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On the Benard Problem with Voight Regularization

Meryem KAYA[1,♠], A. Okay ÇELEBİ[2]

[1]*Department of Mathematics, Faculty of Science, Gazi University, 06500, Beşevler, Ankara, Turkey*
[2]*Department of Mathematics, Faculty of Arts and Science, Yeditepe University, 34755, Istanbul, Turkey*

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ABSTRACT

In this paper we consider the Benard problem involving some regularizing terms. Using maximum principle which is given by Foias, Manley and Temam in [4] we prove the existence-uniqueness of weak solution and the global attractor has a finite fractal dimension.

Keywords: Benard Problem, weak solution, global attractor

1. INTRODUCTION

In this article we consider the following system of equations in $\Omega = (0, L_1) \times (0, L_2) \times (0,1)$

(1.1) $\frac{\partial u}{\partial t} - \nu\Delta u - \mu\Delta u_t + (u.\nabla)u + \nabla p = e_3(T - T_1)$, in $\Omega \times (0, \tau)$

(1.2) $\frac{\partial T}{\partial t} - \kappa\Delta T - \varkappa\Delta T_t + (u.\nabla)T = 0$ in $\Omega \times (0, \tau)$

(1.3) $\nabla. u = 0$ in Ω

where $\tau > 0$, e_3 is the third component of the canonical basis of $\mathbb{R}^3$. u, p and T are the velocity, pressure and temperature of the fluid respectively and ν, μ, κ and $\varkappa$ are positive constants. Now we state boundary conditions for (1.1)-(1.3).

(1.4) $u = 0$ at $x_3 = 0$, $x_3 = 1$

(1.5) $T = T_0$ at $x_3 = 0$, $T = T_1 = T_0 - 1$ at $x_3 = 1$,

(1.6) $p, u, T, \frac{\partial u}{\partial x_i}, \frac{\partial T}{\partial x_i}$ $(1 \le i \le 2)$ are periodic in the x_i directions which means that

$$\varphi(x,t) = \varphi(x + L_i e_i, t), \qquad i = 1,2 \ \forall x \in \mathbb{R}^3, \qquad \forall t > 0$$

for a generic function φ

As in [4] we can convert this system of equations into

(1.7) $\frac{\partial u}{\partial t} - \nu\Delta u - \mu\Delta u_t + (u.\nabla)u + \nabla p = e_3\theta$ in $\Omega \times (0, \tau)$

♠Corresponding author, e-mail: *meryemk@gazi.edu.tr*

(1.8) $\frac{\partial\theta}{\partial t}-\kappa\Delta\theta-\varkappa\Delta\theta_t+(u.\nabla)\theta-u_3=0 \qquad in \;\; \Omega\times(0,\tau)$
(1.9) $\nabla.u=0 \; in \; \Omega$

where u_3 is the third component of u. Firstly we impose the boundary conditions as

(1.10) $p,\; u,\; \theta, \frac{\partial u}{\partial x_i}, \; \frac{\partial\theta}{\partial x_i}$ $(1\le i\le 2)$ are periodic in the x_i directions,
And

(1.11) $u \;=\; 0,\; \theta=0$ at $x_3=0,\; x_3=1$

Secondly the initial conditions are given by

(1.12) $u(x,0)=u_0(x),\; \theta(x,0)=\theta_0(x).$

The Benard problem in the absence of the use of regularization terms has been previously studied by many authors [1], [2], [4], [10], [18]. They reported several mathematical difficulties about on it. In general the lack of uniqueness and continuity of weak solutions are the main difficulties to define a semigroup for such a system in 3D. As stated by Oskolkov [16], Voight has suggested a model which is obtained from the Navier-Stokes system describing the flow of an incompressible Newtonian fluid by adding an regularizing term. By adding such a term to the Navier -Stokes equation, he was able to prove the uniqueness and continuity of weak solutions without using any restriction. For the motivation of using such terms we may consult with the articles [16] (see also [7]-[9], [11]). We have used same idea for the Benard problem. Thanks to the Voight regularization we have better estimates for the solutions. In our case, we prove the uniqueness and continuity of weak solutions.

This article is organized as follows: In Section 2 mathematical framework of the investigation is given from [4], [18], [19]. In Section 3 we prove the existence-uniqueness and the continuity of the weak solution of (1.7)-(1.12). In Section 4 it is shown that the semigroup generated by the system (1.7)-(1.12) has a global attractor. In the last Section we give an estimate of the dimension of the global attractor.

2. PRELIMINARIES

In this section we employ the standard notations and the usual function spaces (see, [19]). We introduce the Hilbert spaces,

$$H_1=\left\{v\in\left(L^2(\Omega)\right)^3:\nabla.v=0, v_{x_3=1}=v_{x_3=0}=0, v_{x_i=0}=v_{x_i=L_i}\; i=1{,}2\right\},$$
$$H_2=L^2(\Omega)\quad H=H_1\times H_2$$

We use the notation (.,.) for the inner products in H, H_1, H_2 and the corresponding norms denoted by $|.|$.

$$V_2=\left\{u\in H^1(\Omega): u_{x_3=1}=u_{x_3=0}=0, u_{x_i=0}=u_{x_i=L_i}\; i=1{,}2\right\},$$

where $H^1(\Omega)$ is the spaces of functions u and whose first order distributional derivatives are in $L^2(\Omega)$.

$$V_1=\{v\in(V^2)^3, \qquad \nabla.v=0\},$$
$$V=V_1\times V_2.$$

The inner product and the norm in V_2 are given by

$$\big((u,v)\big)=\int_\Omega \nabla u\,\nabla v dx \quad \forall u,v\in V_2, \quad \|u\|=\big((u,u)\big)^{\frac{1}{2}}, \qquad \forall u\in V_2$$

For simplicity we use the same symbols $\|.\|$, $((.,.))$ to denote the inner product and norm in V_1 and V. Let A_i be an unbounded linear operator from $D(A_i)$ into H_i defined by

$$(A_i u,v)=\big((u,v)\big) \quad \forall u,v\in D(A_i),\;\; i=1{,}2,$$
$$D(A)=D(A_1)\times D(A_2)$$

where

$D(A_1)=(H^2(\Omega))^3\cap V_1,\; D(A_2)=H^2(\Omega)\cap V_2$.

Let $u,\;\theta$ be a solution of the problem (1.7)-(1.12) η and ψ be test functions in V_1, V_2. We multiply (1.7) and (1.8) by η and ψ respectively and integrate over Ω, to get

(2.1)
$$\frac{d}{dt}[(u,\eta)+\mu(\nabla u,\nabla\eta)]+\nu(\nabla u,\nabla\eta)+b_1(u,u,\eta)=(e_3\theta,\eta)$$
(2.2)
$$\frac{d}{dt}[(\theta,\psi)+\varkappa(\nabla\theta,\nabla\psi)+\kappa(\nabla\theta,\nabla\psi)]+b_2(u,\theta,\psi)=(u_3,\psi)$$

where

$$b_1(u,v,w)=\sum_{i,j=1}^{3}\int_{\Omega} u_i\frac{\partial v_j}{\partial x_i}w_j dx,\qquad \forall u,v,w\in(H^1(\Omega))^3$$
$$b_2(\varphi,\psi,\phi)=\sum_{i=1}^{3}\int_{\Omega} \varphi_i\frac{\partial \psi}{\partial x_i}\phi dx,\qquad \forall\varphi\in\left(H^1(\Omega)\right)^3,\forall\,\psi,\phi\in H^1(\Omega)$$

are the trilinear forms on V_1 and V_2.

Now we give the definition of weak solutions for the problem (1.7)-(1.12).

Definition 1. Let $u_0\in V_1,\ \theta_0\in V_2$ and $\tau>0$. $u\in L^2(0,\tau,V_1)\cap L^\infty(0,\tau,V_1),\ \theta\in L^2(0,\tau,V_2)\cap L^\infty(0,\tau,V_2)$ is said to be weak solution to (1.7)- (1.12) in the interval $[0,\tau]$ if u,θ satisfy (2.1), (2.2) for any test functions η,ψ where the derivative with respect to t is in the distribution sense.

3. THE EXISTENCE, UNIQUENESS AND CONTINUITY RESULT

Firstly we give a proof of the existence of the weak solutions. Afterwards the uniqueness and continuity of the solutions in V will be obtained.

3.1. The Existence Of Weak Solutions

Theorem 1*. Let $(u_0,\theta_0)\in V$ and, $\tau>0$. The problem (1.7)-(1.12) has at least one weak solution.*

Proof*.* The existence of a weak solution of this problem is obtained by the well known method of the Feado- Galerkin approximation [10] (see also [3], [5], [12], [15], [17], [18]). Let $\{w_i\}\subset D(A_1),\ \{\widetilde{w_i}\}\subset D(A_2)$ be the orthonormal basis of H_1 and H_2 respectively. For each m we define an approximate solution

$$u_m=\sum_{i=1}^{m}\phi_{im}(t)w_i\ ,\theta_m=\sum_{i=1}^{m}\tilde{\phi}_{im}(t)\,\widetilde{w_i}\ ,u_m^3=\sum_{i=1}^{m}\phi_{im}^3(t)w_i \tag{3.1}$$

where u_m^3 is a third component of u_m. Subsituting (3.1) in the equation (2.1)-(2.2) with $\eta=w_j\ ,\psi=\widetilde{w_k}$ we get the following equations

$$\sum_{i=1}^{m}(w_i,w_j)\,\phi'_{im}+\nu\sum_{i=1}^{m}(\nabla w_i,\nabla w_j)\,\phi_{im}(t)+\mu\sum_{i=1}^{m}(\nabla w_i,\nabla w_j)\phi'_{im}(t)+\sum_{i=1}^{m}(w_i.\nabla w_l,w_j)\phi_{im}(t)\phi_{lm}(t)=\sum_{i=1}^{m}(e_3\tilde{\phi}_{im}\widetilde{w_i},w_j) \tag{3.2}$$

$$\sum_{i=1}^{m}(\widetilde{w_i},\widetilde{w_k})\,\tilde{\phi}'_{im}+\kappa\sum_{i=1}^{m}(\nabla\widetilde{w_i},\nabla\widetilde{w_k})\,\tilde{\phi}_{im}(t)+\varkappa\sum_{i=1}^{m}(\nabla\widetilde{w_i},\nabla\widetilde{w_k})\widetilde{\phi'}_{im}(t)+\sum_{l,i=1}^{m}(w_i.\nabla\widetilde{w_l},\widetilde{w_k})\phi_{im}\tilde{\phi}_{lm}=\sum_{i=1}^{m}(w_i,\widetilde{w_k})\,\phi_{im}^3. \tag{3.3}$$

The (3.2), (3.3) give a system of nonlinear differential equations for $\phi_{im}(t)$, and $\tilde{\phi}_{im}(t)$ $i=1,2,\dots,m$, with the initial conditions

$$(u_0,w_i)=\phi_{im}(0),\qquad(\theta_0,\widetilde{w_i})=\tilde{\phi}_{im}(0)$$

This initial value problem has a maximal solution defined on some interval $[0,t_m]$. The a priori estimate which we are going to prove, enables us to take $t_m=\tau$.

Now we prove a priori estimates. Let us multiply the equations (3.2) and (3.3) by $\phi_{jm}(t)$ and $\tilde{\phi}_{km}(t)$ respectively, and summing with respect to j and k we have

(3.4) $$\frac{1}{2}\frac{d}{dt}(\|u_m\|^2 + \mu\|\nabla u_m\|^2) + \nu\|\nabla u_m\|^2 = (e_3\theta_m, u_m)$$

(3.5) $$\frac{1}{2}\frac{d}{dt}(\|\theta_m\|^2 + \varkappa\|\nabla\theta_m\|^2) + \kappa\|\nabla\theta_m\|^2 = (u_m^3, \theta_m)$$

where we used $b_1(u_m, u_m, u_m) = 0$, $b_1(u_m, \theta_m, \theta_m) = 0$. Using Poincaré, Cauchy-Schwarz and Gronwall inequalities, we obtain

$$(u_m, \theta_m) \in L^2(0,\tau,V) \cap L^\infty(0,\tau,V)$$

and

$$(u_m, \theta_m) \in L^2(0,\tau,H) \cap L^\infty(0,\tau,H).$$

From the above results we can choose subsequences of $\{u^m\}$ and $\{\theta^m\}$, which we denote by the same symbols, such that

(3.6) $u^m \to u$ weakly in $L^2(0,\mathrm{T},V_1)$, weakly* in $L^\infty(0,\mathrm{T},V_1)$,
(3.7) $\theta_m \to \theta$ weakly in $L^2(0,\mathrm{T},V_2)$, weakly* in $L^\infty(0,\mathrm{T},V_2)$.

By the similar technique in the proof of Theorem 3.1 in the book by Temam [18] and used Theorem 2.2, in it, we can select subsequences which we denote by the same symbol such that

$u^m \to u$ strongly in $L^2(0,\mathrm{T},H_1)$,
$\theta_m \to \theta$ strongly in $L^2(0,\mathrm{T},H_2)$.

Using the above results and (3.6), (3.7), we see that u and θ satisfy (2.1), (2.2) for any $\eta = w_j$, $\psi = w_k$. By a continuity argument (2.1), (2.2) are satisfied for any $\eta \in V_1$, $\psi \in V_2$. Besides the initial conditions are also satisfied. Hence there exists at least one weak solution for the (1.7)-(1.12.)

3.2. Uniqueness of Weak Solutions

Knowing that the problem we are dealing with has at least one weak solution, we will prove that it is unique.

Theorem 2. *Let $(u_0, \theta_0) \in V$ and $\tau > 0$. The problem (1.7)-(1.12) has a unique weak solution.*

Proof. Let (v_1, w_1) and (v_2, w_2) be any two solutions of (1.7)-(1.12) and $u = v_1 - v_2$ and $\theta = w_1 - w_2$. Then u and θ satisfy

(3.8) $$\frac{\partial u}{\partial t} - \nu\Delta u - \mu\Delta u_t + (v_1.\nabla)v_1 - (v_2.\nabla)v_2 + \nabla p = e_3\theta$$

(3.9) $$\frac{\partial\theta}{\partial t} - \kappa\Delta\theta - \varkappa\Delta\theta_t + (v_1.\nabla)w_1 - (v_2.\nabla)w_2 - u_3 = 0$$

(3.10) $$u(x,0) = 0, \qquad \theta(x,0) = 0$$

Taking the scalar product of (3.8) and (3.9) with u and θ respectively, we obtain

(3.11) $$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \mu\|\nabla u\|^2) + \nu\|\nabla u\|^2 = b(u, v_1, u) + (e_3\theta, u)$$

(3.12) $$\frac{1}{2}\frac{d}{dt}(\|\theta\|^2 + \varkappa\|\nabla\theta\|^2) + \kappa\|\nabla\theta\|^2 = b(u, w_1, \theta) + (u_3, \theta)$$

Since $v_1 \in L^\infty(0,\tau,V_1)$ and $w_1 \in L^\infty(0,\tau,V_2)$, using Ladyhenskaya, Hölder and Young inequalities we get

(3.13) $$\frac{d}{dt}(\|u\|^2 + \mu\|\nabla u\|^2 + \|\theta\|^2 + \varkappa\|\nabla\theta\|^2) + 2\nu\|\nabla u\|^2 + 2\kappa\|\nabla\theta\|^2 \\ \leq \left[\frac{3\nu\beta_1}{2\mu}\mu\|\nabla u\|^2 + \left(\frac{81\beta_1}{2\nu^3} + 2\right)\|u\|^2\right] + \left[\left(\frac{27\beta_1}{2\nu^3} + 2\right)\|\theta\|^2 + \frac{\beta_{1\nu}}{2\varkappa}\varkappa\|\nabla\theta\|^2\right]$$

where $\beta_1 = \max\{ess\sup\|\nabla v_1\|^2,\ ess\sup\|\nabla w_1\|^2\}$. Dropping the second and third terms of the left hand side of (3.13) and choosing

$$\beta_2 = \max\left\{\frac{3\nu\beta_1}{2\mu}, \frac{81\beta_1}{2\nu^3} + 2, \frac{27\beta_1}{2\nu^3} + 2, \frac{\beta_{1\nu}}{2\varkappa}\right\}$$

we obtain

$$\frac{d}{dt}(\|u\|^2+\mu\|\nabla u\|^2+\|\theta\|^2+\varkappa\|\nabla\theta\|^2)\le\beta_2(\|u\|^2+\mu\|\nabla u\|^2+\|\theta\|^2+\varkappa\|\nabla\theta\|^2)$$

which gives

$$\|u\|^2+\mu\|\nabla u\|^2+\|\theta\|^2+\varkappa\|\nabla\theta\|^2\le 0$$

and hence $v_1=v_2$ and $w_1=w_2$.

3.3. Continuity of the weak solutions

The continuity of weak solutions is given by the following theorem.

Theorem 3. *Let $u_0\in H^2\cap V_1,\ \theta_0\in H^2\cap V_2$. The solution u and θ of the problem (1.7)-(1.12) is in $C([0,\tau];V)$*

Proof. Differentiating the equations (1.7) and (1.8) with respect to t and multiplying these equations in L^2 with u_t,θ_t, respectively, we get

$$\frac{d}{dt}(\|u_t\|^2+\mu\|\nabla u_t\|^2)+2\nu\|\nabla u_t\|^2=2(e_3\theta,u_t)+2b(u,u_t,u_t)$$
$$\frac{d}{dt}(\|\theta_t\|^2+\varkappa\|\nabla\theta_t\|^2)+2\kappa\|\nabla\theta_t\|^2=2(u_3,\theta_t)+2b(u,\theta_t,\theta_t).$$

Majorizing right hand sides of these equations by using some well-known inequalities and adding these inequalities up, we write

(3.14)

$$\frac{d}{dt}(\|u_t\|^2+\mu\|\nabla u_t\|^2+\|\theta_t\|^2+\varkappa\|\nabla\theta_t\|^2)+2\nu\|u_t\|^2+2\kappa\|\nabla\theta_t\|^2$$
$$\le\|u_t\|^2+\|\theta_t\|^2+\frac{\beta_3}{\mu}\mu\|\nabla u_t\|^2+\frac{\beta_3}{\varkappa}\varkappa\|\nabla\theta_t\|^2$$

where

$$\beta_3=\max\{ess\,sup\|\nabla u\|^2,c(\Omega)\}\ ;$$

$c(\Omega)$ is constant. Dropping the last two terms of left hand side of the inequality (3.14) we write

$$\frac{d}{dt}(\|u_t\|^2+\mu\|\nabla u_t\|^2+\|\theta_t\|^2+\varkappa\|\nabla\theta_t\|^2)\le\beta_4(\|u_t\|^2+\mu\|\nabla u_t\|^2+\|\theta_t\|^2+\varkappa\|\nabla\theta_t\|^2)$$

where

$$\beta_4=\max\left\{1,\frac{\beta_3}{\mu},\frac{\beta_3}{\varkappa}\right\}.$$

From this inequality we easily obtain the following estimates

$$\|u_t\|^2+\mu\|\nabla u_t\|^2+\|\theta_t\|^2+\varkappa\|\nabla\theta_t\|^2\le(\|u_t(x,0)\|^2+\mu\|\nabla u_t(x,0)\|^2+\|\theta_t(x,0)\|^2+\varkappa\|\nabla\theta_t(x,0)\|^2)e^{\gamma\tau}$$

We find the initial values $u_t(x,0),\nabla u_t(x,0),\theta_t(x,0),\nabla\theta_t(x,0)$ from the following boundary-value problem

$$-\mu\Delta u_t(x,0)+u_t(x,0)+\nabla p(x,0)=\nu\Delta u_0-(u_0.\nabla)u_0+e_3\theta_0\equiv F_1(x),\ u_t(x,0)=0\ \ in\ \partial\Omega$$

$$-\varkappa\Delta\theta_t(x,0)+\theta_t(x,0)=\kappa\Delta u_0-(u_0.\nabla)\theta_0+u_3(x,0)\equiv F_2(x),\qquad \theta_t(x,0)=0\ in\ \partial\Omega$$

as given by Oskolkov ([16], p.444). If for all $\tau,u_0\in W_2^2\cap V_1$ and $\theta_0\in W_2^2\cap V_2$, then $F_1(x)\in L^2,\ F_2(x)\in L^2$. Solving this problem we get

$$\|u_t\|^2+\mu\|\nabla u_t\|^2+\|\theta_t\|^2+\varkappa\|\nabla\theta_t\|^2\le c$$

and

$$\int_0^\tau\|u_t\|^2+\mu\|\nabla u_t\|^2+\|\theta_t\|^2+\varkappa\|\nabla\theta_t\|^2\le c\tau.$$

We can conclude from this inequality that $(u_t,\theta_t)\in L^2(0,\tau,V)$. It is also known that $(u,\theta)\in L^2(0,\tau,V)$. Hence we obtain $(u,\theta)\in C([0,\tau],V)$ (see [17], p. 190).

Taking into account the above results, the semigroup $S(t)$ is well defined from V into V.

4. ABSORBING SET AND ATTRACTOR

In this section we will show that the semigroup $S(t)$ generated by (1.7)-(1.12) has an absorbing set and a global attractor in V exists. First we need to give an estimates for the temperature in the system (1.7)-(1.12). To do this we use the Lemma which is given by Foias, Manley and Temam (see [4], p 945). The Lemma they have stated is:

Lemma 1. *We assume that u and θ satisfy (1.7)-(1.12) and that*

(4.1) $$-1 \leq \theta(x,0) \leq 1 \quad a.e.t.$$

Then

(4.2) $$-1 \leq \theta(x,t) \leq 1 \quad a.e. \;\; x \in \Omega \;\; a.e.t.$$

If $\{u,\theta\}$ are defined for all $t > 0$ and (4.1) is not assumed, then

(4.3) $$\theta(.,t) = \overline{\theta}(.,t) + \overline{\theta}(.,t)$$

where $-1 \leq \overline{\theta}(x,t) \leq 1$ a.e.t, and

(4.4) $$\overline{\theta}(.,t) \to 0 \;\; in\; H_2 \left(= L^2(\Omega)\right) \;\; as\; t \to \infty.$$

It is easy to observe that the above Lemma for their own system of equations is also true for our system. The difference in the proof is given in the following. We can state the following estimates for θ, which will be used in the sequel.

(4.5) $\|\theta(t)\| \leq |\Omega|^{\frac{1}{2}} + \{\|(\theta-1)_+(0)\| + \|\nabla(\theta-1)_+(0)\| + \|(\theta+1)_-(0)\| + \|\nabla(\theta+1)_-(0)\|\}^{\frac{1}{2}} e^{-\frac{\kappa\beta t}{2}}$,

(4.6) $\|\theta\|_\infty \leq |\Omega|^{\frac{1}{2}} + \{\|(\theta-1)_+(0)\| + \|\nabla(\theta-1)_+(0)\| + \|(\theta+1)_-(0)\| + \|\nabla(\theta+1)_-(0)\|\}^{\frac{1}{2}}$

(4.7) $\lim_{t\to\infty} sup|\theta(t)| \leq |\Omega|^{\frac{1}{2}}$.

Now we give a proof of the existence of an absorbing set *V*. Multiplying (1.7) with u and integrating over Ω and using the property of the trilinear form

(4.8) $$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \mu\|\nabla u\|^2) + \nu\|\nabla u\|^2 = (e_3\theta, u).$$

Using Cauchy-Schwarz and Young inequalities we majorize right-hand side of (4.8) to obtain

$$\frac{d}{dt}(\|u\|^2 + \mu\|\nabla u\|^2) + 2\nu\|\nabla u\|^2 \leq \frac{1}{\nu}\|\theta\|^2 + \nu\|u\|^2.$$

Employing Poincaré inequality we write

(4.9) $$\frac{d}{dt}(\|u\|^2 + \mu\|\nabla u\|^2) + \beta_5(\|u\|^2 + \mu\|\nabla u\|^2) \leq \frac{1}{\nu}\|\theta\|^2$$

where $\beta_5 = \min\left\{\nu, \frac{\nu}{\mu}\right\}$. From (4.9):

(4.10) $$\|u\|^2 + \mu\|\nabla u\|^2 \leq (\|u_0\|^2 + \mu\|\nabla u_0\|^2)e^{-\beta_5 t} + \frac{1}{\nu}\|\theta\|_\infty^2\left(1 - e^{-\beta_5 t}\right),$$

(4.11) $$\lim_{t\to\infty} \sup|u(t)| \leq \frac{|\Omega|^{\frac{1}{2}}}{\nu}, \qquad \lim_{t\to\infty} \sup|\nabla u(t)| \leq \frac{|\Omega|^{\frac{1}{2}}}{\nu\mu}.$$

Similarly multiplying (1.8) with θ and integrating over Ω and using (4.10) we obtain

(4.12) $$\lim_{t\to\infty} \sup|\nabla\theta(t)| \leq M(\|\nabla u_0\|, \kappa, \varkappa, \nu),$$

From (4.11) and (4.12) we conclude the existence of the absorbing set

$$B = \left\{(u,\theta) \in V, \qquad \|\nabla u\|^2 \leq \frac{|\Omega|^{\frac{1}{2}}}{\nu\mu}, \qquad \|\nabla\theta\|^2 \leq M(\|\nabla u_0\|, \kappa, \varkappa, \nu)\right\}$$

for the semigroup $S(t)$.

We now proceed to prove the compactness of the semigroup $S(t)$. First recall the following theorems.

Theorem 4. *(See e.g.*[6], [13]*) If a semigroup $S(t),\ t\in\mathbb{R}^+$ acts on Banach space X, and $S(t)$ can be decomposed in the sum $W(t)+U(t), W(t)\ t\in\mathbb{R}^+$ is a family of operators such that*

$$\|W(t)(B)\|_X\le m_1(t)m_2(\|B\|_X)$$

where $m_k:\mathbb{R}^+\to\mathbb{R}^+$ are continuous for $k=\ 1,2$ and $m_1(t)\to 0$ when $t\to+\infty$,
$\|B\|_X=Sup_{x\in B}\|x\|_X$ while $U(t)$ for all $t\in\mathbb{R}^+$ maps bounded sets into precompact sets; then $S\ (t)$ is asymptotically compact semigroup.

Theorem 5. *([6],[13])Let $S(t):X\to X, t\in\mathbb{R}^+$ be a continuous bounded point dissipative asymptotically compact semigroup. Then for this semigroup there exist a nonempty minimal global attractor M. It is compact, invariant and connected.*

Now we will prove that semigroup $S(t)$ is asymptotically compact. Using Theorem 4, it is clear that the solution $\big(u(t),\theta(t)\big)$ can be decomposed as

$$u(t)=y(t)+z(t),\qquad \theta=v(t)+w(t)$$

where $\big(y(t),v(t)\big)$ satisfy

(4.13) $$y_t+\nu A_1y+\mu A_1y_t=e_3v$$

(4.14) $$v_t+\kappa A_2v+\varkappa A_1v_t=y_3$$

(4.15) $$y(0)=u_0,\qquad v(0)=\theta_0$$

with in which A_1 and A_2 are the operators previously defined. $(z\ (t),w\ (t))$ satisfy the following equations

(4.16) $$z_t+\nu A_1z+\mu A_1z_t=e_3w-B(u,u)$$

(4.17) $$w_t+\kappa A_2w+\varkappa A_2w_t=z_3-B(u,\theta)$$

(4.18) $$z(0)=0,\qquad w(0)=0$$

where $B(u,u)=P_1(u.\nabla v), B(u,\theta)=P_2(u.\nabla\theta)$, P_i are the projections from $L^2(\Omega)$ onto $H_i(i=1{,}2)$.

The semigroup $S(t)$ has the representation

$$S\ (t)=\xi(t)+\zeta(t)$$

where $\xi(t)$ is the semigroup generated by (4.13)-(4.15), $\zeta(t)$ is a solution of problem (4.16)-(4.18). Multiplying the equation (4.13), (4.14) by y and v respectively and integrating over Ω and using some well known inequalities we get

$$\frac{d}{dt}(\|y\|^2+\mu\|\nabla y\|^2)+\nu\|\nabla y\|^2\le\frac{1}{\nu}\|\nabla v\|^2$$
$$\frac{d}{dt}(\|v\|^2+\varkappa\|\nabla v\|^2)+\kappa\|\nabla v\|^2\le\frac{1}{\nu}\|\nabla y\|^2.$$

Let us add the above two inequalities up and use Poincaré inequality to get

$$\frac{d}{dt}(\|y\|^2+\mu\|\nabla y\|^2+\|v\|^2+\varkappa\|\nabla v\|^2)+\gamma_1(\|y\|^2+\mu\|\nabla y\|^2+\|v\|^2+\varkappa\|\nabla v\|^2)\le 0$$

where

$$\gamma_1=\min\left(\frac{\nu}{2},\frac{\kappa}{2},\frac{\nu\kappa-2}{2\mu\kappa},\frac{\nu\kappa-2}{2\varkappa\nu}\right)>0.$$

Thus we find

$$\|y\|^2+\mu\|\nabla y\|^2+\|v\|^2+\varkappa\|\nabla v\|^2\le e^{-\gamma t}[\|u_0\|^2+\mu\|\nabla u_0\|^2+\|\theta_0\|^2+\varkappa\|\nabla\theta_0\|^2].$$

Since $(u_0,\theta_0)\in V$, the semigroup $\xi(t):V\to V$ is exponentially contractive.

Now we want to show the operator $\zeta(t):V\to V$ which is semigroup generated by the system (4.16)-(4.18) is compact. First recall the following proposition from [9].

Proposition 1. *Let $s\in\mathbb{R}$. If $z_0\in V_s,\ g\in L^2([0,\tau);V_{s-2})$ then the linear problem*

$$z_t + \nu A_1 z + \mu A_1 z_t = g(t), \qquad z(0) = z_0$$

has a unique weak solution which belong to $C([0,\tau); V_s)$ *and the following inequality holds*

$$sup_{t\in[0,T)} \|z(t)\| \leq c\|g\|_{L^2([0,\tau);V_{s-2})}, \qquad s \in \mathbb{R}.$$

To use the Proposition 1, let us define

$$g_1 = e_3 - B(u,u), \qquad g_2 = z_3 - B(u,\theta); \;\; G = (g_1, g_2)$$

We will show

$$G = (g_1, g_2) \in \; L^2\left([0,\tau); V_{-\frac{1}{2}}\right).$$

But it was shown in [9] that

(4.19) $\qquad B(u,u) \in L^\infty\left(\mathbb{R}^+; (V_1)_{-\frac{1}{2}}\right).$

It is easy to Show

(4.20) $\qquad B(u,\theta) \in L^\infty\left(\mathbb{R}^+; (V_2)_{-\frac{1}{2}}\right)$

by a similar computation. We have shown previously that

(4.21) $\qquad (u,\theta) \in L^\infty(0,\tau; V).$

Using (4.19)-(4.21) we conclude

$$G = (g_1, g_2) \in \; L^2\left([0,\tau); V_{-\frac{1}{2}}\right)$$

Hence considering Proposition 1 (z,w) belongs to $C\left([0,\tau); V_{\frac{3}{2}}\right)$. Since the embedding $V_{\frac{3}{2}} \subset V$ is compact, $\zeta(t)$ is a compact operator for all $t > 0$. And from the Theorem 4 we obtain $S(t)$ is asymptotically compact semigroup. From the Theorem 5 we can say that the semigroup has a compact attractor.

5. ESTIMATE OF THE FRACTAL DIMENSION OF THE ATTRACTOR

Now we want to estimate the dimension of the global attractor. First, we recall the following theorem.

Theorem 6. [14] *Let* B *be a bounded set in a Hilbert space* H, *and let there be defined a map* $V: B \to H$ *such that* $B \subset V(B)$ *and for all* $v, \tilde{v} \in B$

(5.1) $\qquad \|V(v) - V(\tilde{v})\|_H \leq l\|v - \tilde{v}\|_H$

and

(5.2) $\qquad \|Q_N V(v) - Q_N V(\tilde{v})\|_H \leq \delta\|v - \tilde{v}\|_H, \qquad \delta<1$

where Q_N *is the orthogonal projection of* H *onto the subspace* $H_N^\perp$ *of codimension N. Then for the fractal dimension of B the inequality*

(5.3) $\qquad d_F(B) \leq N\, log\frac{8\kappa l^2}{1-\delta^2} \,/\, log\frac{2}{1-\delta^2}$

is true, where κ *is Gaussian constant.*

Let us start to find the estimate of the dimension of the global attractor M. Let (u_1, θ_1) and (u_2, θ_2) be two solutions of the problem $(1.7) - (1.12)$ with

$$u_1(x,0) = u_1^0(x), \; u_2(x,0) = u_2^0(x), \; \theta^1(x,0) = \theta_1^0(x), \; \theta_2(x,0) = \theta_2^0(x)$$

in M. Then from the Theorem 5, it follows that $(u(.,t)\; v(.,t)) \in M$ for all $t \in \mathbb{R}^+$. Let us denote $\varphi = u_1 - u_2$, $\phi = \theta_1 - \theta_2$ which satisfy the equations

(5.4)
$$\varphi_t - \nu\Delta\varphi - \mu\Delta\varphi_t + (\varphi.\nabla)u_1 + u_2\nabla\varphi + \nabla p = e_3\phi$$

(5.5)
$$\phi_t - \kappa\Delta\phi - \varkappa\Delta\phi_t + \varphi\nabla\theta_1 - u_2\nabla\phi = \varphi_3$$

where φ_3 is the third component of φ. Multiplying the equations (5.4) and (5.5) with the φ and ϕ respectively, we obtain.

(5.6)
$$\frac{1}{2}\frac{d}{dt}(\|\varphi\|^2 + \mu\|\nabla\varphi\|^2) + \nu\|\nabla\varphi\|^2 = b(\varphi, u_1, \varphi) + (e_3\phi, \varphi)$$

(5.7)
$$\frac{1}{2}\frac{d}{dt}(\|\phi\|^2 + \varkappa\|\nabla\phi\|^2) + \kappa\|\nabla\phi\|^2 = b(\varphi, \phi, \theta_1) + (\varphi_3, \phi)$$

The terms of the type $b(u, v, w)$ in the equations (5.6)and (5.7) can be estimated as

(5.8) $|b(\varphi, \varphi, u_1)| \leq c\|\nabla\varphi\|\|\varphi\| \leq \frac{1}{2}c^2\|\varphi\|^2 + \frac{1}{2}\|\nabla\varphi\|^2$

(5.9) $|b(\varphi, \phi, \theta_1)| \leq c\|\nabla\phi\|\|\varphi\| \leq \frac{1}{2}c^2\|\varphi\|^2 + \frac{1}{2}\|\nabla\phi\|^2$

where c is a constant which correspond to

(5.10) $sup_{x\in\Omega}|u_1(x)| \leq c, sup_{x\in\Omega}|\theta_1(x)| \leq c \; for \; the \; elements \; (u_1, \theta_1) \; in \; M.$

Using (5.8)-(5.9), Schwarz inequality and Young inequality in (5.6) and (5.7) we get

$$\frac{1}{2}\frac{d}{dt}(\|\varphi\|^2 + \mu\|\nabla\varphi\|^2) + \nu\|\nabla\varphi\|^2 \leq \frac{1}{2}c^2\|\varphi\|^2 + \frac{1}{2}\|\nabla\varphi\|^2 + \frac{1}{2}\|\phi\|^2 + \frac{1}{2}\|\varphi\|^2$$
$$\frac{1}{2}\frac{d}{dt}(\|\phi\|^2 + \varkappa\|\nabla\phi\|^2) + \kappa\|\nabla\phi\|^2 \leq \frac{1}{2}c^2\|\varphi\|^2 + \frac{1}{2}\|\nabla\phi\|^2 + \frac{1}{2}\|\phi\|^2 + \frac{1}{2}\|\varphi\|^2$$

From these inequalities we obtain

(5.11)
$$\frac{d}{dt}(\|\varphi\|^2 + \mu\|\nabla\varphi\|^2 + \|\phi\|^2 + \varkappa\|\nabla\phi\|^2) + 2\nu\|\nabla\varphi\|^2 + 2\kappa\|\nabla\phi\|^2 \leq (2 + 2c^2)\|\varphi\|^2 + 2\|\phi\|^2 + \frac{\mu\|\varphi\|^2}{\mu} + \frac{\varkappa\|\nabla\phi\|^2}{\varkappa}$$

Choosing

$$k_1 = max(2 + 2c^2, \frac{1}{\mu}, \frac{1}{\varkappa})$$

and dropping last two terms in the left hand side of (5.11) we write

(5.12)
$$\frac{d}{dt}(\|\varphi\|^2 + \mu\|\nabla\varphi\|^2 + \|\phi\|^2 + \varkappa\|\nabla\phi\|^2) \leq k_1(\|\varphi\|^2 + \mu\|\nabla\varphi\|^2 + \|\phi\|^2 + \varkappa\|\nabla\phi\|^2).$$

Integrating the equation (5.12), we get

(5.13) $\|\varphi\|^2 + \mu\|\nabla\varphi\|^2 + \|\phi\|^2 + \varkappa\|\nabla\phi\|^2 \leq (\|\varphi(0)\|^2 + \mu\|\nabla\varphi(0)\|^2 + \|\phi(0)\|^2 + \varkappa\|\nabla\phi(0)\|^2)e^{k_1 t}$

Using Poincaré inequality in (5.13) we write

(5.14) $\|\nabla\varphi\|^2 + \|\nabla\phi\|^2 \leq 2((1+\mu)\|\nabla\varphi(0)\|^2 + (1+\varkappa)\|\nabla\phi(0)\|^2)^{\frac{1}{2}}e^{\frac{k_1 t}{2}}.$

Let P_N be the orthogonal projection of V onto V_N which is spanned by the first N basis elements of V. Multiplying the equation (1.7), (1.8) in $L^2(\Omega)$ by $Q_N(\varphi)=(I-P_N)\varphi$ and $Q_N(\phi)=(I-P_N)\phi$ respectively we obtain

(5.15)
$$\frac{1}{2}\frac{d}{dt}(\|Q_N\varphi\|^2+\mu\|\nabla Q_N\varphi\|^2)+\nu\|\nabla Q_N\varphi\|^2\le b(\varphi,Q_N\varphi,u_1)+b(u_2,Q_N\varphi,\varphi)+(e_3\phi,\ Q_N\varphi)$$

(5.16)
$$\frac{1}{2}\frac{d}{dt}(\|Q_N\phi\|^2+\varkappa\|\nabla Q_N\phi\|^2)+\kappa\|\nabla Q_N\phi\|^2\le b(\varphi,Q_N\phi,\theta_1)+b(u_2,Q_N\phi,\phi)+(\varphi^3,\ Q_N\phi).$$

The first two terms of the right hand side of (5.15) and (5.16) may be estimated as in the computations given for (5.8) and (5.9) :

(5.17) $$|b(\varphi,Q_N\varphi,u_1)+b(u_2,Q_N\varphi,\varphi)|\le 2c\|\nabla Q_N\varphi\|\|\varphi\|$$

(5.18) $$|b(\varphi,\ Q_N\phi,\theta_1)+b(u_2,\ Q_N\phi,\ \phi)|\le 2c\|\nabla Q_N\phi\|(\|\varphi\|+\|\phi\|)$$

where c is a constant same as in (5.10). Using Schwarz and Young inequalities together with (5.17), (5.18) in (5.15), (5.16), we get

(5.19)
$$\frac{d}{dt}(\|Q_N\varphi\|^2+\mu\|\nabla Q_N\varphi\|^2)+2\nu\|\nabla Q_N\varphi\|^2\le c^2\|\varphi\|^2+2\|\nabla Q_N\varphi\|^2+\frac{1}{2}\|\phi\|^2$$

(5.20)
$$\frac{d}{dt}(\|Q_N\phi\|^2+\varkappa\|\nabla Q_N\phi\|^2)+2\kappa\|\nabla Q_N\phi\|^2\le(c^2+1)\|\varphi\|^2+2\|\phi\|^2+3\|\nabla Q_N\phi\|^2$$

Employing

$$\|\nabla Q_N\psi\|\le\lambda_{N+1}^{-\frac{1}{2}}\|\nabla\ Q_N\psi\|\qquad\forall\ \psi\in V_N^{\perp},$$

in (5.19) and (5.20) and summing them up we write

(5.21)
$$\begin{gathered}\frac{d}{dt}(\|Q_N\varphi\|^2+\mu\|\nabla Q_N\varphi\|^2+\|Q_N\phi\|^2+\varkappa\|\nabla Q_N\phi\|^2)+\\ k_2(\|Q_N\varphi\|^2+\mu\|\nabla Q_N\varphi\|^2+\|Q_N\phi\|^2+\varkappa\|\nabla Q_N\phi\|^2)\\ \le(2c^2+1)\|\varphi\|^2+3\|\varphi\|^2\end{gathered}$$

where

$$k_2=\min(\nu\lambda_{N+!},\frac{\nu-2}{\mu},\ \kappa\lambda_{N+!},\frac{\kappa-3)}{\varkappa})>0.$$

We integrate (5.21) to get

$$\begin{gathered}\|Q_N\varphi\|^2+\mu\|\nabla Q_N\varphi\|^2+\|Q_N\phi\|^2+\varkappa\|\nabla Q_N\phi\|^2\le\\ (\|Q_N\varphi(0)\|^2+\mu\|\nabla Q_N\varphi(0)\|^2+\|Q_N\phi(0)\|^2+\varkappa\|\nabla Q_N\phi(0)\|^2)e^{-k_2t}\\ +(2c^2+1)\,e^{-k_2t}\int_0^t\|\varphi(\tau)\|^2\,e^{-k_2\tau}\,d\tau+3\,e^{-k_2t}\int_0^t\|\phi(\tau)\|^2\,e^{-k_2\tau}\,d\tau\end{gathered}$$

In the last inequality using the majorant in (5.14) for $\|\varphi(\tau)\|^2$ and $\|\phi(\tau)\|^2$ and after some calculations we obtain,

(5.22) $$\|\nabla Q_N\varphi\|^2+\|\nabla Q_N\phi\|^2\le 2((1+\mu)\,\|\nabla\varphi(0)\|^2+(1+\varkappa)\|\nabla\phi(0)\|^2)e^{-k_2t}$$
$$\{1+\frac{e^{(k_1+k_2)t}-1}{k_1+k_2}(2c^2+4)\}.$$

Now we select t_0 and N such that

$$2e^{-k_2t_0}\{1+\frac{e^{(k_1+k_2)t_0-1}}{k_1+k_2}(2c^2+4)\}\le\delta<1.$$

So the conditions of the Theorem 6 are satisfied. Thus we have established the following theorem.

Theorem 7. *Let the conditions of Theorem 6 be satisfied and $\nu>2$, $\kappa>3$. Then the attractor of the semigroup*

$$S(t):V\to V$$

has a finite fractal dimension.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Birnir, B and Svanstedt, N., Existence theory and strong attractors for the Rayleigh-Benard problem with a large aspect ratio, Discrete and Continuous Dynamical Systems,Vol.10 no1&2, 53-74, (2004).

[2] Cabral, M., Rosa, R. and Temam, R., Existence and dimension of the attractor for Benard problem on channel-like domains, Discrete and Continuous Dynamical Systems, vol.10,N.1&2, 89-116 (2004).

[3] Constantin, P. and Foias, C., Navier-Stokes equations, Chicago Lectures in Mathematics, The University of Chicago, (1988).

[4] Foias, C., Manley, O. and Temam, R., Attractors for the Benard problem: existence and physical bounds on their fractal dimension, Nonlinear Analysis Theory, Methods and Applications, Vol.11.No.8, 939-967 (1987).

[5] Galdi, G. P., Lectures in Mathematical Fluid Dynamics, Birkhäuser-Verlag, (2000).

[6] Hale, J.K., Asymptotic Behaviour of dissipative Systems, Mathematical Survey and Monographs, 25 American Mathematical Society, Providence, RI, (1988).

[7] Kalantorov, V.K., Attractors for some nonlinear problems of physics, Zap. Nauchn. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 152, 50-54, (1986).

[8] Kalantarov, V.K., Levant, B. and Titi, E. S., Gevrey regularity for the attractor of 3D Navier-Stokes-Voight equations, Journal of Nonlinear Science,19, 133-152, (2009).

[9] Kalantarov, V.K. and Titi, E.S., Global attractors and determining modes for the 3D Navier-Stokes-Voight equations, Chin. Ann. Math. Ser.B 30, no 6, 697-714 (2009) .

[10] Kapusyan, O.V., Melnik, V.S., and Valero, J., A Weak Attractor and Properties of solutions for the three-dimensional Benard Problem, Discrete and Continuous Dynamical Systems, Vol.18, N 2&3, 449-481, (2007).

[11] Khouider, B. and Titi, E.S. An Inviscid regularization for the surface quasi geostrophic equation, Comm. Pure Appl. Math., 61 No 10, 1331-1346, (2008).

[12] Ladyzhenskaya, O.A., The Mathematical Theory of viscous Incompressible Flow, Gordon and Breach, (1969).

[13] Ladyhenskaya, O.A., On the determination of minimal global attractors for the Navier Stokes and other partial differential equations, Uspekhi Math. Nauk, 42:6, 25-60, (1987), Russian Math. Survey, 42:6, 27-73 (1987).

[14] Ladyhenskaya, O.A., Finite dimensionality of bounded invariant sets for Navier-Stokes systems and other dissipative systems, Zap. Nauchn. Sem. LOMI 115, 137-155, (1982), J.Soviet Math., 28, 714-726, (1985).

[15] Morimoto, H., Non-stationary Boussinesq equations, J. Fac. Sci. Univ. Tokyo, Sect. IA. Math. 39, 61-75 (1992).

[16] Oskolkov, A.P., The uniqueness and Global Solvability of boundary-value problems for the equations of motion for aqueous solutions of polymers, Journal Soviet Mathematics, 8, No 4, 427-455, (1977).

[17] Robinson, J.C., Infinite-dimensional Dynamical Systems, Cambridge University Press "Text in Applied Mathematics" Series, (2001).

[18] Temam, R., Navier-Stokes equations: Theory and numerical analysis, North-Holland-Amsterdam, (1984).

[19] Temam, R., Infinite Dimensional Dynamical System in Mechanic and Physics, Springer Verlag, (1997).