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On the Norms of Some Special Matrices with the Harmonic Fibonacci Numbers

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ABSTRACT

The aim of this paper is to study norms of some circulant matrices and some special matrices, which entries consist of harmonic Fibonacci numbers.

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1. INTRODUCTION

The Fibonacci sequence plays an important role in applied mathematics, number theory and many other areas. The Fibonacci sequence is defined by the following recurrence relation, for $n \geq 1$;

$$F_{n+1} = F_n + F_{n-1}$$

where $F_0 = 0$, $F_1 = 1$. In [1], Tuglu et al. investigated finite sum of the reciprocal Fibonacci numbers

$$\mathrm{F}_n = \sum_{k=1}^{n} \frac{1}{F_k}$$

which is called harmonic Fibonacci numbers. Then the authors gave some combinatorics properties of harmonic Fibonacci numbers as follows:

$$\sum_{k=0}^{n-1} \binom{k}{m} \mathrm{F}_k = \binom{n}{m+1} \mathrm{F}_n - \sum_{k=0}^{n-1} \binom{k+1}{m+1} \frac{1}{F_{k+1}}, \quad (1.1)$$

$$\sum_{k=0}^{n-1} \frac{\mathrm{F}_k}{k+1} = H_n \mathrm{F}_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}}, \quad (1.2)$$

$$\sum_{k=1}^{n} \mathrm{F}_k = (n+1)\mathrm{F}_{n+1} - \sum_{k=0}^{n} \frac{k+1}{F_{k+1}}, \quad (1.3)$$

$$\sum_{k=1}^{n} \mathrm{F}_k^2 = (n+1)\mathrm{F}_{n+1}^2 - \sum_{k=0}^{n} \frac{k+1}{F_{k+1}} \left(2\mathrm{F}_k + \frac{1}{F_{k+1}}\right), \quad (1.4)$$

where m is a nonnegative integer and H_n is the n^{th} harmonic number.

Recently, there have been many papers on the norms of some special matrices [2-6]. For example in [2], Solak has given some bounds for the circulant matrices with classical Fibonacci and Lucas numbers entries. In [3], Kocer et al. obtained norms of circulant and semicirculant matrices with Horadams numbers. In [4], Shen gave upper and lower bounds for the Toeplitz matrices involving $k-$Fibonacci and $k-$Lucas numbers. In

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[5], Bahşi and Solak defined special matrices with harmonic numbers and they gave spectral norms, Euclidean norms and determinants of these matrices. In [6], Zhou et al. gave spectral norms of circulant matrices involving binomial coefficients and harmonic numbers.

Motivated by the above papers, we investigate spectral norms of circulant matrices involving harmonic Fibonacci numbers. Then we give Euclidean norms of some special matrices with harmonic Fibonacci numbers. Now we give some definitions and lemmas related to our study.

Definition 1. [7] A circulant matrix is an $n \times n$ matrix with the following form:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

Obviously, the circulant matrix C is determined by its first row elements $c_0, c_1, c_2, \ldots, c_{n-1}$, thus we denote $C = Circ(c_0, c_1, c_2, \ldots, c_{n-1})$. Let $A = (a_{ij})$ be any $m \times n$ matrix. The well known Euclidean norm of matrix A is

$$\|A\|_E = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2},$$

and also the spectral norm of A matrix is

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(A^H A)},$$

where $\lambda_i(A^H A)$ is an eigenvalue of $A^H A$ and A^H is conjugate transpose of matrix A. Then the following inequality holds:

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E . \quad (1.5)$$

In [8], finite difference of $f(x)$ is defined as

$$\Delta f(x) = f(x+1) - f(x) .$$

Δ operator has an inverse, the anti-difference operator $\sum$ defined as follows. Let a and b are integers with $b \ge a$. If $\Delta f(x) = g(x)$ then

$$\sum_a^b g(x)\delta_x = \sum_{x=a}^{b-1} g(x) = f(b) - f(a) .$$

Anti-difference operator has some properties as follows:

$$\sum_a^b u(x)\Delta v(x)\delta_x = u(x)v(x)|_a^{b+1} - \sum_a^b v(x+1)\Delta u(x)\delta_x \quad (1.6)$$

and for $m \ne -1$,

$$\sum x^{\underline{m}}\,\delta_x = \frac{x^{\underline{m+1}}}{m+1},$$

where $x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$.

2. MAIN RESULTS

Theorem 1. For m nonnegative integer, the spectral norm of the matrix

$$A = Circ\left(\binom{0}{m}\mathrm{F}_0, \binom{1}{m}\mathrm{F}_1, \ldots, \binom{n-1}{m}\mathrm{F}_{n-1}\right)$$

is

$$\|A\|_2 = \binom{n}{m+1}\mathrm{F}_n - \sum_{k=0}^{n-1}\binom{k+1}{m+1}\frac{1}{F_{k+1}} .$$

Proof. Since the circulant matrix A is normal, its spectral norm equal to spectral radius. Furthermore, A is irreducible and its entries are nonnegative the spectral radius of the A matrix is equal to its Perron roots. Let υ denote all ones vectors of order n $(\upsilon = (1,1,\ldots,1)^T)$. Then

$$A\upsilon = \left(\sum_{k=0}^{n-1}\binom{k}{m}\mathrm{F}_k\right)\upsilon .$$

As, $\sum_{k=0}^{n-1}\binom{k}{m}\mathrm{F}_k$ is an eigenvalue of A associated with a positive eigenvector, it is necessarily the Perron value of A. Therefore from the (1.1), we have

$$\|A\|_2 = \binom{n}{m+1}\mathrm{F}_n - \sum_{k=0}^{n-1}\binom{k+1}{m+1}\frac{1}{F_{k+1}} .$$

Corollary 1. We have

$$\sqrt{\sum_{k=0}^{n-1}\binom{k}{m}^2\mathrm{F}_k^2} \le \binom{n}{m+1}\mathrm{F}_n - \sum_{k=0}^{n-1}\binom{k+1}{m+1}\frac{1}{F_{k+1}} \le \sqrt{n\sum_{k=0}^{n-1}\binom{k}{m}^2\mathrm{F}_k^2} .$$

Proof. The proof is trivial from the definition of the Euclidean norm and the relation between Euclidean norm and spectral norm in (1.5).

Theorem 2. The spectral norm of the matrix

$$B = Circ\left(\frac{\mathrm{F}_0}{1}, \frac{\mathrm{F}_1}{2}, \ldots, \frac{\mathrm{F}_{n-1}}{n}\right)$$

is

$$\|B\|_2 = H_n \, \mathrm{F}_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}}.$$

Proof. Analysis similar to that in the proof of Theorem 1 shows that

$$\|B\|_2 = \sum_{k=0}^{n-1} \frac{\mathrm{F}_k}{k+1}$$

and from the (1.2)

$$\|B\|_2 = H_n \, \mathrm{F}_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}} .$$

Corollary 2. We have

$$\sqrt{\sum_{k=0}^{n-1} \frac{\mathrm{F}_k^2}{(k+1)^2}} \le H_n \, \mathrm{F}_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}} \le \sqrt{n \sum_{k=0}^{n-1} \frac{\mathrm{F}_k^2}{(k+1)^2}} .$$

Proof. The proof is trivial from the definition of the Euclidean norm and the relation between Euclidean norm and spectral norm in (1.5).

Lemma 1. For the harmonic Fibonacci numbers, we have

$$\sum_{k=1}^{n} k \, \mathrm{F}_k^2 = \frac{(n+1)^{\underline{2}}}{2} \, \mathrm{F}_{n+1}^2 - \frac{1}{2} \sum_{k=1}^{n} \frac{(k+1)^{\underline{2}}}{F_{k+1}} \left(2 \, \mathrm{F}_k + \frac{1}{F_{k+1}} \right) .$$

Proof. The proof is based on the properties of difference operator. Let $u(k) = \mathrm{F}_k^2$ and $\Delta v(k) = k^{\underline{1}}$ be in (1.6). Then we obtain $\Delta u(k) = \frac{1}{F_{k+1}} \left(2 \, \mathrm{F}_k + \frac{1}{F_{k+1}} \right)$ and $v(k) = \frac{k^{\underline{2}}}{2}$. By using the equation (1.6), we have

$$\sum_{k=1}^{n} k \, \mathrm{F}_k^2 = \frac{(n+1)^{\underline{2}}}{2} \, \mathrm{F}_{n+1}^2 - \frac{1}{2} \sum_{k=1}^{n} \frac{(k+1)^{\underline{2}}}{F_{k+1}} \left(2 \, \mathrm{F}_k + \frac{1}{F_{k+1}} \right).$$

Lemma 2. For the harmonic Fibonacci numbers, we have

$$\sum_{k=1}^{n-1} (n-k) \, \mathrm{F}_{n+k}^2 = \frac{(n+1)^{\underline{2}}}{2} \, \mathrm{F}_{2n}^2 - n \, \mathrm{F}_{n+1}^2 - \frac{1}{2} \sum_{k=1}^{n-1} \frac{(k+1)(2n-k)}{F_{n+k+1}} \left(2 \, \mathrm{F}_{n+k} + \frac{1}{F_{n+k+1}} \right)$$

Proof. Repeated application of equation (1.6) enables us to write $u(k) = \mathrm{F}_{n+k}^2$ and $\Delta v(k) = n-k$. Then we obtain $\Delta u(k) = \frac{1}{F_{n+k+1}} \left(2 \, \mathrm{F}_{n+k} + \frac{1}{F_{n+k+1}} \right)$ and $v(k) = nk - \frac{k^{\underline{2}}}{2}$. By using the equation (1.6), we have

$$\sum_{k=1}^{n-1} (n-k) \, \mathrm{F}_{n+k}^2 = \frac{(n+1)^{\underline{2}}}{2} \, \mathrm{F}_{2n}^2 - n \, \mathrm{F}_{n+1}^2 - \frac{1}{2} \sum_{k=1}^{n-1} \frac{(k+1)(2n-k)}{F_{n+k+1}} \left(2 \, \mathrm{F}_{n+k} + \frac{1}{F_{n+k+1}} \right) .$$

Lemma 3. For the harmonic Fibonacci numbers, we have

$$\sum_{k=1}^{n-1} 2^{k-1} \, \mathrm{F}_k^2 = 2^{n-1} \, \mathrm{F}_n^2 - \sum_{k=0}^{n-1} \frac{2^k}{F_{k+1}} \left(2 \, \mathrm{F}_k + \frac{1}{F_{k+1}} \right).$$

Proof. Let $u(k) = \mathrm{F}_k^2$ and $\Delta v(k) = 2^{k-1}$ be in (1.6). Then we obtain $\Delta u(k) = \frac{1}{F_{k+1}} \left(2 \, \mathrm{F}_k + \frac{1}{F_{k+1}} \right)$ and $v(k) = 2^{k-1}$. By using the equation (1.6), we have

$$\sum_{k=1}^{n-1} 2^{k-1} \, \mathrm{F}_k^2 = 2^{n-1} \, \mathrm{F}_n^2 - \sum_{k=0}^{n-1} \frac{2^k}{F_{k+1}} \left(2 \, \mathrm{F}_k + \frac{1}{F_{k+1}} \right).$$

Definition 2. Let $P = \left[\mathrm{F}_i \right]_{i=1}^{n}$, $H = \left[\mathrm{F}_{i+j-1} \right]_{i,j=1}^{n}$ and $R = \left[\sqrt{\binom{i-1}{n-j}} \, \mathrm{F}_i \right]_{i=1}^{n}$ are matrices, which entries consist of harmonic Fibonacci numbers such that these matrices are

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & 2 & \cdots & 2 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \cdots & \frac{5}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathrm{F}_n & \mathrm{F}_n & \mathrm{F}_n & \cdots & \mathrm{F}_n \end{pmatrix}, \tag{2.1}$$

$$H = \begin{pmatrix} \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 & \cdots & \mathrm{F}_n \\ \mathrm{F}_2 & \mathrm{F}_3 & \mathrm{F}_4 & \cdots & \mathrm{F}_{n+1} \\ \mathrm{F}_3 & \mathrm{F}_4 & \mathrm{F}_5 & \cdots & \mathrm{F}_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathrm{F}_n & \mathrm{F}_{n+1} & \mathrm{F}_{n+2} & \cdots & \mathrm{F}_{2n+1} \end{pmatrix}, \tag{2.2}$$

$$R = \begin{pmatrix} 0 & \cdots & 0 & 0 & \mathrm{F}_1 \\ 0 & \cdots & 0 & \mathrm{F}_2 & \mathrm{F}_2 \\ 0 & \cdots & \mathrm{F}_3 & \sqrt{2}\,\mathrm{F}_3 & \mathrm{F}_3 \\ \vdots & & \vdots & \vdots & \vdots \\ \mathrm{F}_n & \cdots & \sqrt{\binom{n-1}{2}} \, \mathrm{F}_n & \sqrt{\binom{n-1}{1}} \, \mathrm{F}_n & \mathrm{F}_n \end{pmatrix}. \tag{2.3}$$

Now, we give some theorems on the norms of these matrices by using the difference operator.

Theorem 3. The eigenvalues of the $n \times n$ matrix P are

$$\lambda_1 = (n+1)\,\mathrm{F}_{n+1} - \sum_{k=0}^{n} \frac{k+1}{F_{k+1}},$$

and

$$\lambda_m = 0$$

for $m = 2,3,\dots,n$.

Proof. The eigenvalues of the matrix P are roots of the $|\lambda I - P| = 0$ such that

$$|\lambda I - P| = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -2 & \lambda-2 & -2 & \cdots & -2 \\ -\frac{5}{2} & -\frac{5}{2} & \lambda-\frac{5}{2} & \cdots & -\frac{5}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ -\mathrm{F}_n & -\mathrm{F}_n & -\mathrm{F}_n & \cdots & \lambda-\mathrm{F}_n \end{vmatrix}$$

From the properties of determinant we can easy see

$$|\lambda I - P| = \begin{vmatrix} \lambda-1 & -\lambda & -\lambda & \cdots & -\lambda \\ -2 & \lambda & 0 & \cdots & 0 \\ -\frac{5}{2} & -\frac{5}{2} & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\mathrm{F}_n & 0 & 0 & \cdots & \lambda \end{vmatrix}.$$

We calculate the determinant, we obtain

$$|\lambda I - P| = \lambda^{n-1}\left(\lambda - 1 - 2 - \frac{5}{2} - \cdots - \mathrm{F}_n\right).$$

If we solve the characteristic equation

$$\lambda^{n-1}\left(\lambda - 1 - 2 - \frac{5}{2} - \cdots - \mathrm{F}_n\right) = 0.$$

The eigenvalues of the matrix P are

$$\lambda_1 = 1 + 2 + \frac{5}{2} - \cdots + \mathrm{F}_n$$

$$= \sum_{k=1}^{n} \mathrm{F}_k$$

$$= (n+1)\,\mathrm{F}_n - \sum_{k=0}^{n} \frac{k+1}{F_{k+1}}$$

and

$$\lambda_m = 0$$

for $m = 2,3,\dots,n$.

Theorem 4. The Euclidean norm of the matrix P is

$$\|P\|_E = \sqrt{n(n+1)\,\mathrm{F}_{n+1}^2 - n\sum_{k=0}^{n} \frac{k+1}{F_{k+1}}\left(2\,\mathrm{F}_k + \frac{1}{F_{k+1}}\right)}.$$

Proof. Definition of the Euclidean norm and from the (1.4), we have

$$\|P\|_E = \sqrt{n\sum_{k=1}^{n} \mathrm{F}_k^2}$$

$$= \sqrt{n(n+1)\,\mathrm{F}_{n+1}^2 - n\sum_{k=0}^{n} \frac{k+1}{F_{k+1}}\left(2\,\mathrm{F}_k + \frac{1}{F_{k+1}}\right)}.$$

Theorem 5. The spectral norm P is

$$\|P\|_2 = \sqrt{n(n+1)\,\mathrm{F}_{n+1}^2 - n\sum_{k=0}^{n} \frac{k+1}{F_{k+1}}\left(2\,\mathrm{F}_k + \frac{1}{F_{k+1}}\right)}.$$

Proof. The spectral norm of P is

$$\|P\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(P^H P)}$$

where $\lambda_i(P^H P)$ are eigenvalues of $P^H P$ and P^H is conjugate transpose of P. Therefore

$$P^H P = \begin{pmatrix} \sum_{k=1}^{n} \mathrm{F}_k^2 & \sum_{k=1}^{n} \mathrm{F}_k^2 & \cdots & \sum_{k=1}^{n} \mathrm{F}_k^2 \\ \sum_{k=1}^{n} \mathrm{F}_k^2 & \sum_{k=1}^{n} \mathrm{F}_k^2 & \cdots & \sum_{k=1}^{n} \mathrm{F}_k^2 \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^{n} \mathrm{F}_k^2 & \sum_{k=1}^{n} \mathrm{F}_k^2 & \cdots & \sum_{k=1}^{n} \mathrm{F}_k^2 \end{pmatrix}$$

The eigenvalues of the matrix $P^H P$ are

$$\lambda_1 = \sum_{k=1}^{n} \mathrm{F}_k^2$$

and

$$\lambda_m = 0$$

where $m = 2,3,\dots,n$. From (1.4) we obtain

$$\|P\|_2 = \sqrt{n(n+1)\,\mathrm{F}_{n+1}^2 - n\sum_{k=0}^{n} \frac{k+1}{F_{k+1}}\left(2\,\mathrm{F}_k + \frac{1}{F_{k+1}}\right)}.$$

Theorem 6. For the Euclidean norm of the H matrix

$$\|H\|_E^2 = \frac{(n+1)^{\underline{2}}}{2} \mathrm{F}_{2n}^2 + \frac{n^{\underline{2}}}{2} \mathrm{F}_{n+1}^2 - \frac{1}{2}\left[\frac{(n+1)^{\underline{2}}}{F_{n+1}}\left(2\,\mathrm{F}_n + \frac{1}{F_{n+1}}\right)\right] - \frac{1}{2}\sum_{k=1}^{n-1}(k+1)\left[\frac{k}{F_{k+1}}\left(2\,\mathrm{F}_k + \frac{1}{F_{k+1}}\right) + \frac{2n-k}{F_{n+k+1}}\left(2\,\mathrm{F}_{n+k} + \frac{1}{F_{n+k+1}}\right)\right]$$

is valid.

Proof. Definition of Euclidean norm

$$\|H\|_E^2 = \sum_{k=1}^{n} k\,\mathrm{F}_k^2 + \sum_{k=1}^{n-1}(n-k)\,\mathrm{F}_{n+k}^2 .$$

Therefore from the Lemma 1 and Lemma 2

$$\|H\|_E^2 = \frac{(n+1)^{\underline{2}}}{2} \mathrm{F}_{2n}^2 + \frac{n^{\underline{2}}}{2} \mathrm{F}_{n+1}^2 - \frac{1}{2}\left[\frac{(n+1)^{\underline{2}}}{F_{n+1}}\left(2\,\mathrm{F}_n + \frac{1}{F_{n+1}}\right)\right] - \frac{1}{2}\sum_{k=1}^{n-1}(k+1)\left[\frac{k}{F_{k+1}}\left(2\,\mathrm{F}_k + \frac{1}{F_{k+1}}\right) + \frac{2n-k}{F_{n+k+1}}\left(2\,\mathrm{F}_{n+k} + \frac{1}{F_{n+k+1}}\right)\right].$$

Theorem 7. The Euclidean norm of the matrix R is

$$\|R\|_E = \sqrt{2^{n-1}\,\mathrm{F}_n^2 - \sum_{k=0}^{n-1}\frac{2^k}{F_{k+1}}\left(2\,\mathrm{F}_k + \frac{1}{F_{k+1}}\right)}.$$

Proof. From the definition of Euclidean norm

$$\|\mathrm{R}\|_{\mathrm{E}}^2 = \sum_{k=1}^{n-1} 2^{k-1}\,\mathrm{F}_k^2 .$$

By using the Lemma 3, we obtain

$$\|R\|_E = \sqrt{2^{n-1}\,\mathrm{F}_n^2 - \sum_{k=0}^{n-1}\frac{2^k}{F_{k+1}}\left(2\,\mathrm{F}_k + \frac{1}{F_{k+1}}\right)}.$$

CONFLICT OF INTEREST

The authors declare that there is no conflict of interests regarding the publication of this paper.

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