_n)+G(Tx_n,Tx_n,Tx^*)]$$
$$\leq s[G(x^*,x^*,x_{n+1})+\lambda G(x_n,x_n,x^*)]$$
$$\ll s\left[\left(\frac{c}{2s}\right)+\lambda\left(\frac{c}{2s\lambda}\right)\right]=c,$$

for each $n>n_0$. Then by Lemma 2.19 (PT4), we deduce that $G(x^*,x^*,Tx^*)=\theta$, i.e. $Tx^*=x^*$ and so x^* is fixed point of T. Now we show that the fixed point is unique. If there is another fixed point y^*, by the given condition (3.5), we have

(3.8) $$G(x^*,\ y^*,\ y^*)=G(Tx^*,\ Ty^*,T\ y^*)$$
$$\leq\lambda G(x^*,\ y^*,\ y^*)$$

By Lemma 2.19 (PT6), we have $x^*=y^*$. The proof is completed.

Next example illustrates Theorem 3.2.

Example 3.3 Let $X=[0,+\infty)$ and $\mathbb{E}=C_R^1[0,1]$ with the norm $\|x\|=\|x\|_\infty+\|x'\|_\infty$ and consider

$$P=\{x\in\mathbb{E}: x(t)\geq 0 \text{ on } [0,1]\}.$$

Define $G:X\times X\times X\to\mathbb{E}$ by

$$G(x,y,z)=\max\{|x-y|^p,|y-z|^p,|z-x|^p\}\,\varphi,$$

$\forall\, x,y,z\in X$, with $p\geq 1$, where $\varphi:[0,1]\to\mathbb{R}$ such that $\varphi(t)=e^t$ .Then (X,G) is a complete G_b-cone metric space with the coefficient $s=2^{p-1}$. Let us define $T:X\to X$ as $Tx=\frac{1}{2}x-\frac{1}{4}x^2$for all $x\in X$. Therefore

$$G(Tx,Ty,Tz)$$
$$=\max\{|Tx-Ty|^p,|Ty-Tz|^p,|Tz-Tx|^p\}\,e^t,$$
$$=\max\left\{\left|\left(\frac{1}{2}x-\frac{1}{4}x^2\right)-\left(\frac{1}{2}y-\frac{1}{4}y^2\right)\right|^p,\right.$$
$$\left|\left(\frac{1}{2}y-\frac{1}{4}y^2\right)-\left(\frac{1}{2}z-\frac{1}{4}z^2\right)\right|^p,$$
$$\left.\left|\left(\frac{1}{2}z-\frac{1}{4}z^2\right)-\left(\frac{1}{2}x-\frac{1}{4}x^2\right)\right|^p\right\}e^t$$
$$=\max\left\{\left|\frac{1}{2}(x-y)-\frac{1}{4}(x-y)(x+y)\right|^p,\right.$$
$$\left|\frac{1}{2}(y-z)-\frac{1}{4}(y-z)(y+z)\right|^p,$$
$$\left.\left|\frac{1}{2}(z-x)-\frac{1}{4}(z-x)(z+x)\right|^p\right\}e^t$$
$$=\max\left\{|x-y|^p.\left|\frac{1}{2}-\frac{1}{4}(x+y)\right|^p,\right.$$
$$|y-z|^p.\left|\frac{1}{2}-\frac{1}{4}(y+z)\right|^p,$$
$$\left.|z-x|^p.\left|\frac{1}{2}-\frac{1}{4}(z+x)\right|^p\right\}e^t$$
$$\leq\frac{1}{2^p}\max\{|x-y|^p,|y-z|^p,|z-x|^p\}e^t$$
$$\leq\frac{1}{2^p}G(x,y,z).$$

Here $0\in X$ is the unique fixed point of T.

Theorem 3.4 Let (X,G) be a complete G_b-cone metric space with the coefficient $s\geq 1$ relative to a solid cone P and let $a_i\geq 0$, $(i=1,2,3,4)$ be constants with $s(a_1+a_2)+a_3+a_4<1$. Suppose the mapping $T:X\to X$ satisfies the contractive condition

(3.9) $$G(Tx,Ty,Tz)\leq a_1G(x,y,z)+a_2G(x,Tx,Tx)$$
$$+a_3G(y,Ty,Ty)+a_4G(z,Tz,Tz)$$

$\forall\, x,y,z\in X$. Then T has a unique fixed point in X. Furthermore, theiterative sequence $(T^n x)$ converges to the fixed point.

Proof Choose $x_0\in X$. We construct the iterative sequence (x_n), where $x_n=Tx_{n-1}, n\geq 1$, i.e. $x_{n+1}=Tx_n=T^{n+1}x_0$. From (3.9), we have

$$(3.10)\quad \begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\le a_1 G(x_{n-1}, x_n, x_n) \\ &\quad + a_2 G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \\ &\quad + a_3 G(x_n, Tx_n, Tx_n) \\ &\quad + a_4 G(x_n, Tx_n, Tx_n) \\ &\le a_1 G(x_{n-1}, x_n, x_n) \\ &\quad + a_2 G(x_{n-1}, x_n, x_n) \\ &\quad + a_3 G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + a_4 G(x_n, x_{n+1}, x_{n+1}) \end{aligned}$$

Thus, we have

$$(3.11)\quad \begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\le \left(\frac{a_1+a_2}{1-a_3-a_4}\right) G(x_{n-1}, x_n, x_n) \\ &= \lambda G(x_{n-1}, x_n, x_n) \end{aligned}$$

where $\lambda = \frac{a_1+a_2}{1-a_3-a_4} < \frac{1}{s}$. So, by Lemma 3.1, (x_n) is G_b-Cauchy sequences in (G, X). Since (X, G) is a complete G_b-cone metric space, for any $c \in \mathbb{E}$ with $c \gg \theta$, there exists $x^* \in X$ such that

$$(3.12)\quad \begin{aligned} G(x_n, x_n, x^*) &\ll \frac{(1-sa_4)c}{3sa_1}, \\ G(x_{n+1}, x^*, x^*) &\ll \frac{(1-sa_4)c}{3s}, \\ G(x_n, x_{n+1}, x_{n+1}) &\ll \frac{(1-sa_4)c}{3s(a_2+a_3)} \end{aligned}$$

for all $n > n_0$. Hence

$$(3.13)\quad \begin{aligned} &G(x^*, x^*, Tx^*) \\ &\le s[G(x^*, x^*, Tx_n) + G(Tx_n, Tx_n, Tx^*)] \\ &\le sG(x^*, x^*, x_{n+1}) \\ &\quad + sa_1 G(x_n, x_n, x^*) \\ &\quad + sa_2 G(x_n, Tx_n, Tx_n) \\ &\quad + sa_3 G(x_n, Tx_n, Tx_n) \\ &\quad + sa_4 G(x^*, Tx^*, Tx^*) \\ &\le sG(x^*, x^*, x_{n+1}) \\ &\quad + sa_1 G(x_n, x_n, x^*) \\ &\quad + sa_2 G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + sa_3 G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + sa_4 G(x^*, x^*, Tx^*) \end{aligned}$$

Thus, we obtain

$$(3.14)\quad \begin{aligned} G(x^*, x^*, Tx^*) &\le \frac{s}{(1-sa_4)} G(x^*, x^*, x_{n+1}) \\ &\quad + \frac{sa_1}{(1-sa_4)} G(x_n, x_n, x^*) \\ &\quad + \frac{s(a_2+a_3)}{(1-sa_4)} G(x_n, x_{n+1}, x_{n+1}) \\ &\ll \frac{s}{(1-sa_4)} \frac{(1-sa_4)c}{3s} \\ &\quad + \frac{sa_1}{(1-sa_4)} \frac{(1-sa_4)c}{3sa_1} \\ &\quad + \frac{s(a_2+a_3)}{(1-sa_4)} \frac{(1-sa_4)c}{3s(a_2+a_3)} \\ &= c \end{aligned}$$

for each $n > n_0$. Then by Lemma 2.19 (PT4), we deduce that $G(x^*, x^*, Tx^*) = \theta$, i.e. $Tx^* = x^*$ and so x^* is fixed point of T. Now we show that the fixed point is unique. If there is another fixed point y^*, by the given condition (3.9), we have

$$(3.15)\quad \begin{aligned} G(x^*, y^*, y^*) &= G(Tx^*, Ty^*, Ty^*) \\ &\le a_1 G(x^*, y^*, y^*) \\ &\quad + a_2 G(x^*, x^*, Tx^*) \\ &\quad + a_3 G(y^*, y^*, Ty^*) \\ &\quad + a_4 G(y^*, y^*, Ty^*) \\ &\le a_1 G(x^*, y^*, y^*) \\ &\quad + a_2 G(x^*, x^*, x^*) \\ &\quad + a_3 G(y^*, y^*, y^*) \\ &\quad + a_4 G(y^*, y^*, y^*) \\ &= a_1 G(x^*, y^*, y^*) \end{aligned}$$

By Lemma 2.19 (PT6), we have $x^* = y^*$. The proof is completed. ■

Theorem 3.5 Let (X, G) be a complete G_b-cone metric space with the coefficient $s \ge 1$ relative to a solid cone P and let $a_i \ge 0$, $(i = 1,2,3,4)$ be constants with $s(a_1 + a_2 + a_3) + a_4 < 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$(3.16)\quad \begin{aligned} G(Tx, Ty, Tz) &\le a_1 G(x, y, z) + a_2 G(x, x, Tx) \\ &\quad + a_3 G(y, y, Ty) + a_4 G(z, z, Tz) \end{aligned}$$

$\forall\, x, y, z \in X$. Then T has a unique fixed point in X. Furthermore, the iterative sequence $(T^n x)$ converges to the fixed point.

Proof Choose $x_0 \in X$. We construct the iterative sequence (x_n), where $x_n = Tx_{n-1}, n \ge 1$, i.e. $x_{n+1} = Tx_n = T^{n+1}x_0$. From (3.16), we have

$$(3.17)\quad \begin{aligned} G(x_n, x_n, x_{n+1}) &= G(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\le a_1 G(x_{n-1}, x_{n-1}, x_n) \\ &\quad + a_2 G(x_{n-1}, x_{n-1}, Tx_{n-1}) \\ &\quad + a_3 G(x_{n-1}, x_{n-1}, Tx_{n-1}) \\ &\quad + a_4 G(x_n, x_n, Tx_n) \\ &\le a_1 G(x_{n-1}, x_{n-1}, x_n) \\ &\quad + a_2 G(x_{n-1}, x_{n-1}, x_n) \\ &\quad + a_3 G(x_{n-1}, x_{n-1}, x_n) \\ &\quad + a_4 G(x_n, x_n, x_{n+1}) \end{aligned}$$

Thus, we have

(3.18) $G(x_n, x_n, x_{n+1}) \preceq \left(\frac{a_1+a_2+a_3}{1-a_4}\right) G(x_{n-1}, x_{n-1}, x_n)$

$$= \lambda G(x_{n-1}, x_{n-1}, x_n)$$

where $\lambda = \frac{a_1+a_2+a_3}{1-a_4} < \frac{1}{s}$. So, by Lemma 3.1, (x_n) is Cauchy sequences in (G, X). Since (X, G) is a complete G_b-cone metric space, for any $c \in \mathbb{E}$ with $c \gg \theta$, there exists $x^* \in X$ such that

(3.19) $G(x_n, x_n, x^*) \ll \frac{(1-sa_4)c}{3sa_1}$,

$$G(x_{n+1}, x^*, x^*) \ll \frac{(1-sa_4)c}{3s}$$

$$G(x_n, x_n, x_{n+1}) \ll \frac{(1-sa_4)c}{3s(a_2+a_3)}$$

for all $n > n_0$. Hence

(3.20) $G(x^*, x^*, Tx^*)$

$$\begin{aligned}
&\preceq s[G(x^*, x^*, Tx_n) + G(Tx_n, Tx_n, Tx^*)]\\
&\preceq sG(x^*, x^*, x_{n+1})\\
&+sa_1 G(x_n, x_n, x^*)\\
&+sa_2 G(x_n, x_n, Tx_n)\\
&+sa_3 G(x_n, x_n, Tx_n)\\
&+sa_4 G(x^*, x^*, Tx^*)\\
&\preceq sG(x^*, x^*, x_{n+1})\\
&+sa_1 G(x_n, x_n, x^*)\\
&+sa_2 G(x_n, x_n, x_{n+1})\\
&+sa_4 G(x_n, x_n, x_{n+1})\\
&+sa_4 G(x^*, x^*, Tx^*)
\end{aligned}$$

Thus, we obtain

(3.21) $G(x^*, x^*, Tx^*) \preceq \frac{s}{(1-sa_4)} G(x^*, x^*, x_{n+1})$

$$\begin{aligned}
&+\frac{sa_1}{(1-sa_4)} G(x_n, x_n, x^*)\\
&+\frac{s(a_2+a_3)}{(1-sa_4)} G(x_n, x_n, x_{n+1})\\
&\ll \frac{s}{(1-sa_4)} \frac{(1-sa_4)c}{3s}\\
&+\frac{sa_1}{(1-sa_4)} \frac{(1-sa_4)c}{3sa_1}\\
&+\frac{s(a_2+a_3)}{(1-sa_4)} \frac{(1-sa_4)c}{3s(a_2+a_3)}\\
&= c
\end{aligned}$$

for each $n > n_0$. Then by Lemma 2.19 (PT4), we deduce that $G(x^*, x^*, Tx^*) = \theta$, i.e. $Tx^* = x^*$ and so x^* is fixed point of T. Now we show that the fixed point is unique. If there is another fixed point y^*, by the given condition (3.16), we obtain

(3.22) $G(x^*, y^*, y^*) = G(Tx^*, Ty^*, Ty^*)$

$$\begin{aligned}
&\preceq a_1 G(x^*, y^*, y^*)\\
&+a_2 G(x^*, x^*, Tx^*)\\
&+a_3 G(y^*, y^*, Ty^*)\\
&+a_4 G(y^*, y^*, Ty^*)\\
&\preceq a_1 G(x^*, y^*, y^*)\\
&+a_2 G(x^*, x^*, x^*)\\
&+a_3 G(y^*, y^*, y^*)\\
&+a_4 G(y^*, y^*, y^*)\\
&= a_1 G(x^*, y^*, y^*)
\end{aligned}$$

By Lemma 2.19 (PT6), we have $x^* = y^*$. The proof is completed.

Next examples illustrate Theorem 3.4 and Theorem 3.5.

Example 3.6 Let $X = [0,1]$, $\mathbb{E} = \mathbb{R}^2$ and $P = \{(x, y) \in \mathbb{E} : x \geq 0, y \geq 0\}$. Define $G: X \times X \times X \to \mathbb{E}$ by

$$\begin{aligned}
G(x,y,z) = \max\{&(|x-y|^2, |x-y|^2),\\
&(|y-z|^2, |y-z|^2),\\
&(|z-x|^2, |z-x|^2)\}
\end{aligned}$$

Then, it is easy to see that (X, G) is a G_b-cone metric space with the coefficient $s = 2$. But it is not a G-cone metric space since the rectangle inequality is not satisfyied. Let us define $T : X \to X$as $Tx = \frac{1}{6}x^2$ for all $x \in X$.Therefore

$$\begin{aligned}
G(Tx, Ty, Tz) &= \max\{(|Tx-Ty|^2, |Tx-Ty|^2),\\
&\quad ,(|Ty-Tz|^2, |Ty-Tz|^2)\\
&\quad ,(|Tz-Tx|^2, |Tz-Tx|^2)\}\\
&= \max\left\{\left(\left|\frac{1}{6}x^2 - \frac{1}{6}y^2\right|^2, \left|\frac{1}{6}x^2 - \frac{1}{6}y^2\right|^2\right),\right.\\
&\quad ,\left(\left|\frac{1}{6}y^2 - \frac{1}{6}z^2\right|^2, \left|\frac{1}{6}y^2 - \frac{1}{6}z^2\right|^2\right)\\
&\quad \left.,\left(\left|\frac{1}{6}z^2 - \frac{1}{6}x^2\right|^2, \left|\frac{1}{6}z^2 - \frac{1}{6}x^2\right|^2\right)\right\}\\
&= \max\left\{\frac{1}{36}|x+y|^2(|x-y|^2, |x-y|^2),\right.\\
&\quad ,\frac{1}{36}|y+z|^2(|y-z|^2, |y-z|^2)\\
&\quad ,\frac{1}{36}|z+x|^2(|z-x|^2, |z-x|^2)\}\\
&\preceq \frac{1}{9}\max\{(|x-y|^2, |x-y|^2)\\
&\quad ,(|y-z|^2, |y-z|^2)\\
&\quad ,(|z-x|^2, |z-x|^2)\}\\
&\preceq \frac{1}{9} G(x,y,z)
\end{aligned}$$

where $a_1 = \frac{1}{9}, a_2 = a_3 = a_4 = 0$ with $s(a_1 + a_2 + a_3) + a_4 = 0.222 < 1$ and $s(a_1 + a_2) + a_3 + a_4 = 0.222 < 1$. It is clear thatthe conditions of Theorem 3.4 and Theorem 3.5 are satisfied.Therefore, T has a fixed point $x = 0$.

5. COMMON FIXED POINT THEOREM

Now, we give a common fixed theorem for two weakly compatible self-mappings satisfying the contractive condition in G_b-cone metric spaces without the assumption of normality.

We need the following definition:

Definition 3.7 (see [29-30]) Let S and T be two self-mappings on a nonempty set X. Then S and T are said to be weakly compatible if they commute at all of their coincidence points; that is, $Sx = Tx$ for some $x \in X$ and then $STx = TSx$.

Lemma 3.8 (see [30]) Let S and T be weakly compatible self-mappings of a nonempty set X. If S and T have a unique point of coincidence $w = Sx = Tx$, then w is the unique common fixed point of S and T.

Now, our common fixed point theorem as follows.

Theorem 3.9 Let (X, G) be a cone $\mathrm{G_b}$-metric space with the coefficient $\mathrm{s} \geq 1$ relative to a solid cone P. Let $\mathrm{S}, \mathrm{T}: \mathrm{X} \to \mathrm{X}$ be two mappings and assume that there exist non-negative constants $\mathrm{a_i} \in [0,1)$, $\mathrm{i} = 1,2,3,4,5,6,7$ with

$$3sa_1 + (s+2)\sum_{i=2}^{4} a_i + (s^2+s+1)\sum_{i=5}^{7} a_i < 3$$

such that the following contractive condition holds for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$:

(3.23) $$\begin{aligned} \mathrm{G}(\mathrm{Sx}, \mathrm{Sy}, \mathrm{Sz}) \preceq\ & \mathrm{a_1 G}(\mathrm{Tx}, \mathrm{Ty}, \mathrm{Tz}) + \mathrm{a_2 G}(\mathrm{Tx}, \mathrm{Sx}, \mathrm{Sx}) \\ & + \mathrm{a_3 G}(\mathrm{Ty}, \mathrm{Sy}, \mathrm{Sy}) + \mathrm{a_4 G}(\mathrm{Tz}, \mathrm{Sz}, \mathrm{Sz}) \\ & + \mathrm{a_5 G}(\mathrm{Tx}, \mathrm{Sy}, \mathrm{Sy}) + \mathrm{a_6 G}(\mathrm{Ty}, \mathrm{Sz}, \mathrm{Sz}) \\ & + \mathrm{a_7 G}(\mathrm{Tz}, \mathrm{Sx}, \mathrm{Sx}) \end{aligned}$$

If the range of S contains the range of T and $\mathrm{T}(\mathrm{X})$ or $\mathrm{S}(\mathrm{X})$ is a complete subspace of X, then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible. Then S and T have a unique common fixed point in X.

Proof For an arbitrary $\mathrm{x_0} \in \mathrm{X}$, since because $\mathrm{S}(\mathrm{X}) \subset \mathrm{T}(\mathrm{X})$, there exists an $x_1 \in X$ such that $S\mathrm{x_0} = \mathrm{Tx_1}$. By induction, a sequence $(\mathrm{x_n})$ can be chosen such that $S\mathrm{x_n} = \mathrm{Tx_{n+1}}$ $(\mathrm{n} \geq 1)$. If $T\mathrm{x_{n_0-1}} = T\mathrm{x_{n_0}} = S\mathrm{x_{n_0-1}}$ for some natural number $\mathrm{n_0}$, then $\mathrm{x_{n_0-1}}$ is a coincidence point of S and T in X. Suppose that $T\mathrm{x_{n-1}} = T\mathrm{x_n}$ for all $\mathrm{n} \geq 1$. Thus, by (3.23) for any $n \in \mathbb{N}$, we have

(3.24) $$\begin{aligned} \mathrm{G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) &= \mathrm{G}(\mathrm{Sx_{n-1}}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &\preceq \mathrm{a_1 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &+ \mathrm{a_2 G}(\mathrm{Tx_{n-1}}, \mathrm{Sx_{n-1}}, \mathrm{Sx_{n-1}}) \\ &+ \mathrm{a_3 G}(\mathrm{Tx_n}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &+ \mathrm{a_4 G}(\mathrm{Tx_n}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &+ \mathrm{a_5 G}(\mathrm{Tx_{n-1}}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &+ \mathrm{a_6 G}(\mathrm{Tx_n}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &+ \mathrm{a_7 G}(\mathrm{Tx_n}, \mathrm{Sx_{n-1}}, \mathrm{Sx_{n-1}}) \\ &= \mathrm{a_1 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &+ \mathrm{a_2 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &+ \mathrm{a_3 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_4 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_5 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_6 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_7 G}(\mathrm{Tx_n}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &\preceq \mathrm{a_1 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &+ \mathrm{a_2 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &+ \mathrm{a_3 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_4 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_5 s}[\mathrm{G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &+ \mathrm{G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}})] \\ &+ \mathrm{a_6 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \end{aligned}$$

Set

(3.25) $$G_n = \mathrm{G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}).$$

Thus, from (3.24) we have

(3.26) $$\begin{aligned} G_n \preceq\ & (\mathrm{a_1} + \mathrm{a_2} + \mathrm{sa_5})G_{n-1} \\ & + (\mathrm{a_3} + \mathrm{a_4} + \mathrm{sa_5} + \mathrm{a_6})G_n \end{aligned}$$

Similarly,

(3.27) $$\begin{aligned} \mathrm{G}(\mathrm{Tx_{n+1}}, \mathrm{Tx_n}, \mathrm{Tx_{n+1}}) &= \mathrm{G}(\mathrm{Sx_n}, \mathrm{Sx_{n-1}}, \mathrm{Sx_n}) \\ &\preceq \mathrm{a_1 G}(\mathrm{Tx_n}, \mathrm{Tx_{n-1}}, \mathrm{Tx_n}) \\ &+ \mathrm{a_2 G}(\mathrm{Tx_n}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &+ \mathrm{a_3 G}(\mathrm{Tx_{n-1}}, \mathrm{Sx_{n-1}}, \mathrm{Sx_{n-1}}) \\ &+ \mathrm{a_4 G}(\mathrm{Tx_n}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &+ \mathrm{a_5 G}(\mathrm{Tx_n}, \mathrm{Sx_{n-1}}, \mathrm{Sx_{n-1}}) \\ &+ \mathrm{a_6 G}(\mathrm{Tx_{n-1}}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &+ \mathrm{a_7 G}(\mathrm{Tx_n}, \mathrm{Sx_n}, \mathrm{Sx_n}) \\ &= \mathrm{a_1 G}(\mathrm{Tx_n}, \mathrm{Tx_{n-1}}, \mathrm{Tx_n}) \\ &+ \mathrm{a_2 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_3 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &+ \mathrm{a_4 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}} \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_5 G}(\mathrm{Tx_n}, \mathrm{Tx_n}, \mathrm{Tx_n}) \\ &+ \mathrm{a_6 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_7 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &\preceq \mathrm{a_1 G}(\mathrm{Tx_n}, \mathrm{Tx_{n-1}}, \mathrm{Tx_n}) \\ &+ \mathrm{a_2 G}(\mathrm{Tx_n}, \mathrm{Tx_{n+1}}, \mathrm{Tx_{n+1}}) \\ &+ \mathrm{a_3 G}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}, \mathrm{Tx_n}) \end{aligned}$$

$+a_4G(Tx_n, Tx_{n+1}Tx_{n+1})$

$+a_6s[G(Tx_{n-1}, Tx_n, Tx_n)$

$+G(Tx_n, Tx_{n+1}, Tx_{n+1})]$

$+a_7G(Tx_n, Tx_{n+1}, Tx_{n+1})$

Thus,

(3.28) $G_n \preceq (a_1 + a_3 + sa_6)G_{n-1}$

$+(a_2 + a_4 + sa_6 + a_7)G_n$

and

(3.29) $G(Tx_{n+1}, Tx_{n+1}, Tx_n)$

$= G(Sx_n, Sx_n, Sx_{n-1})$

$\preceq a_1G(Tx_n, Tx_n, Tx_{n-1})$

$+a_2G(Tx_n, Sx_n, Sx_n)$

$+a_3G(Tx_n, Sx_n, Sx_n)$

$+a_4G(Tx_{n-1}, Sx_{n-1}, Sx_{n-1})$

$+a_5G(Tx_n, Sx_n, Sx_n)$

$+a_6G(Tx_n, Sx_{n-1}, Sx_{n-1})$

$+a_7G(Tx_{n-1}, Sx_n, Sx_n)$

$= a_1G(Tx_n, Tx_n, Tx_{n-1})$

$+a_2G(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+a_3G(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+a_4G(Tx_{n-1}, Tx_n, Tx_n)$

$+a_5G(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+a_6G(Tx_n, Tx_n, Tx_n)$

$+a_7G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1})$

$\preceq a_1G(Tx_n, Tx_n, Tx_{n-1})$

$+a_2G(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+a_3G(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+a_4G(Tx_{n-1}, Tx_n, Tx_n)$

$+a_5G(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+a_7s[G(Tx_{n-1}, Tx_n, Tx_n)$

$+G(Tx_n, Tx_{n+1}, Tx_{n+1})]$

Thus,

(3.30) $G_n \preceq (a_1 + a_4 + sa_7)G_{n-1}$

$+(a_2 + a_3 + a_5 + sa_7)G_n$

Adding (3.26), (3.28) and (3.30), we obtain

$3G_n \preceq [3a_1 + a_2 + a_3 + a_4 + s(a_5 + a_6 + a_7)]G_{n-1}$

$+[2(a_2 + a_3 + a_4) + (s+1)(a_5 + a_6 + a_7)]G_n$

Thus

(3.31) $G_n \preceq \left(\frac{3a_1+\sum_{i=2}^{4} a_i+s\sum_{i=5}^{7} a_i}{3-2\sum_{i=2}^{4} a_i-(s+1)\sum_{i=5}^{7} a_i}\right) G_{n-1}$

Since

$$3sa_1 + (s+2)\sum_{i=2}^{4} a_i + (s^2+s+1)\sum_{i=5}^{7} a_i < 3,$$

we have

(3.32) $G_n \preceq \lambda G_{n-1} \preceq \lambda^2 G_{n-2} \preceq \cdots \preceq \lambda^n G_0$

where $\lambda = \frac{3a_1+\sum_{i=2}^{4} a_i+s\sum_{i=5}^{7} a_i}{3-2\sum_{i=2}^{4} a_i-(s+1)\sum_{i=5}^{7} a_i}$. Obviously, $\lambda \in \left[0, \frac{1}{s}\right)$.

Thus, setting any positive integers m and n, we have

(3.33) $G(Tx_n, Tx_{n+m}, Tx_{n+m})$

$\preceq sG(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+sG(Tx_{n+1}, Tx_{n+m}, Tx_{n+m})$

$\preceq sG(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+s^2G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2})$

$+s^2G(Tx_{n+2}, Tx_{n+m}, Tx_{n+m})$

$\preceq sG(Tx_n, Tx_{n+1}, Tx_{n+1})$

$+s^2G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2})$

$+s^3G(Tx_{n+2}, Tx_{n+3}, Tx_{n+3}) + \cdots$

$+s^{m-1}G(Tx_{n+m-2}, Tx_{n+m-1}, Tx_{n+m-1})$

$+s^{m-1}G(Tx_{n+m-1}, Tx_{n+m}, Tx_{n+m})$

$\preceq (s\lambda^n + s^2\lambda^{n+1} + \cdots + s^m\lambda^{n+m-1})G_0$

$= \frac{s\lambda^n[1-(s\lambda)^m]}{1-s\lambda} G_0$

$\preceq \frac{s\lambda^n}{1-s\lambda} G_0$

Since $\lambda \in \left[0, \frac{1}{s}\right)$, we notice that $\frac{s\lambda^n}{1-s\lambda} G(Tx_0, Tx_1, Tx_1) = \frac{s\lambda^n}{1-s\lambda} G_0 \to \theta$ as $n \to +\infty$ for any $m \in \mathbb{N}$. By Lemma 2.19 (PT7), for $c \in intP$, we can choose $n_0 \in \mathbb{N}$ such that $\frac{s\lambda^n}{1-s\lambda} G_0 \ll c$ for all $n > n_0$. Thus, for each $c \in intP$, $G(Tx_n, Tx_{n+m}, Tx_{n+m}) \ll c$ for all $n > n_0$, $m \geq 1$. Therefore (Tx_n) is a G_b-cone Cauchy sequence in $T(X)$. If $T(X) \subset X$ is complete, there exists $q \in T(X)$ and $p \in X$ such that $Tx_n \to q$ as $n \to +\infty$ and $Tp = q$. (If $S(X) \subset X$ is complete, there exists $q \in S(X)$ such that $Sx_n \to q$ as $n \to +\infty$. Since $S(X) \subset T(X)$, we can find $p \in X$ such that $Tp = q$.)

Now, we shall show that $Sp = q$. From (3.23), we have

(3.33) $G(Tx_{n+2}, Sp, Sp)$

$= G(Sx_{n+1}, Sp, Sp)$

$\preceq a_1G(Tx_{n+1}, Tp, Tp)$

$+a_2G(Tx_{n+1}, Sx_{n+1}, Sx_{n+1})$

$+a_3G(Tp, Sp, Sp)$

$+a_4G(Tp, Sp, Sp)$

$+a_5G(Tx_{n+1}, Sp, Sp)$

$+a_6G(Tp, Sp, Sp)$

$+a_7G(Tp, Sx_{n+1}, Sx_{n+1})$

$$\begin{aligned}
&\preceq a_1G(Tx_{n+1},Tp,Tp)\\
&+a_2G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&+a_3G(Tp,Sp,Sp)\\
&+a_4G(Tp,Sp,Sp)\\
&+a_5G(Tx_{n+1},Sp,Sp)\\
&+a_6G(Tp,Sp,Sp)\\
&+a_7G(Tp,Tx_{n+2},Tx_{n+2})\\
&=(a_1+a_5)G(Tx_{n+1},q,q)\\
&+a_2G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&+(a_3+a_4+a_6)G(q,Sp,Sp)\\
&+a_7G(q,Tx_{n+2},Tx_{n+2})
\end{aligned}$$

Similarly,

(3.34) $$\begin{aligned}
&G(Sp,Tx_{n+2},Sp)\\
&=G(Sp,Sx_{n+1},Sp)\\
&\preceq a_1G(Tp,Tx_{n+1},Tp)\\
&+a_2G(Tp,Sp,Sp)\\
&+a_3G(Tx_{n+1},Sx_{n+1},Sx_{n+1})\\
&+a_4G(Tp,Sp,Sp)\\
&+a_5G(Tp,Sx_{n+1},Sx_{n+1})\\
&+a_6G(Tx_{n+1},Sp,Sp)\\
&+a_7G(Tp,Sp,Sp)\\
&=a_1G(Tp,Tx_{n+1},Tp)\\
&+a_2G(Tp,Sp,Sp)\\
&+a_3G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&+a_4G(Tp,Sp,Sp)\\
&+a_5G(Tp,Tx_{n+2},Tx_{n+2})\\
&+a_6G(Tx_{n+1},Sp,Sp)\\
&+a_7G(Tp,Sp,Sp)\\
&=(a_1+a_6)G(q,Tx_{n+1},q)\\
&+(a_2+a_4+a_7)G(q,Sp,Sp)\\
&+a_3G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&+a_5G(q,Tx_{n+2},Tx_{n+2})
\end{aligned}$$

and

(3.35) $$\begin{aligned}
&G(Sp,Sp,Tx_{n+2})\\
&=G(Sp,Sp,Sx_{n+1})\\
&\preceq a_1G(Tp,Tp,Tx_{n+1})\\
&+a_2G(Tp,Sp,Sp)\\
&+a_3G(Tp,Sp,Sp)\\
&+a_4G(Tx_{n+1},Sx_{n+1},Sx_{n+1})\\
&+a_5G(Tp,Sp,Sp)\\
&+a_6G(Tp,Sx_{n+1},Sx_{n+1})\\
&+a_7G(Tx_{n+1},Sp,Sp)\\
&=a_1G(Tp,Tp,Tx_{n+1})\\
&+a_2G(Tp,Sp,Sp)\\
&+a_3G(Tp,Sp,Sp)\\
&+a_4G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&+a_5G(Tp,Sp,Sp)\\
&+a_6G(Tp,Tx_{n+2},Tx_{n+2})\\
&+a_7G(Tx_{n+1},Sp,Sp)\\
&=(a_1+a_7)G(q,q,Tx_{n+1})\\
&+(a_2+a_3+a_5)G(q,Sp,Sp)\\
&+a_4G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&+a_6G(q,Tx_{n+2},Tx_{n+2})
\end{aligned}$$

Adding from (3.33) to (3.35), we obtain that

$$\begin{aligned}
3G(Tx_{n+2},Sp,Sp) &\preceq \left(3a_1+\sum_{i=5}^{7}a_i\right)G(Tx_{n+1},q,q)\\
&+\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)G(q,Sp,Sp)\\
&+\left(\sum_{i=2}^{4}a_i\right)G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&+\left(\sum_{i=5}^{7}a_i\right)G(q,Tx_{n+2},Tx_{n+2})\\
&\preceq \left(3a_1+\sum_{i=5}^{7}a_i\right)s[G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&\qquad +G(Tx_{n+2},q,q)]\\
&+\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)s[G(q,Tx_{n+2},Tx_{n+2})\\
&\qquad +G(Tx_{n+2},Sp,Sp)]\\
&+\left(\sum_{i=2}^{4}a_i\right)G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\
&+\left(\sum_{i=5}^{7}a_i\right)G(q,Tx_{n+2},Tx_{n+2})
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)G(Tx_{n+2},Sp,Sp)\\
&\preceq s\left(3a_1+\sum_{i=5}^{7}a_i\right)G(Tx_{n+2},q,q)
\end{aligned}$$

$$+\left(2s\sum_{i=2}^{4}a_i+(s+1)\sum_{i=5}^{7}a_i\right)G(q,Tx_{n+2},Tx_{n+2})$$
$$+\left(3sa_1+s\sum_{i=5}^{7}a_i+\sum_{i=2}^{4}a_i\right)G(Tx_{n+1},Tx_{n+2},Tx_{n+2})$$

Since

$$0\le 2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i<\frac{3}{s}$$

We have

(3.36) $G(Tx_{n+2},Sp,Sp)$

$$\preceq \frac{s\left(3a_1+\sum_{i=5}^{7}a_i\right)}{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)}G(Tx_{n+1},q,q)$$
$$+\frac{\left(2s\sum_{i=2}^{4}a_i+(s+1)\sum_{i=5}^{7}a_i\right)}{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)}G(q,Tx_{n+2},Tx_{n+2})$$
$$+\frac{\left(3sa_1+s\sum_{i=5}^{7}a_i+\sum_{i=2}^{4}a_i\right)}{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)}G(Tx_{n+1},Tx_{n+2},Tx_{n+2})$$

Since (Tx_n) is a G_b-cone Cauchy sequence in $T(X)$ and $Tx_n\to q$ as $n\to+\infty$, for any $c\in intP$, we can choose $n_1\in\mathbb{N}$ such that for all $n>n_1$,

$$G(Tx_{n+1},q,q)\ll\frac{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)c}{3s\left(3a_1+\sum_{i=5}^{7}a_i\right)},$$
$$G(q,Tx_{n+2},Tx_{n+2})\ll\frac{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)c}{3\left(2s\sum_{i=2}^{4}a_i+(s+1)\sum_{i=5}^{7}a_i\right)},$$
$$G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\ll\frac{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)c}{3\left(3sa_1+s\sum_{i=5}^{7}a_i+\sum_{i=2}^{4}a_i\right)}.$$

Thus, from (3.36), for any $c\in intP$, we have

$G(Tx_{n+2},Sp,Sp)$

$$\ll\frac{s\left(3a_1+\sum_{i=5}^{7}a_i\right)}{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)}\frac{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)c}{3s\left(3a_1+\sum_{i=5}^{7}a_i\right)}$$
$$+\frac{\left(2s\sum_{i=2}^{4}a_i+(s+1)\sum_{i=5}^{7}a_i\right)}{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)}\frac{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)c}{3\left(2s\sum_{i=2}^{4}a_i+(s+1)\sum_{i=5}^{7}a_i\right)}$$
$$+\frac{\left(3sa_1+s\sum_{i=5}^{7}a_i+\sum_{i=2}^{4}a_i\right)}{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)}\frac{\left(3-s\left(2\sum_{i=2}^{4}a_i+\sum_{i=5}^{7}a_i\right)\right)c}{3\left(3sa_1+s\sum_{i=5}^{7}a_i+\sum_{i=2}^{4}a_i\right)}$$
$$=c$$

for all $n>n_1$. Therefore, by Lemma 2.19 (PT7), we have $Tx_{n+2}\to Sp$ and then from Lemma 2.13, $Tp=Sp=q$. Assume that there exist u,v in X such that $Tu=Su=v$. From (2.23), we have

(3.37) $G(Tu,Tp,Tp)$
$$\begin{aligned}&=G(Su,Sp,Sp)\\&\preceq a_1G(Tu,Tp,Tp)+a_2G(Tu,Su,Su)\\&\quad+a_3G(Tp,Sp,Sp)+a_4G(Tp,Sp,Sp)\\&\quad+a_5G(Tu,Sp,Sp)+a_6G(Tp,Sp,Sp)\\&\quad+a_7G(Tp,Su,Su)\\&=a_1G(Tu,Tp,Tp)+a_5G(Tu,Tp,Tp)\\&\quad+a_7G(Tp,Tu,Tu)\\&\preceq(a_1+a_5+2sa_7)G(Tu,Tp,Tp)\end{aligned}$$

(3.38) $G(Tp,Tu,Tp)$
$$\begin{aligned}&=G(Sp,Su,Sp)\\&\preceq a_1G(Tp,Tu,Tp)+a_2G(Tp,Sp,Sp)\\&\quad+a_3G(Tu,Su,Su)+a_4G(Tp,Sp,Sp)\\&\quad+a_5G(Tp,Su,Su)+a_6G(Tu,Sp,Sp)\\&\quad+a_7G(Tp,Sp,Sp)\\&=a_1G(Tp,Tu,Tp)+a_5G(Tp,Tu,Tu)\\&\quad+a_6G(Tu,Tp,Tp)\\&\preceq(a_1+2sa_5+a_6)G(Tu,Tp,Tp)\end{aligned}$$

(3.39) $G(Tp,Tp,Tu)$
$$\begin{aligned}&=G(Sp,Sp,Su)\\&\preceq a_1G(Tp,Tp,Tu)+a_2G(Tp,Sp,Sp)\\&\quad+a_3G(Tp,Sp,Sp)+a_4G(Tu,Su,Su)\\&\quad+a_5G(Tp,Sp,Sp)+a_6G(Tp,Su,Su)\\&\quad+a_7G(Tu,Sp,Sp)\\&=a_1G(Tp,Tp,Su)+a_6G(Tp,Tu,Tu)\\&\quad+a_7G(Su,Sp,Sp)\\&\preceq(a_1+2sa_6+a_7)G(Tu,Tp,Tp)\end{aligned}$$

Adding from (3.37) to (3.39), we have

(3.40) $3G(Tp,Tp,Tu)$

$$\preceq\left[3a_1+(1+2s)\sum_{i=5}^{7}a_i\right]G(Tp,Tp,Tu)$$

Hence

(3.41) $G(Tp,Tp,Tu)$

$$\preceq\left[a_1+\left(\frac{1+2s}{3}\right)\sum_{i=5}^{7}a_i\right]G(Tp,Tp,Tu)$$

Since $1+2s\le s^2+s+1$ because $s\ge1$ and then

$$3a_1+(1+2s)\sum_{i=5}^{7}a_i<3$$

That is,

$$a_1+\left(\frac{1+2s}{3}\right)\sum_{i=5}^{7}a_i<1$$

Thus, by Lemma 2.19 (PT7), we can obtain that

$$G(Tp,Tp,Tu)=\theta,$$
$$i.e.\ v=Tp=Tu=q.$$

Moreover, the mappings S and T are weakly compatible, by Lemma 3.8, we know that q is the unique common fixed point of S and T.

Example 3.10 Let $X = [1, +\infty)$, $\mathbb{E} = C^1_{\mathbb{R}}[0,1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider

$$P = \{\varphi \in \mathbb{E} : \varphi \geq 0\} \subset \mathbb{E}.$$

Define $G: X \times X \times X \to \mathbb{E}$ by

$$G(x, y, z) = \max\{|x - y|^2, |y - z|^2, |z - x|^2\}e^t,$$

$\forall\, x, y, z \in X$. Then (X, G) is a complete G_b-cone metric space with the coefficient $s = 2$, but it is not a cone metric space. We consider the functions $S, T: X \to X$ defined by

$$Sx = \frac{1}{6}\ln x + 1 \text{ and } Tx = \ln x + 1.$$

Obviously, $S(X) \subset T(X)$ is a complete subspace of X.Here

$$G(Tx, Ty, Tz) = \max\{|Tx - Ty|^2, |Ty - Tz|^2, |Tz - Tx|^2\}e^t$$
$$= \max\{|\ln x - \ln y|^2, |\ln y - \ln z|^2, |\ln z - \ln x|^2\}e^t$$

$$G(Tx, Sx, Sx) = |Tx - Sx|^2 e^t = \left|\ln x - \frac{1}{6}\ln x\right|^2 e^t = \frac{25}{36}(\ln x)^2 e^t$$

$$G(Ty, Sy, Sy) = |Ty - Sy|^2 e^t = \left|\ln y - \frac{1}{6}\ln y\right|^2 e^t = \frac{25}{36}(\ln y)^2 e^t$$

$$G(Tz, Sz, Sz) = |Tz - Sz|^2 e^t = \left|\ln z - \frac{1}{6}\ln z\right|^2 e^t = \frac{25}{36}(\ln z)^2 e^t$$

$$G(Tx, Sy, Sy) = |Tx - Sy|^2 e^t = \left|\ln x - \frac{1}{6}\ln y\right|^2 e^t$$

$$G(Ty, Sz, Sz) = |Ty - Sz|^2 e^t = \left|\ln y - \frac{1}{6}\ln z\right|^2 e^t$$

$$G(Tz, Sx, Sx) = |Tz - Sx|^2 e^t = \left|\ln z - \frac{1}{6}\ln x\right|^2 e^t$$

Now,

$$G(Sx, Sy, Sz) = \max\{|Sx - Sy|^2, |Sy - Sz|^2, |Sz - Sx|^2\}e^t$$
$$= \max\left\{\left|\frac{1}{6}\ln x - \frac{1}{6}\ln y\right|^2, \left|\frac{1}{6}\ln y - \frac{1}{6}\ln z\right|^2, \left|\frac{1}{6}\ln z - \frac{1}{6}\ln x\right|^2\right\}e^t$$
$$\leq \frac{1}{36}\Big[\max\{|\ln x - \ln y|^2, |\ln y - \ln z|^2, +|\ln z - \ln x|^2\} + (\ln x)^2 + (\ln x)^2 + (\ln x)^2 + \left|\ln x - \frac{1}{6}\ln y\right|^2 + \left|\ln y - \frac{1}{6}\ln z\right|^2 + \left|\ln z - \frac{1}{6}\ln x\right|^2\Big]e^t$$
$$\leq \frac{1}{36}\max\{|\ln x - \ln y|^2, |\ln y - \ln z|^2, +|\ln z - \ln x|^2\}e^t + \frac{1}{25}\frac{25}{36}(\ln x)^2 e^t + \frac{1}{25}\frac{25}{36}(\ln x)^2 e^t + \frac{1}{25}\frac{25}{36}(\ln x)^2 e^t$$
$$+\frac{1}{36}\left|\ln x - \frac{1}{6}\ln y\right|^2 e^t + \frac{1}{36}\left|\ln y - \frac{1}{6}\ln z\right|^2 e^t + \frac{1}{36}\left|\ln z - \frac{1}{6}\ln x\right|^2 e^t$$
$$= a_1 G(Tx, Ty, Tz) + a_2 G(Tx, Sx, Sx) + a_3 G(Ty, Sy, Sy) + a_4 G(Tz, Sz, Sz) + a_5 G(Tx, Sy, Sy) + a_6 G(Ty, Sz, Sz) + a_7 G(Tz, Sx, Sx)$$

where

$$a_1 = a_5 = a_6 = a_7 = \frac{1}{36},$$
$$a_2 = a_3 = a_4 = \frac{1}{25}.$$

and

$$3sa_1 + (s+2)\sum_{i=2}^{4} a_i + (s^2 + s + 1)\sum_{i=5}^{7} a_i = 1.23 < 3$$

Also $S1 = T1 \Rightarrow ST1 = TS1$, that is, the pair (S, T) is weakly compatible. It is clear that the conditions of Theorem 3.9 are satisfied. Here $x^* = 1$is a unique common fixed point of S and T. ■

6. CONCLUSION

In this paper, introduced the concept of G_b-cone metric space and we described some properties of such metric. Also, we established some fixed point and common fixed theorems for contraction mappings in G_b-cone metric spaces using the idea of weakly compatible mappings. Also, presented examples are showing that our results are real generalization of known ones in fixed point theory. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

AUTHOR'S CONTRIBUTIONS

Both authors contributed equally and significantly to writing this paper. Both authors read and approved the final manuscript.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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