Gazi University Journal of Science
GU J Sci
28(4):689-694 (2015)

A New Implicit Block Method for Solving Second Order Ordinary Differential Equations Directly

Zurni OMAR[1], John Olusola KUBOYE[1,♠]

[1] *Department of Mathematics, School of Quantitative Sciences, University Utara Malaysia Sintox, Kedah, Malaysia*

Received: 01/09/2015 *Accepted: 24/11/2015*

ABSTRACT

This article considers the derivation of an implicit block method for the solution of initial value problems of ordinary differential equations directly. The method of interpolation and collocation is adopted in developing the method where approximated power series of the form $y(x)=\sum_{j=0}^{k+2} a_j x^j$ is used as an interpolation polynomial and its second derivative is collocated at the selected grid points where *k*=5. The method developed is zero stable, consistent and convergent. The generated numerical results show that the new method is better when compared with the existing methods of the same step-length in terms of error.

Keywords: *Power series, Interpolation, Collocation, Block Method, Ordinary Differential Equations.*

1. INTRODUCTION

The mathematical formulation of physical phenomena in the field of science and engineering in most cases lead to differential equations which are the building blocks of mathematical modeling (see[1]). This paper focuses on solving second order initial value problems of ordinary differential equations (ODEs) of the form

$$y''=f(x,y,y') \quad y(x_0)=a, y'(x_0)=b, \quad a\le x\le b \qquad (1)$$

The solution of equation (1) can be achieved either by the use of direct method proposed in [2 – 6] or reduction to its equivalent system of first order equations and then suitable numerical for first order will be used to solve the resulting equations [6]. However, these methods compute numerical solution of ODEs at one point at a time which reduces the accuracy of a method.

Block method for solving ODEs concurrently was introduced by Milne 1967 whereby it was previously used as a starting value for predictor-corrector algorithm and later adopted as full method. Some researchers such as Omar [1], Badmus and Yahaya [9], Mohammed [7] and Mohammed and Adeniyi [8] developed block methods for direct solution of second order ODEs. It is observed that the accuracy of the methods is not encouraging.

In order to improve the accuracy of the existing methods, this paper presents new implicit block method for solving second order ODEs directly. Interpolation and collocation approach is adopted in developing the

♠Corresponding author, e-mail: kubbysholly2007@yahoo.com

method. The points of interpolation based on the order of differential equation are made at the two points prior to the last two points while collocation points are chosen at all grid points within the interval of integration.

2. DERIVATION OF THE METHOD

Power series of the form

$$y(x)=\sum_{j=0}^{k+2} a_j x^j \tag{2}$$

is considered as an approximate solution to equation (1). The first and second derivatives of (2) give

$$y'(x)=\sum_{j=1}^{k+2} j a_j x^{j-1} \tag{3}$$

$$y''(x)=\sum_{j=2}^{k+2} j(j-1) a_j x^{j-2} \tag{4}$$

Equation (2) is interpolated at $x=x_{n+i}, i=2{,}3$ and (4) is collocated at $x=x_{n+c}, c=0(1)5$.

Therefore, interpolation and collocation equations at the selected grid points produce

$$\sum_{j=0}^{k+2} a_j x^j = y_{n+i} \tag{5}$$

$$\sum_{j=0}^{k+2} j(j-1) a_j x^{j-2} = f_{n+c} \tag{6}$$

Using Gaussian elimination method, the unknown coefficients $a_j{}'s$ in equations (5) and (6) can be obtained. Substituting the $a_j{}'s$ into (2), this gives a continuous implicit scheme of the form

$$y(z)=\sum_{j=2}^{k-2} \alpha_j(z) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(z) f_{n+j} \tag{7}$$

where $x = zh + x_n + 4h$,

$$\begin{pmatrix}\alpha_2(z)\\ \alpha_3(z)\end{pmatrix}=\begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}z^0\\ z^1\end{pmatrix} \tag{8}$$

$$\begin{pmatrix}\beta_0(z)\\ \beta_1(z)\\ \beta_2(z)\\ \beta_3(z)\\ \beta_4(z)\\ \beta_5(z)\end{pmatrix} = E\begin{pmatrix}z^0\\ z^1\\ z^2\\ z^3\\ z^4\\ z^5\\ z^6\\ z^7\end{pmatrix} \tag{9}$$

where

$$E=\begin{pmatrix}
0 & \frac{-120}{30240} & 0 & \frac{252}{30240} & \frac{105}{30240} & \frac{-63}{30240} & \frac{-42}{30240} & \frac{-6}{30240}\\
0 & \frac{705}{30240} & 0 & \frac{-1680}{30240} & \frac{-630}{30240} & \frac{441}{30240} & \frac{252}{30240} & \frac{30}{30240}\\
\frac{504}{5040} & \frac{52}{5040} & 0 & \frac{840}{5040} & \frac{245}{5040} & \frac{-231}{5040} & \frac{-98}{5040} & \frac{-10}{5040}\\
\frac{4074}{5040} & \frac{5429}{\frac{105}{30240}} & 0 & \frac{-1680}{5040} & \frac{-70}{5040} & \frac{357}{5040} & \frac{112}{5040} & \frac{10}{5040}\\
\frac{3024}{30240} & \frac{12336}{30240} & \frac{15120}{30240} & \frac{5460}{30240} & \frac{-1575}{30240} & \frac{-1575}{30240} & \frac{-378}{30240} & \frac{-10}{30240}\\
\frac{-42}{10080} & \frac{-149}{10080} & 0 & \frac{336}{10080} & \frac{350}{10080} & \frac{147}{10080} & \frac{28}{10080} & \frac{2}{10080}
\end{pmatrix}$$

Equations (8) and (9) are evaluated at the non-interpolating points and this gives

$$J\begin{pmatrix}y_n\\ y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\end{pmatrix} = h^2 G\begin{pmatrix}f_n\\ f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\\ f_{n+5}\end{pmatrix} \tag{10}$$

The values of J and G are given as

$$J=\begin{pmatrix}240 & 0 & -720 & 480 & 0 & 0\\ 0 & 240 & -480 & 240 & 0 & 0\\ 0 & 0 & 240 & -480 & 240 & 0\\ 0 & 0 & 480 & -720 & 0 & 240\end{pmatrix} \text{ and}$$

$$G=\begin{pmatrix}16 & 257 & 392 & 62 & -8 & 1\\ -1 & 24 & 194 & 24 & -1 & 0\\ 0 & -1 & 24 & 194 & 24 & -1\\ 1 & -8 & 62 & 392 & 257 & 16\end{pmatrix}$$

The derivative of (8) and (9) are evaluated at all the grid points. This gives the derivative of (10). Therefore, combining equation (10) and its derivative in a matrix form, then multiply *y* and *f* functions by the inverse of the coefficients of $y_{n+m}, m = 1(1)5$ to give a block of the form:

$$AY_M = by_n + chy_n' + dh^2 f(y_n) + eh^2 F(Y_N) \quad (11)$$

where

$$Y_M = [y_{n+1}, y_{n+2}, \cdots, y_{n+k}]^T, y_n = [y_{n-(k-1)}, y_{n-(k-2)}, \cdots, y_n]^T,$$

$$y_n' = [y'_{n-(k-1)}, y'_{n-(k-2)}, \cdots, y'_n]^T, f(y_n) = [f_{n-(k-1)}, f_{n-(k-2)}, \cdots, f_n]^T$$
$$F(Y_M) = [f_{n+1}, f_{n+2}, \cdots, f_{n+k}]^T,$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{4924}{20160} \\ 0 & 0 & 0 & 0 & \frac{355}{630} \\ 0 & 0 & 0 & 0 & \frac{8856}{10080} \\ 0 & 0 & 0 & 0 & \frac{752}{630} \\ 0 & 0 & 0 & 0 & \frac{1525}{10080} \end{pmatrix},$$

$$c = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{4924}{20160} \\ 0 & 0 & 0 & 0 & \frac{355}{630} \\ 0 & 0 & 0 & 0 & \frac{8856}{10080} \\ 0 & 0 & 0 & 0 & \frac{752}{630} \\ 0 & 0 & 0 & 0 & \frac{1525}{10080} \end{pmatrix},$$

$$e = \begin{pmatrix} \frac{8630}{20160} & \frac{-6088}{20160} & \frac{3764}{20160} & \frac{-1364}{20160} & \frac{214}{20160} \\ \frac{1088}{630} & \frac{-370}{630} & \frac{272}{630} & \frac{-101}{630} & \frac{16}{630} \\ \frac{8856}{10080} & \frac{31509}{10080} & \frac{-6}{10080} & 0 & \frac{8856}{10080} \\ \frac{2848}{630} & \frac{352}{630} & \frac{1216}{630} & \frac{-160}{630} & \frac{32}{630} \\ \frac{59375}{10080} & \frac{12500}{10080} & \frac{31250}{10080} & \frac{6250}{10080} & \frac{1375}{10080} \end{pmatrix}$$

The derivative of block method (11) gives

$$\begin{pmatrix} y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \\ y'_{n+5} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y'_n + Hh \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix}$$

where

$$H = \begin{pmatrix} \frac{1425}{4320} & \frac{4281}{4320} & \frac{-2394}{4320} & \frac{1446}{4320} & \frac{-519}{4320} & \frac{81}{4320} \\ \frac{84}{270} & \frac{387}{270} & \frac{42}{270} & \frac{4}{270} & \frac{-18}{270} & \frac{3}{270} \\ \frac{153}{480} & \frac{657}{480} & \frac{342}{480} & \frac{342}{480} & \frac{63}{480} & \frac{9}{480} \\ \frac{42}{135} & \frac{192}{135} & \frac{72}{135} & \frac{192}{135} & \frac{42}{135} & \frac{42}{135} \\ \frac{285}{864} & \frac{1125}{864} & \frac{750}{864} & \frac{750}{864} & \frac{1125}{864} & \frac{285}{864} \end{pmatrix}$$

3. PROPERTIES OF THE BLOCK METHOD

3.1. Order of the Block Method

The linear difference operator of the block (11) can be defined as

$$L\{y(x):h\} = AY_M - by_n - chy_n' - dh^2 f(y_n) - eh^2 F(Y_N) \quad (12)$$

Expanding Y_M, $f(y_n)$ and $F(Y_N)$ in Taylor series, this produces

$$L\{y(x):h\} = c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) + \\ + \cdots + c_p h^p y^{(p)}(x) + c_{p+1} h^{p+1} y^{(p+1)}(x) + \cdots$$

Definition: The block method (11) together with the associated linear difference operator (12) are said to have order p if $c_0 = c_1 = \cdots c_{p+1} = 0$ and $c_{p+2} \neq 0$. The value c_{p+2} is called the error constant and the truncation error is given as

$$T_{n+k} = c_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3})$$

Therefore, from our computation, the block method (11) has a uniform order $[6,6,6,6,6]^T$ with error constants $\left(\frac{-118}{14345}, \frac{-19}{945}, \frac{-141}{4480}, \frac{-8}{189}, \frac{-653}{11489}\right)^T$

3.2. Zero Stability

The block method (11) is said to be *zero-stable* as $h \to 0$ if its first characteristic polynomial $\rho(z) = \det[Az - b] = 0$ satisfy the condition that $|z| \le 1$ and those roots $|z| = 1$ have multiplicity not greater than the order of the differential equation. Therefore, for the new method we have

$$\rho(z) = z^4 (z-1) = 0$$

$$z = 0,0,0,0,1$$

Hence, the block method (11) is zero stable and consistent since the order is greater than one. Furthermore, since the method is zero-stable and consistent, it implies that the method is convergent (see [5]).

3.3. Interval of Absolute Stability

Using the boundary locus method proposed by Lambert [6]. The new method (11) is absolutely stable within the interval of (-22.07, 0). This represented in the diagram below

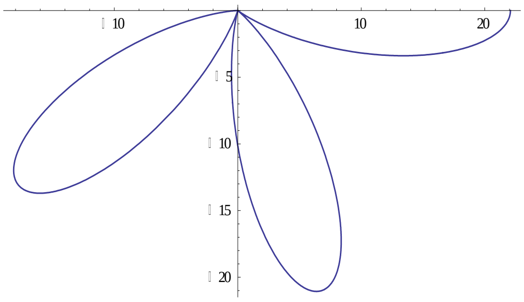

Figure 1: Region of absolute stability for the new block method.

4. NUMERICAL EXPERIMENTS

In this section, the accuracy of the new method is tested with three second order initial value problems and the obtained results are compared with the existing methods. These are shown in Tables 1, 2 and 3 below.

Problem 1: $y'' = y', \quad y(0) = 0, y'(0) = -1, \; h = 0.1$

Exact Solution: $y(x) = 1 - e^x$

The problem 1 above was solved by Mohammed [7] and Mohammed and Adeniyi [8]. We applied the new method to the same problem for the purpose of comparison. The results are shown in Table 1 below

Table 1*: Results of the new method compared with Mohammed [7] and Mohammed and Adeniyi [8]*

\

x	Exact Solution	Computed Solution	Error in new Method, *k*=5	Error in Mohammed& Adeniyi [8], *k*=5	Error in Mohammed [7], *k*=5
0.1	-0.105170918076	-0.105170918075	2.508826E-13	2.004000000E-07	2.198000000E-05
0.2	-0.221402758160	-0.221402758095	6.493175E-11	5.386000000E-07	6.070400000E-06
0.3	-0.349858807576	-0.349858805893	1.683146E-09	8.840000000E-07	1.005100000E-05
0.4	-0.491824697641	-0.491824680635	1.700635E-08	1.229700000E-06	1.402530000E-05
0.5	-0.648721270700	-0.648721168155	1.025454E-07	1.575200000E-06	1.799340000E-05
0.6	-0.822118800391	-0.822116241680	2.558711E-06	1.920400000E-06	2.161620000E-05
0.7	-1.013752707470	-1.013747434171	5.273300E-06	2.506000000E-06	2.799300000E-05
0.8	-1.225540928492	-1.225532652558	8.275935E-06	3.106000000E-06	3.456100000E-05
0.9	-1.459603111157	-1.459591494483	1.161667E-05	3.705000000E-06	4.111400000E-05
0.1	-1.718281828459	-1.718266406589	1.542187E-05	4.304000000E-06	4.765600000E-05

Problem2: $y'' = -y + 2\cos x, \; y(0) = 1, \; y'(0) = 0,$
$0 \le x \le 1$

Exact Solution: $y(x) = \cos x + x \sin x$

This problem 2 was solved by Omar [1] in which maximum errors were selected. Our new method was applied to the same differential problem and selection of maximum errors was also considered. The results are shown in Table 2 below:

The following notations are also used in Tables 2.

S2PEB Sequential Implementation of the 2-point Explicit Block Method

P2PEB Explicit Parallel Implementation of the 2-point Block Method

S3PEB Sequential Implementation of the 3-point Explicit Block Method

P3PEB Explicit Parallel Implementation of the 3-point Block Method

Table 2*: Comparison of the new method with Omar [1] block method*

h-values	New Method	Omar [1]	Number of Steps	Error in new Method, *k*=5	Error in Omar [1] *k*=5
10^{-2}	5-Step Method	S2PEB	53	4.886702E-12	1.43153E-03
		P2PEB	53	4.886702E-12	1.43153E-03
		S3PEB	36	1.187872E-11	1.43153E-03
		P3PEB	36	1.187872E-11	1.43153E-03
10^{-3}	5-Step Method	S2PEB	503	4.518608E-14	1.43166E-04
		P2PEB	503	4.518608E-14	1.43166E-04
		S3PEB	336	2.220446E-16	1.43166E-04
		P3PEB	336	2.220446E-16	1.43166E-04
10^{-4}	5-Step Method	S2PEB	5003	2.101652E-13	1.43167E-05
		P2PEB	5003	2.101652E-13	1.43167E-05
		S3PEB	3336	3.197442E-14	1.43167E-05
		P3PEB	3336	3.197442E-14	1.43167E-05
10^{-5}	5-Step Method	S2PEB	50003	6.672440E-14	1.43167E-06
		P2PEB	50003	6.672440E-14	1.43167E-06
		S3PEB	33336	9.459100E-14	1.43167E-06
		P3PEB	33336	9.459100E-14	1.43167E-06

Problem 3: $y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right) y = 0,\ y(0) = 1,\ y'(1) = 0,\ h = \frac{0.1}{32}$

Exact Solution: $y(x) = \dfrac{5x^3 - 2}{3x^4}$

Badmus and Yahaya [9] applied their developed method to the problem above. The same problem was also considered by the new method. The generated results are shown in Table 3 below.

Table 3*: Results of the new block method compared with Yahaya and Badmus [9]*

x-values	Exact Solution	Computed Solution	Error in new Method, *k*=5	Error in Badmus andYahaya [9] *k* =5
0.003125	1.003076525857696400	1.003076525857696100	2.220446E-16	3.8354E-05
0.00625	1.006057503083516400	1.006057503083515900	4.440892E-16	7.5004E-05
0.009375	1.008944995088837600	1.008944995088838700	1.110223E-15	1.0592E-04
0.0125	1.011741018167988400	1.011741018167986500	1.998401E-15	1.35476E-04
0.015625	1.014447542686413900	1.014447542686407700	6.217249E-15	1.55567E-04
0.01875	1.017066494235672400	1.017066494235681100	8.659740E-15	1.86372E-04
0.025	1.011741018167988400	1.011741018167981600	6.883383E-15	1.96055E-04
0.028125	1.024416518738402700	1.024416518738479100	7.638334E-14	2.21045E-04
0.03125	1.026703577500806200	1.026703577500870800	6.461498E-14	2.05628E-04

5. CONCLUSION

We have proposed a five-step block method of order six for direct solution of second order initial value problems of ODEs. The new method was applied to some second initial value problems and the results generated are compared with the existing methods. It can be observed from the above Tables 1-3 that the new method gives better approximation than the existing methods of the same step-length *k=5*.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interests regarding the publication of this paper.

REFERENCES

[1]. Z. Omar. Developing Parallel 3-Point Implicit Block Method for solving second Order Ordinary Differential Equations Directly. *IJMS, 2004,* 11(1), 91 – 103.

[2]. C. W. Gear. The numerical integration of ordinary differential equations. *Math. Comp*., 21 (1966), 146 – 156.

[3]. C. W. Gear. The automatic integration of ordinary differential equations. *Communications of the ACM,* 1971, 14(3), 176-179.

[4]. C. W. Gear. The stability of numerical methods for second order ordinary differential equations. *SIAM J. Numer. Analy, 1987,* 15(1), 118 – 197.

[5]. P. Henrici. Discrete variable methods in ordinary differential equations, 1962.

[6]. J. D. Lambert. *Computational Methods in Ordinary Differential Equations. Introductory Mathematics for Scientists and Engineers*. Wiley, 1973

[7]. U. Mohammed. A class of implicit five-step block method for general second order ordinary differential equations. *Journal of Nigerian Mathematical Society (JNMS).* 30 (2011), 25 – 39

[8]. U. Mohammed and R. B. Adeniyi R.B. Derivation of block hybrid backward difference formula (HBDF) through multistep collocation for solving second order differential equations. *The Pacific Journal of Science and Technology*, 2014, 15(3), (89 – 95).

[9]. A .M. Badmus and Y. A. Yahaya. An accurate uniform order 6 block method for direct solution of general second order ordinary differential equations. *Pacific Journal of Science and Technology, 2009, 10*(2), 248-254.