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Homothetic Motions at E_4^8 with Split Octonions

Mehdi JAFARI[1,♠]

[1]*Department of Mathematics, University College of Science and Technology Elm o Fan, Urmia, Iran*

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ABSTRACT

In this paper, a matrix which is similar to Hamilton operators has been developed for split-octonions in eight dimensional semi-Euclidean space E_4^8 and a new motion has been defined by this matrix. It is shown that this is a homothetic motion. Furthermore, it is found that the motion defined by a regular curve of order *r* has only one acceleration center of order (*r*-1) at every instant *t*.

Keyword: Acceleration center, split octonion, Hamilton motion, pole points, semi-orthogonal matrix

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1. INTRODUCTION

In mathematics, the split octonions are an 8-dimensional nonassociative algebra over the real numbers. Unlike the standard octonions, they contain non-zero elements which are non-invertible. Also the signatures of their quadratic forms differ: the split octonions have a split-signature (4,4) whereas the octonions have a positive-definite signature (8,0) [7]. A formulation of the Maxwell equations in terms of the split octonions is presented in [5]. In the previous work, we studied split octonions, their mathematical properties, and how they can be used to rotate objects in eight dimensional semi-Euclidean E_4^8 [2]. In [8], Hamilton motion has been defined in four-dimensional Euclidean space E^4. With the aid of the Hamilton operators, real octonions have been expressed in terms of 8×8 matrices. These matrices are determined a homothetic motions in 8-dimensional Euclidean space E^8 [9]. Recently, the homothetic motions in different spaces are investigated *e.g.* [1,3,4].

It is shown that this study can be repeated for split octonoin, which is a homothetic motion in 8-dimensional semi-Euclidean space and this homothetic motion satisfied all of the properties in [9].

♠Corresponding author, e-mail: mjafari@science.ankara.edu.tr, mj_msc@yahoo.com

2. PRELIMINARIES

Definition 1. E^8 with the metric tensor

$$\langle u,v\rangle = \sum_{i=1}^{4} u_i v_i - \sum_{j=5}^{8} u_j v_j, \qquad u,v \in E^8,$$

is called semi-Euclidean space and is denoted by E_4^8 where 4 is called the index of metric.

Definition 2. A vector $u \in E_4^8$ is called

Space-like if $\langle u,u\rangle < 0$ *or* $u = 0$,
Time-like if $\langle u,u\rangle > 0$,
Light-like if $\langle u,u\rangle = 0$, $u \neq 0$.

A matrix A is called a semi-orthogonal matrix if $A\varepsilon A^T = A^T \varepsilon A = \varepsilon$ and $\det A = 1$ where

$$\varepsilon = \begin{bmatrix} I_4 & 0 \\ 0 & -I_4 \end{bmatrix}.$$

Definition 3. A split octonion *x* has an expression of the form

$$x = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7$$

with real coefficients $\{a_i\}$. A split octonion *x* can also be written as

$$x = (a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3) + (a_4 + a_5 e_1 + a_6 e_2 + a_7 e_3) e_4 = q + q' e,$$

where $e^2 = 1$ and

$$q, q' \in H = \left\{ q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \;\middle|\; e_1^2 = e_2^2 = e_3^2 = -1 , a_i \in R \right\},$$

the real quaternion division algebra. The octonionic units $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ satisfy the equalities that are given in the table below;

1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	1	$-e_1$	$-e_2$	$-e_3$
e_5	e_4	$-e_7$	e_6	e_1	1	e_3	$-e_2$
e_6	e_7	e_4	$-e_5$	e_2	$-e_3$	1	e_1
e_7	$-e_6$	e_5	e_4	e_3	e_2	$-e_1$	1

The set of all split octonions is denoted by O'. By linearity, multiplication of split octonion can be described by a matrix-vector product as

$$x.\omega=\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & a_4 & a_5 & a_6 & a_7 \\ a_1 & a_0 & -a_3 & a_2 & a_5 & -a_4 & -a_7 & a_6 \\ a_2 & a_3 & a_0 & -a_1 & a_6 & a_7 & -a_4 & -a_5 \\ a_3 & -a_2 & a_1 & a_0 & a_7 & -a_6 & a_5 & -a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix},$$

where $x,\omega\in\mathrm{O}'$.

The algebra O' is not associative, since

$$e_1(e_2e_4)=e_1e_6=-e_7,$$
$$(e_1e_2)e_4=e_3e_4=e_7.$$

But it has the property of *alternativity*, that is, any two elements in it generate an associate subalgebra isomorphic to an algebra $\mathrm{R},\mathrm{C},\mathrm{C}',\mathrm{H},\mathrm{H}',\mathrm{H}^0,\mathrm{H}'^0,\mathrm{H}^{00}$.

The subalgebra with basis e_0,e_1,e_2,e_3 is isomorphic to the algebra H of quaternions, and the algebra with basis

$$e_0,e_1,e_4,e_5;\quad e_0,e_1,e_6,e_7;\quad e_0,e_2,e_5,e_7;\quad e_0,e_2,e_4,e_5;\quad e_0,e_3,e_4,e_7;\quad e_0,e_3,e_5,e_6$$

are isomorphic to the algebra H' of split quaternions. The subalgebras with bases e_0,e_1,e_2+e_4,e_3+e_5 and $e_0,e_4,e_1+e_7,-e_3-e_5$ are isomorphic to H^0 and H'^0, the subalgebra with basis

$$e_0,\frac{e_1+e_7}{2},\frac{e_2+e_4}{2},\frac{e_3+e_5}{2}$$

is isomorphic to H^{00} [3].

It is useful, therefore, to define the following terms:

The *conjugate* of x is

$$\bar{x}=a_0\,e_0-(a_1\,e_1+a_2\,e_2+a_3\,e_3+a_4\,e_4+a_5\,e_5+a_6\,e_6+a_7\,e_7),$$

The *norm* of x is

$$N_x=|x|^2=\bar{x}\,x=\sum_{i=0}^{3}a_i^2-\sum_{i=4}^{7}a_i^2.$$

The modulus $|x|$ of a split octonion *x*, like the modulus of a split complex number, or split quaternion, can be real or imaginary and can be equal to 0 for $x\neq 0$ [2].

A split octonion *x* is timelike, spacelike or lightlike, if $N_x>0, N_x<0$ or $N_x=0,$ respectively. If $N_x=1,$ then x is called a unit split octonion.

The *inverse* of x with $N_x\neq 0,$ is

$$x^{-1}=\frac{1}{N_x}\bar{x}.$$

Theorem 1. The algebra of real Zorn vector-matrices is isomorphic to the algebra O' of split-octonions.

Proof: Real Zorn vector-matrices are linear combinations of the basis vector-matrices

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & e_1\\ e_1 & 0\end{bmatrix}, \begin{bmatrix}0 & e_2\\ e_2 & 0\end{bmatrix}, \begin{bmatrix}0 & e_3\\ e_3 & 0\end{bmatrix},$$
$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & -e_1\\ e_1 & 0\end{bmatrix}, \begin{bmatrix}0 & e_2\\ -e_2 & 0\end{bmatrix}, \begin{bmatrix}0 & -e_3\\ e_3 & 0\end{bmatrix},$$

whose multiplication rules coincide with the multiplication rules of the basis elements

$$\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

of the algebra O'. Hence, we obtain the isomorphism of the algebra of real Zorn vector-matrices and the algebra O' [6].

Definition 4. Let

$$x = a_0\, e_0 + a_1\, e_1 + a_2\, e_2 + a_3\, e_3 + a_4\, e_4 + a_5\, e_5 + a_6\, e_6 + a_7\, e_7,$$

be a split octonion and $\varphi_x : O' \to O'$ defined as follows:

$$\varphi_x(\omega) = x\,\omega, \qquad \omega \in O'.$$

The Hamilton's operator φ_x, could be represented as the matrix;

$$\overset{+}{H}(x) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & a_4 & a_5 & a_6 & a_7 \\ a_1 & a_0 & -a_3 & a_2 & a_5 & -a_4 & -a_7 & a_6 \\ a_2 & a_3 & a_0 & -a_1 & a_6 & a_7 & -a_4 & -a_5 \\ a_3 & -a_2 & a_1 & a_0 & a_7 & -a_6 & a_5 & -a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}, \qquad (1)$$

or equality

$$\varphi_x = \begin{bmatrix} H^+(q) & N^T \\ N & H^-(q) \end{bmatrix},$$

where $\overset{+}{H}$, $\overset{-}{H}$ are Hamilton operators for quaternions and N is a 4×4 matrix. By using the definition of $\overset{+}{H}$ the multiplication of the two split-octonions x, ω is given by $x\,\omega = \overset{+}{H}(x)\,\omega.$

Theorem 2. Let $x, \omega \in O'$ and $\lambda \in R$ be given. Then

1. $x = \omega \Leftrightarrow \overset{+}{H}(x) = \overset{+}{H}(\omega).$
2. $\overset{+}{H}(x+\omega) = \overset{+}{H}(x) + \overset{+}{H}(\omega), \quad \overset{+}{H}(\lambda x) = \lambda \overset{+}{H}(x).$
3. $\overset{+}{H}(\overline{x}) = \left[\overset{+}{H}(x)\right]^T, \quad \overset{+}{H}(1) = I_4.$
4. $\det \overset{+}{H}(x) = (N_x)^8.$
5. $\operatorname{tr} \overset{+}{H}(x) = 8a_0.$

Proof: Follows from a direct verification. ■

Theorem 3. Let x be a unit split octonion. Matrix generated by operator $\overset{+}{H}$ is a semi-orthogonal matrix, *i.e.*

$$\overset{+}{H}(x)\,\varepsilon\left[\overset{+}{H}(x)\right]^T=\left[\overset{+}{H}(x)\right]^T\varepsilon\,\overset{+}{H}(x)=\varepsilon.$$

where $\varepsilon=\begin{bmatrix} I_4 & 0 \\ 0 & -I_4 \end{bmatrix}$.

3. HOMOTHETIC MOTIONS IN SEMI-EUCLIDEAN SPACE E_4^8

Let us consider the following curve:

$$\alpha : I\subset \mathbb{R}\to E_4^8,$$

defined by $\alpha(t)=\left(a_0(t),a_1(t),a_2(t),a_3(t),a_4(t),a_5(t),a_6(t),a_7(t)\right)$ for every $t\in I$.

We suppose that the unit velocity curve $\alpha(t)$ is differentiable regular curve of order r. Let position vector of the curve be time-like. The operator *B*, corresponding to $a(t)$ is defined by the following matrix;

$$B=\overset{+}{H}\left[\alpha(t)\right]=\begin{bmatrix}
a_0(t) & -a_1(t) & -a_2(t) & -a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t)\\
a_1(t) & a_0(t) & -a_3(t) & a_2(t) & a_5(t) & -a_4(t) & -a_7(t) & a_6(t)\\
a_2(t) & a_3(t) & a_0(t) & -a_1(t) & a_6(t) & a_7(t) & -a_4(t) & -a_5(t)\\
a_3(t) & -a_2(t) & a_1(t) & a_0(t) & a_7(t) & -a_6(t) & a_5(t) & -a_4(t)\\
a_4(t) & a_5(t) & a_6(t) & a_7(t) & a_0(t) & -a_1(t) & -a_2(t) & -a_3(t)\\
a_5(t) & -a_4(t) & a_7(t) & -a_6(t) & a_1(t) & a_0(t) & a_3(t) & -a_2(t)\\
a_6(t) & -a_7(t) & -a_4(t) & a_5(t) & a_2(t) & -a_3(t) & a_0(t) & a_1(t)\\
a_7(t) & a_6(t) & -a_5(t) & -a_4(t) & a_3(t) & a_2(t) & -a_1(t) & a_0(t)
\end{bmatrix} \quad (2)$$

Definition 5. The 1-parameter Hamilton motions of a body in E_4^8 are generated by transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix}=\begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix}$$

or equivalently

$$Y=BX+C \quad (3)$$

where $B=\overset{+}{H}\left[\alpha(t)\right]$ and Y , X and C are $n\times 1$ real matrices. Y and X correspond to the position vectors of the same point *P*.

Theorem 4. The Hamilton motion determined by equation (3) in semi-Euclidean space E_4^8 is a homothetic motion.

Proof: Because $\alpha(t)$ does not pass through the origin, the matrix B can be represented as

$$B=h\begin{bmatrix} a_0(t)/h & -a_1(t)/h & -a_2(t)/h & -a_3(t)/h & a_4(t)/h & a_5(t)/h & a_6(t)/h & a_7(t)/h \\ a_1(t)/h & a_0(t)/h & -a_3(t)/h & a_2(t)/h & a_5(t)/h & -a_4(t)/h & -a_7(t)/h & a_6(t)/h \\ a_2(t)/h & a_3(t)/h & a_0(t)/h & -a_1(t)/h & a_6(t)/h & a_7(t)/h & -a_4(t)/h & -a_5(t)/h \\ a_3(t)/h & -a_2(t)/h & a_1(t)/h & a_0(t)/h & a_7(t)/h & -a_6(t)/h & a_5(t)/h & -a_4(t)/h \\ a_4(t)/h & a_5(t)/h & a_6(t)/h & a_7(t)/h & a_0(t)/h & -a_1(t)/h & -a_2(t)/h & -a_3(t)/h \\ a_5(t)/h & -a_4(t)/h & a_7(t)/h & -a_6(t)/h & a_1(t)/h & a_0(t)/h & a_3(t)/h & -a_2(t)/h \\ a_6(t)/h & -a_7(t)/h & -a_4(t)/h & a_5(t)/h & a_2(t)/h & -a_3(t)/h & a_0(t)/h & a_1(t)/h \\ a_7(t)/h & a_6(t)/h & -a_5(t)/h & -a_4(t)/h & a_3(t)/h & a_2(t)/h & -a_1(t)/h & a_0(t)/h \end{bmatrix}=hA$$

where $h:I\subset \mathrm{R}\to \mathrm{R},$

$$t\to h(t)=\left|\alpha(t)\right|=\sqrt{\left|a_0^2(t)+a_1^2(t)+a_2^2(t)+a_3^2(t)-a_4^2(t)-a_5^2(t)-a_6^2(t)-a_7^2(t)\right|}.$$

So, we find $A\varepsilon A^T=\varepsilon$ and $\det A=1$, thus B is a homothetic matrix and equation (3) determines a homothetic motion.

Example 1. Let $\alpha:I\subset \mathrm{R}\to \mathrm{E}_4^8$ be a curve given by $\alpha(t)=(t,\sinh t,-t,-2,\cosh t,t,-t,1)$. Since $\left|\dot{\alpha}(t)\right|=1,$ then $\alpha(t)$ is a unit velocity curve. Because $\alpha(t)$ does not pass through the origin, the matrix B can be represented as

$$B=\begin{bmatrix} t & -\sinh t & t & 2 & \cosh t & t & -t & 1 \\ \sinh t & t & 2 & -t & t & -\cosh t & -1 & -t \\ -t & -2 & t & -\sinh t & -t & 1 & -\cosh t & -t \\ -2 & t & \sinh t & t & 1 & t & t & -\cosh t \\ \cosh t & t & -t & 1 & t & -\sinh t & t & 2 \\ t & -\cosh t & 1 & t & \sinh t & t & -2 & t \\ -t & -1 & -\cosh t & t & -t & 2 & t & \sinh t \\ 1 & -t & -t & -\cosh t & -2 & -t & -\sinh t & t \end{bmatrix}$$
$$=\sqrt{2}A,$$

where

$h(t)=\left|\alpha(t)\right|=\sqrt{\left|t^2+\sinh^2 t+t^2+4-\cosh^2 t-t^2-t^2-1\right|}=\sqrt{2}.$ We find $A\varepsilon A^T=A^T\varepsilon A=\varepsilon,$ $\det A=1,$ where $\varepsilon=\begin{bmatrix} I_4 & 0 \\ 0 & -I_4 \end{bmatrix}.$

Theorem 5. The derivation operator $\dot{B}=\dfrac{dB}{dt}$ of the Hamilton operator $B=hA$ is a semi-orthogonal matrix.

Proof: By (2), $\dot{B}\varepsilon\dot{B}^T=\dot{B}^T\varepsilon\dot{B}=\varepsilon,$ and $\det\dot{B}=1.$ Then theorem is proved.

Colorally 1. In $\mathrm{E}_4^8,$ the motion is a regular motion, and it is independent of h.

4. POLE POINTS AND POLE CURVES OF THE MOTION IN SEMI-EUCLIDEAN SPACE E_4^8

To find the pole points of the Hamilton motion determined by equation (3), we have to solve the equation

$$\dot{B}X+\dot{C}=0. \qquad (4)$$

Any solution of the equation (4) is a pole point of the motion at that instant in $R_\circ$. Since $\dot{B}$ is regular, the equation (4) has only one solution, *i.e.*, $X_\circ = (-\dot{B})^{-1}\dot{C} = 0$ at every instant t. This pole point in the fixed system is

$$X = B(-\dot{B})^{-1}\dot{C} + C.$$

Theorem 6. During the homothetic motion of semi-Euclidean space of 8-dimensions, there is a unique instantaneous pole point at every time t.

5. ACCELERATION CENTERS OF ORDER (*R*-1) OF THE MOTION

Definition 6. The set of zeros of the equation of the sliding acceleration of order r is called the acceleration center of order (*r*-1) [9].

In order to find the acceleration center of order (*r*-1) for the equation (3) according to definition above, we find the solution of the equation

$$B^{(r)}X + C^{(r)} = 0, \qquad (5)$$

Since the curve $\alpha(t)$ is a regular curve of order *r*, then

$$\sum_{i=0}^{3}\left[a_i^{(r)}\right]^2 - \sum_{i=4}^{7}\left[a_i^{(r)}\right]^2 \neq 0,$$

Furthermore,

$$\det B^{(r)} = \left\{\sum_{i=0}^{3}\left[a_i^{(r)}\right]^2 - \sum_{i=4}^{7}\left[a_i^{(r)}\right]^2\right\}^4,$$

then $\det B^{(r)} \neq 0.$ Therefore, matrix $B^{(r)}$ has an inverse, and, by equation (5), the acceleration center of order (*r*-1) at every *t* instant, is

$$x = \left[B^{(r)}\right]^{-1}.\left[-C^{(r)}\right].$$

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Jafari M., Yayli Y., Homothetic motions at $E^4_{\alpha\beta}$, International journal of contemporary mathematical sciences, vol. 5, no. 47: (2010) 2319-2326.

[2] Jafari M., Real matrix representation of split octonions, Submitted for publication.

[3] Kula, L., Yayli, Y., Homothetic motions in semi-Euclidean space E^4_2, Mathematical proceedings of the Royal Irish academy, 105A1 (2005) 9-15.

[4] Kabadayi H., Yayli, Y., Homothetic motions at E^4 with bicomplex numbers, Advances in applied Clifford algebras, 21 (2011) 541-546.

[5] Nurowski P. Split octonions and Maxwell equations, Acta physica polonica A, 116 (2009) 992-993.

[6] Rosenfeld B., Geometry of Lie groups, Kluwer Academic Publishers, Netherlands, 1997.

[7] Split Octonion, http://en.wikipedia.org/wiki/Split-octonion

[8] Yayli, Y., Homothetic motions at E^4, Mechanism and machine theory 27,(1992) 303-305.

[9] Yayli Y., Bukuc B., Homothetic motions at E^8 with Cayley numbers, Mechanism and machine theory, 3 (1995) 417-420.