Journal of Universal Mathematics
Vol.3 No.2 pp.131-136 (2020)
ISSN-2618-5660
DOI: 10.33773/jum.637104

REMARKS ON THE ARITHMETICAL FUNCTION $(a_p(n))$

A. AWEL

Abstract. In this paper, for any arbitrary two primes p and q the relationship between the corresponding arithmetic functions $(a_p(n))$ and $(a_q(n))$ are investigated. Furthermore, a general formula for statistical density of all sets on which the two arithmetic function have the same value is also established.

1. Introduction

In [3] and [6], Fast and Steinhaus introduced the concept of statistical convergence, independently. In [10], Zygmund gave a name ”almost convergence” to the this concept and established a relation between statistical convergence and strong summability of sequences. Especially in [7], Schoenberg gave a matrix characterization of the statistical convergence.

Let K be a subset of the positive integers $\mathbb{N}$ and $K_n := \{k \leq n : k \in K\}$. Natural density of the set K is given by

$$\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n}$$

provided that this limit exists. The symbol $|A|$ denotes the cardinality of the set A.

A real number sequence $x = (x_k)_{k=1}^{\infty}$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set

$$K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has a natural density of zero. In this case, we write $st - \lim x = L$.

An arithmetic function is any real or complex valued function defined on a set of positive integers. In analytic number theory arithmetic functions are simple but very useful tools to understand many advance concepts. There are a lot of arithmetical function but here, we are interested in only the arithmetic function $(a_p(n))$. The main tool here is the concept of the natural (or asymptotic) density. Some more related results about arithmetical functions can be found in [2], [4] and [5], etc.

Date: October, 2019.
2000 *Mathematics Subject Classification.* 40A35, 11B05, 11425.
Key words and phrases. Statistical limit and cluster points, Natural density, Arithmetical function.

The elementary properties of the arithmetic function $(a_p(n))$ is studied in [8] by T. Salat in the sense of natural density. Later on, the same arithmetical function with the perspective of ideal convergence has been investigated in [4].

Very recently the statistical limit and cluster points of the sequence $(a_p(n))$ and some others were studied in [1].

In this paper, by taking different primes p and q the relationship between the arithmetical functions $(a_p(n))$ and $(a_q(n))$ will be studied and a formula will be produced for the natural density of the sets having same value on both functions is established.

Definition 1. Let p be a prime number. The arithmetic function $a_p(n)$ is defined as follows: $a_p(1) = 0$ and if $n \geq 1$, then $a_p(n)$ is the unique positive integer $j \geq 0$ satisfying $p^j|n$ but not $p^{j+1}|n$.

For example for $p = 3$ the sequence $(a_p(n))$ is given as follows:

$$(a_3(n)) = (0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, ...).$$

In the proof of the main results, we need following Lemmas:

Lemma 1.1. [9] *If A and B are two mutually disjoint subsets of the set natural numbers $\mathbb{N}$, then*

$$\delta(A \cup B) = \delta(A) + \delta(B)$$

holds.

Lemma 1.2. [9] *Let $S = \{s_1, s_2, ..., s_n, ..\}$ be a subset of the set of natural numbers $\mathbb{N}$ where $s_1 < s_2 < ... < s_n < ...$ satisfied. Then,*

$$\delta(S) = \lim_{n\to\infty} \frac{|S(s_n)|}{s_n}.$$

2. Main Results

In this section, we will see the key results of the paper for any two primes p and q. First of all, let us defined the following sets which are associated with p and q:

$$K_0 := \{n \in \mathbb{N} : (n, p) = 1 \text{ and } (n, q) = 1\},$$

$$K_1 := \{n \in \mathbb{N} : n = pk_i \text{ or } n = qk_i \text{ where } k_i \in K_0\},$$

$$K_2 := \{n \in \mathbb{N} : n = p^2k_i \text{ or } n = q^2k_i \text{ where } k_i \in K_0\},$$

$$K_3 := \{n \in \mathbb{N} : n = p^3k_i \text{ or } n = q^3k_i \text{ where } k_i \in K_0\},$$

$$\vdots$$

$$K_j := \{n \in \mathbb{N} : n = p^jk_i \text{ or }, n = q^jk_i \text{ where } k_i \in K_0\}$$

and so on.

Also, let us denote the union of these sets by

$$D := \bigcup_{j=1}^{\infty} K_j. \tag{2.1}$$

Lemma 2.1. *For each $i \neq j$, the sets K_i and K_j defined above are mutually disjoint subset of natural numbers.*

Proof. Let $n \in K_i$ be an arbitrary element, then either $n = p^i k_1$ or $n = q^i k_1$ where $k_1 \in K_0$. Let us see each case one by one.

Case 1: Assume that $n = p^i k_1$. For $j \neq i$ suppose that $n \in K_j$, then we have also two cases: either $n = p^j k_2$ or $n = q^j k_2$ where $k_2 \in K_0$. Suppose $n = p^j k_2$.

If $i < j$, then $n = p^j k_2 = p^i p^{j-i} k_2 = p^i k_1$ this implies that $k_1 = p^{j-i} k_2$ and this also in turn implies that k_1 could not be in K_0 which contradicts to our assumption $n \in K_0$. Hence, $n \notin K_j$.

If we also consider $i > j$, then $n = p^i k_1 = p^j p^{i-j} k_1 = p^j k_2$. This implies that $k_2 = p^{i-j} k_1$ and hence k_2 could not be in K_0 which contradicts to our assumption $k_2 \in K_0$.

If $n = q^j k_2$ with $i < j$, then $n = q^j k_2 = q^i q^{j-i} k_2 = q^i k_1$ and this implies that $k_1 = q^{j-i} k_2$. It simply means that k_1 could not be in K_0 which contradicts to our assumption $k_1 \in K_0$. Hence, $n \neq q^j k_2$.

If we also consider $i > j$, then $n = p^i k_1 = p^j p^{i-j} k_1 = p^j k_2$. This implies that $k_2 = p^{i-j} k_1$ and hence k_2 could not be in K_0 which contradicts to our assumption $k_2 \in K_0$. Therefore, $n \notin K_j$.

Case 2: Assume that $n = q^i k_1$.

For $j \neq i$ suppose that $n \in K_j$, then we have also two cases either $n = p^j k_2$ or $n = q^j k_2$ where $k_2 \in K_0$. Suppose $n = p^j k_2$ holds. If $i < j$, then $n = p^j k_2 = p^i p^{j-i} k_2 = p^i k_1$ this implies that $k_1 = p^{j-i} k_2$. This also in turn implies that k_1 could not be in K_0 which contradicts to our assumption $k_1 \in K_0$. Hence, $n \neq p^j k_2$.

If we also consider $i > j$, then $n = p^i k_1 = p^j p^{i-j} k_1 = p^j k_2$. This implies that $k_2 = p^{i-j} k_1$ and hence k_2 could not be in K_0 which contradicts to our assumption $k_2 \in K_0$. Therefore, on each case $n \neq p^i k_1$.

If $n = q^j k_2$ with $i < j$, then $n = q^j k_2 = q^i q^{j-i} k_2 = q^i k_1$. This implies that $k_1 = q^{j-i} k_2$ this simply means that k_1 could not be in K_0 which contradicts to our assumption $n \in K_0$. Hence, $n \neq q^j k_2$.

If we consider $i > j$, then $n = p^i k_1 = p^j p^{i-j} k_1 = p^j k_2$. This implies that $k_2 = p^{i-j} k_1$ and hence k_2 could not be in K_0 which contradicts to our assumption $k_2 \in K_0$. For $k_1 \in K_0$, $n \neq p^i k_1$. Therefore, in all case $n \notin K_j$. Hence, the sets are disjoint and this completes the proof. □

Lemma 2.2. *The set D defined in (2.1) is exactly the same as the set*

$$C := \{n : n = pk \text{ or } n = qk, k \in \mathbb{N}\}.$$

That is, we have $D = C$.

Proof. Let $n \in D$. By Lemma 2.1 the sets K_i's are mutually disjoint. Then, for a fixed $i \in \mathbb{N}$ we have $n \in K_i$. Definition of K_i's, implies two cases $n = p^i k_0$ or $n = q^i k_0$ where $k_0 \in K_0$.

Case 1: $n = p^i k_0$. Now, if $i = 1$ then obviously $n \in C$. If $i > 1$ then $n = p(p^{i-1} k_0)$. This implies that for $k_2 = p^{i-1} k_0$. So, we have $n = pk_2$ which also implies $n \in C$. Hence, $D \subseteq C$.

Case 2: $n = q^i k_0$. Now, if $i = 1$ then obviously $n \in C$. If $i > 1$, then $n = q(q^{i-1} k_0)$. This implies that for $k_2 = q^{i-1} k_0$. So, we have $n = qk_2$ which also implies $n \in C$. Hence, in each case we have $D \subseteq C$.

To prove the converse side, let $n \in C$ then, $n = pk$ where $k \in \mathbb{N}$. Now, if $(p, k) = 1$, then we have $n \in K_1$. If $(p, k) \neq 1$, then $(p, k) = p$ this implies $n = p^2 k_2$. If $(p, k_2) = 1$ then, $n \in K_2$. Since n a is fixed number, then if we continue in this way we will stop at some point $i \in \mathbb{N}$ such that $n = p^i k_i$ and $(k_i, p) = 1$ which implies that $n \in K_i$. Therefore,we have also $C \subseteq D$. This completes the proof. □

Theorem 2.3. *For the set* $K_0 := \{n \in \mathbb{N} : a_p(n) = 1 = a_q(n)\}$, *we have*

$$\delta(K_0) = \frac{(p-1)(q-1)}{pq}.$$

Proof. By Lemma 2.2 we have,

$$D = \bigcup_{i=1}^{\infty} K_i = A \cup B$$

where

$$A = \{n : n = pk,\ k \in \mathbb{N}\} \text{ and } B = \{n : n = qk,\ k \in \mathbb{N}\}.$$

From the definitions of the sets, we have $\mathbb{N} \setminus \{1\} = K_0 \cup D$ and $K_0 \cap D = \emptyset$ hold. Also,

$$1 = \delta(\mathbb{N}) = \delta(D) + \delta(K_0)$$

and

$$\delta(D) = \delta(A \cup B) = \delta(A) + \delta(B) - \delta(A \cap B) = \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}$$

clearly satisfied. Therefore, we have

$$\delta(K_0) = 1 - (\frac{1}{p} + \frac{1}{q} - \frac{1}{pq}) = \frac{pq + 1 - (p+q)}{pq} = \frac{(p-1)(q-1)}{pq}$$

holds. Hence, this completes the proof. □

Theorem 2.4. *For each* $j \in \mathbb{N}$, *the natural density of* K_j *defined above is given by the following formula:*

$$\delta(K_j) = \Big(\frac{1}{p^j} + \frac{1}{q^j}\Big)\frac{(p-1)(q-1)}{pq}$$

Proof. For the sake of convenience let us denote

$$K_j := A_j \cup B_j$$

where

$$A_j = \{n \in \mathbb{N} : n = p^j k_i, k_i \in K_0\} \text{ and } B_j = \{m \in \mathbb{N} : m = q^j k_i, k_i \in K_0\}.$$

It is clear from the definition of the sets for each $p \neq q$,

$$A_j \cap B_j = \emptyset.$$

Let $S_n := \{k_1^0, k_2^0, k_3^0, \dots, k_n^0\}$ be the set of first n elements of K_0 satisfying $k_1 \leq k_2 \leq k_3 \leq \cdots \leq k_n$.

As a result of Theorem 2.3 and definition of density, we have

$$\delta(K_0) = \lim_{n\to\infty} \frac{|K_0 \cap S_n|}{k_n^0} = \frac{(p-1)(q-1)}{pq}$$

and clearly $\delta(K_j) = \delta(A_j) + \delta(B_j)$. For the set

$$A_j := \{n \in \mathbb{N} : n = p^j k,\ k \in K_0\}$$

$j \in \mathbb{N}$, we have

$$|A_j \cap (p^j S_n)| = |K_0 \cap S_n| = n.$$

Hence,

$$\delta(A_j) = \lim_{n\to\infty} \frac{|A_j \cap (p^j S_n)|}{p^j k_n^0} = \frac{1}{p^j} \lim_{n\to\infty} \frac{|K_0 \cap S_n|}{k_n^0} =$$

$$= \frac{1}{p^j}\delta(K_0) = \frac{1}{p^j}\frac{(p-1)(q-1)}{pq}$$

Similarly for the set $B_j := \{n \in \mathbb{N} : n = q^j k \textit{ where } j \in \mathbb{N} \text{ and } k \in K_0\}$. We have

$$|B_j \cap (q^j S_n)| = |K_0 \cap S_n| = n,$$

then

$$\delta(B_j) = \lim_{n\to\infty} \frac{|B_j \cap (q^j S_n)|}{q^j k_n^0} = \frac{1}{q^j} \lim_{n\to\infty} \frac{|K_0 \cap S_n|}{k_n^0}$$

$$= \frac{1}{q^j}\delta(K_0) = \frac{1}{q^j}\frac{(p-1)(q-1)}{pq}$$

satisfied. Since $\delta(K_i) = \delta(A_i \cup B_i) = \delta(A_i) \cup \delta(B_i)$ the result follows immediately. □

3. Application

In this section, for a specific values of p and q, we will denote the relationship between $(a_p(n))$ and $(a_q(n))$ by applying the above main results.

Now, let us consider the primes $p = 3$ and $q = 5$ and the arithmetic functions associated with them:

$$(a_3(n)) = (0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, ...)$$

$$(a_5(n)) = (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, ...)$$

Therefore, associated sets

$$K_0 := \{n \in \mathbb{N} : (n, 3) = 1 \text{ and } (n, 5) = 1\},$$

$$K_1 := \{n \in \mathbb{N} : n = 3k_i \text{ or } n = 5k_i \text{ where } k_i \in K_0\},$$

$$K_2 := \{n \in \mathbb{N} : n = 3^2 k_i \text{ or } n = 5^2 k_i \text{ where } k_i \in K_0\},$$

$$K_3 := \{n \in \mathbb{N} : n = 3^3 k_i \text{ or } n = 5^3 k_i \text{ or } k_i \in K_0\},$$

$$\vdots$$

$$K_j := \{n \in \mathbb{N} : n = 3^j k_i \text{ or } n = 5^j k_i \text{ or } k_i \in K_0\}.$$

Then, by the results of Theorem 2.3 and Theorem 2.4 we have

$$\delta(K_0) = \frac{(5-1)(3-1)}{15} = \frac{8}{15}$$

and for each $j \in \mathbb{N}$

$$\delta(K_j) = \left(\frac{1}{3^j} + \frac{1}{5^j}\right)\frac{8}{15}$$

hold, respectively.

4. Conclusion and Recommendation

For any two distinct prime p and q, we have established a formula for the asymptotic density of the set of points on which the corresponding arithmetic functions $(a_p(n))$ and $(a_q(n))$ have the same value (for any arbitrary number) $m \in \mathbb{N}$

This paper has established a relationship between the arithmetic functions $a_p(n)$ and $a_q(n)$ for two different prime numbers in terms of natural density. Let $p_1, p_2, ..., p_m$ distinct prime numbers and consider the sets

$$K_0 := \{n \in \mathbb{N} : (n, p_i) = 1 \text{ for all } i = 1, 2, ..., m\ \}$$

and for any $j \in \mathbb{N}$

$$K_j := \{n \in \mathbb{N} : n = p_i^j k_l \text{ for any } k_l \in K_0 \text{ and for all } i = 1, 2, ..., m\}.$$

By using similar ways an interested author could extend the result of this paper to find

$$\delta(K_0) =?$$

and

$$\delta(K_j) =?$$

for all $j \in \mathbb{N}$.

5. Acknowledgment

I would like to thank Prof. Dr. M. Küçükaslan for his guidance and valuable contributions during the creation and solution of the problem.

I would also like to thank the referees who contributed to the study with their precious warnings.

References

[1] Abdu Awel and M. Küçükaslan, (2020) A note on statistical limit and cluster points of the arithmetical functions $a_p(n)$, $\gamma(n)$, $\tau(n)$, J. Indones. Math. Soc. (accepted for publication).

[2] V. Bal´az, J. Gogola, T. Visnyai, (2018) I_c^q-convergence of arithmetical functions. J. Number Theory, vol. 183, 74–83

[3] Fast, H., (1951) Sur la convergence statistique., *Colloq.Math*, vol. 2, 241-244.

[4] Fehéra, Z., Lászlóa, B., Mačajb, M., Šalát, T., (2006) Remarks on arithmetical functions $a_p(n)$, $\gamma(n)$, $\tau(n)$, Annales Mathematicae et Informaticae, vol 33, 35-43.

[5] Janos T. T., Ferdinand F., Jozsef B., Laszlo Z., (2020), On $I_{<_q}$ and $I_{\leq_q}$ convergence of arithmetic functions, *Periodica Mathematica Hungarica*, 1-12.

[6] Steinhaus H., (1951) Sur la convergence ordinaire at la convergence asymptotique, Colloc. Math. 2.1, 73-74.

[7] Schoenberg, I. J., (1959) The integrability of certain functions and related summability methods, matrix characterization of statistical convergence, *Amer.Math.*, vol. 66, 361-375.

[8] Šalát, T., (1994) On the function a_p, $p^{a_p(n)} \backslash\backslash\ n\ (n > 1)$, *Mathematica Slovaca*, vol. 44, Number 2, 143-151.

[9] Milan, P., (2017) *Density and related topics*, Mathematics Institute Slovak Acadamic of Sciences.

[10] Zygmund, A., (1979) *Trigonometric series*, 2nd., Ed. Vol. II, Cambridge Univ. press, London and New York.

Mekelle University, Mathematics Department, 231, Mekelle, Ethiopia
Email address: abdua90@gmail.com