Results in Nonlinear Analysis 2 (2019) No. 3, 113–124
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Symmetry Analysis of Time Fractional Convection-reaction-diffusion Equation with a Delay

Kassimu Mpungu[a], Aminu M. Nass[b],

[a]*Department of Mathematics, University of Hafr Al Batin, Hafar Al Batin, 39524, Saudi Arabia.*
[b]*Department of Mathematics, Yusuf Maitama Sule University Kano, Nigeria.*

Abstract

Lie symmetry theory of partial differential equations with both fractional and delay phenomena is considered. A complete group classification of time fractional convection-reaction-diffusion equation with a delay is presented. The Minimal symmetry algebra is found to be one dimensional. The classification is used to find symmetry reductions and exact solutions.

Keywords: Lie symmetries, Fractional Delay, Bäckland Operator, Mittag-Leffler function.
2010 MSC: 35B06, 35R11, 34A08.

1. Introduction

Most of the physical, economical and biological processes are gradual and spontaneous in nature. Therefore, to exhaustively comprehend these processes, differential mathematical models which put into consideration of both the present and the past occurrences are developed to represent and study such processes. In so doing, delay differential equations are developed. To mention but a few, delay differential equations have been applied to: controlled motion of a rigid body, mathematical models of sugar quantity in blood, evolution equations of single species [7]. Other models include; time to maturity and incubation time, for instance in the well known Lotka-Volterra model [53] , delayed feedback in controlled systems [8]. Generally, time delay is observed to have a negative effect on system stability [60].

Whereas the integer-order derivative indicates a variation or certain attribute at particular time, the fractional-order derivative is concerned with the whole time domain and space of the process, for that reason therefore, the theory of fractional order delay differential equations (FDDEs) has become an interesting field of study with applications in fields of science and engineering. [7, 60, 16, 28, 6, 61].

Email addresses: `kmpungu@uhb.edu.sa` (Kassimu Mpungu), `nssami001@myuct.ac.za` (Aminu M. Nass)

Received May 09, 2019, Accepted: October 21, 2019, Online: October 24, 2019.

In this article therefore, we consider a time fractional convection-reaction-diffusion equation with a delay, i.e.

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = f(u, \bar{u}), \quad f_{\bar{u}} \neq 0 \tag{1.1}$$

where $\overline{u} = u(t-s, x)$. In general, a convection-diffusion-reaction model is a mathematical model that describes how the concentration of the substance distributed in the medium changes under the influence of convection, diffusion and reaction processes [26]. The model is developed by balancing three factors, The first process, which is convection, refers to the flow or transfer of materials involved in the process from one region to another depending on flow velocity. On the other hand, diffusion is the movement of a substance from region of high concentration to region of low concentration throughout the physical domain of the problem . Lastly, the reaction term describes possible processes like adsorption, decay and reaction of the substances with other components. These processes when combined together, they form a single model, which explains a physical system that can undergo convection, diffusion and reaction process within a system [9, 26]. The convection-diffusion-reaction equation(1.1) is widely used in science and engineering for instance in modelling evolution of thermal waves in plasma [61]. It is also used to model the Mackey-Glass equation, which simulates a single-species population with age-structure and diffusion [52], and Hematopoiesis model which plays a vital role in investigation on the dynamics of blood cell production [10].

The existence and uniqueness of solutions for a fractional order reaction-diffusion equation with delay by Leray-Schauder's theorem was proved by Ouyang [35], while the existence and uniqueness of mild solutions for a class of nonlinear fractional reaction-diffusion equations was studied by Bo Zhu et al. using the measure of compactness, the theory of resolvent operators, the fixed point theorem and the Banach contraction mapping principle [63]. Some methods have been developed to study fractional differential equations with delay, mostly numerical approach [60, 16, 28, 6, 61].

The modern approach of the applications of the theory of Lie Symmetry remains a powerful tool to study both deterministic and stochastic deferential equations [34, 4, 29, 30, 18, 1, 2, 33, 58, 57, 11, 31, 32]. Not long ago, the Lie group theory was extended to the class of fractional differential equations for the purposes of linearization, reduction in the number of independent variables and finding analytical solutions. Similarly, some work has been done to study the symmetry and invariant solutions of the delay differential equation [48, 5, 59, 19, 3, 47, 25, 17, 12, 13, 14].

Recently, Cheng C. and Yao-Lin J. used Lie group method to derive the invariant solutions for non-linear time-fractional convection-diffusion equations [5]. Several other methods were used by A. D. Polyanin et. al. to study the solutions of different forms of the non-linear time-fractional convection-diffusion equation with delay [37, 38, 39, 40, 41, 42, 43, 44, 45].

In this paper, an extension of Lie group theory to a fractional order delay differential equation has been presented. The rationale of this article is to use classical Lie symmetry theory to present complete group classification of time fractional convection-reaction-diffusion equation with a delay.

2. Preliminaries

There is no unique definition of fractional derivatives [20, 21]. In this paper, we use the version of Riemann-Liouville.

Definition 1.

$$D_t^\alpha u(t,x) = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u}{\partial t^n} & \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\mu,x)}{(t-\mu)^{\alpha+1-n}} d\mu & n-1 < \alpha < n, n \in \mathbb{N} \end{cases} \tag{2.1}$$

where Γ is a gamma function, and D_t^α satisfies the following properties [20, 21, 54]

$$D_t^\alpha t^\varsigma = \frac{\Gamma(\varsigma+1)}{\Gamma(\varsigma+1-\alpha)} t^{\varsigma-\alpha}, \quad \alpha > 0, \quad t > 0, \tag{2.2}$$

$$D_t^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \geq 0, \quad t > 0, \tag{2.3}$$

and

$$D_t^\alpha (g(t)h(t)) = \sum_{n=0}^{+\infty} \binom{\alpha}{n} D_t^{\alpha-n} h D_t^n g. \tag{2.4}$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)}.$$

This article is organized as follows; equivalence Lie group of transformation is given in *Section 2* and admitted Lie group transformation as well as classification are carried out in *Section 3.* Finally, invariant solutions and conclusion are given in *Section 4* and *Section 5* respectively.

3. Equivalence Lie Group of Equation (1.1)

A transformation of both the independent and dependent variables which preserves the differential structure of the equations themselves is referred to as an equivalence transformation. Furthermore, a Lie group formed by a set of equivalence transformations is known as an equivalence Lie group. For simplicity, we start by introducing a new dependent variable v, which is related to u by the formula;

$$v(t,x) = u(t-s,x). \tag{3.1}$$

Therefore, equation (1.1) becomes,

$$u_\alpha + u_x - u_{xx} = f(u,v) \tag{3.2}$$

where $f(u,v)$ is the arbitrary function. To obtain an equivalence Lie group of transformation, we assume that the function f doesn't depend on the independent variables, i.e.

$$f_x = f_t = 0. \tag{3.3}$$

Therefore, the corresponding generator of the equivalence Lie group for (3.2), is as result given by:

$$H = \xi\partial_x + \tau\partial_t + \phi\partial_u + \phi^v\partial_v + \phi^f\partial_f, \tag{3.4}$$

where the coefficients ξ, τ, ϕ, ϕ^v, and ϕ^f are infinitesimals functions depending on variables t, x, u, v and f. The canonical Lie-Bäcklund operator equivalent to generator (3.4) is

$$H^* = \xi\partial_x + \tau\partial_t + \phi^u\partial_u + \phi^{v*}\partial_v + \phi^{f*}\partial_f \tag{3.5}$$

where

$$\phi^u = \phi - \xi u_x - \tau u_t,\ \phi^{v*} = \phi^v - \xi D_x v - \tau D_t v,\ \phi^{f*} = \phi^f - \xi D_x f - \tau D_t f. \tag{3.6}$$

The prolonged operator for the equivalence Lie group is

$$\bar{H}^\alpha = H^* + \phi^{u_x}\partial_{u_x} + \phi^{u_{xx}}\partial_{u_{xx}} + \phi^{u_\alpha}\partial_{u_\alpha}, \tag{3.7}$$

where the coefficients are defined as;

$$\phi^{u_x} = D_x(\phi - \xi u_x - \tau u_t),\ \phi^{u_{xx}} = D_x(\phi^{u_x}),\ \phi^{u_\alpha} = D_t^\alpha(\phi - \xi u_x - \tau u_t). \tag{3.8}$$

The prolongation of the fractional derivative in (3.8) can be expanded using direct calculation to obtain;

$$\begin{aligned}\phi^{u_\alpha} =& \frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi_v \frac{\partial^\alpha v}{\partial t^\alpha} + (\phi_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} - v \frac{\partial^\alpha \phi_v}{\partial t^\alpha} \\ & - \sum_{n=1}^{+\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n} \frac{\partial^n \phi_v}{\partial t^n} \right] D_t^{\alpha-n}(v) \\ & + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n} \frac{\partial^n \phi_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) \\ & - \tau D_t^{\alpha+1}(u) - \xi D_t^\alpha(u_x) + \beta\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}\beta =& \sum_{n=2}^{+\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} [-u]^r \frac{\partial^m}{\partial t^m}(u^{k-r}) \frac{\partial^{n-m+k} \phi}{\partial t^{n-m} \partial u^k} \\ & + \sum_{n=2}^{+\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} [-v]^r \frac{\partial^m}{\partial t^m}(v^{k-r}) \frac{\partial^{n-m+k} \phi}{\partial t^{n-m} \partial v^k}\end{aligned}$$

and $D_t,\ D_x,\ D_t^\alpha,\ D_x^\alpha$ are total derivative.

Remark 1. *It is worth noticing that β in (3.9) becomes zero if the infinitesimal ϕ is linear with respect to the dependent variables u and v. For the detailed proof, we refer the reader to [47, 49, 50, 51].*

We now apply the invariance criteria:

$$\bar{H}^\alpha (u_\alpha + u_x - u_{xx} - f) \Big|_{u_\alpha = f - u_x + u_{xx}} = 0 \tag{3.10}$$

which gives;

$$\phi^{u_\alpha} + \phi^{u_x} - \phi^{u_{xx}} - \phi^{f*} \Big|_{u_\alpha = f - u_x + u_{xx}} = 0. \tag{3.11}$$

Substituting u_α and v_α in the expanded form of (3.11), and then equating the coefficients of various derivatives of u and v to zero, gives a simplified system of determining equations below;

$$\tau_u = \tau_v = \tau_x = \phi_v = \phi_{uu} = \xi_u = \xi_v = \xi_t = 0, \tag{3.12}$$

$$\alpha \tau_t - 2\xi_x = 0, \tag{3.13}$$

$$\alpha \tau_t - \xi_x - 2\phi_{xu} + \xi_{xx} = 0, \tag{3.14}$$

$$\binom{\alpha}{n} \frac{\partial^n \phi_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad \forall n \in \mathbb{N}, \tag{3.15}$$

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} + \phi_u f + \phi_x - \phi_{xx} - \alpha \tau_t f - \phi^f = 0. \tag{3.16}$$

The determining equations corresponding to (3.1) are

$$\begin{aligned}&\phi^v(w(t,x)) - \phi(w(t-s,x)) - v_t(t,x)(\xi(w(t,x))) - \xi(w(t-s,x)) \\ &- v_x(t,x)(\tau(w(t,x))) - \tau(w(t-s,x)) \Big|_{(3.2)} = 0\end{aligned} \tag{3.17}$$

where

$$w(t,x) = (t, x, u(t,x), v(t,x), f(u(t,x), v(t,x)). \tag{3.18}$$

Splitting (3.17) with respect to v_x and v_t we get

$$\xi = \bar{\xi}, \qquad \tau = \bar{\tau}, \qquad \phi^v = \bar{\phi} \tag{3.19}$$

where $\bar{\xi} = \xi(w(t-s,x)),\ \bar{\tau} = \tau(w(t-s,x)),\ \bar{\phi} = \phi(w(t-s,x))$.

From (3.12), (3.13) and (3.19) we have $\tau_t = \xi_x = 0$, consequently, from (3.14) and (3.15)

$$\phi_{ut} = \phi_{ux} = 0.$$

Again the assumption $f_x = f_t = 0$ leads to

$$\phi_t = \phi_x = \phi_x^f = \phi_t^f = 0. \tag{3.20}$$

Hence, the general solution of the system (3.12)-(3.19) and (3.20), using the definition of fractional derivatives becomes:

$$\tau = 0, \qquad \xi = c_1, \qquad \phi = 0, \qquad \phi^f = 0. \tag{3.21}$$

In this section, we apply the techniques used in [27, 24, 23, 15], to find the admitted Lie groups transformation of the time fractional convection-reaction-diffusion equation with a delay (1.1).

Let

$$H = \xi\partial_x + \tau\partial_t + \phi\partial_u, \tag{3.22}$$

be the Lie generator admitted by (1.1), with the infinitesimal ξ, τ, ϕ depend on the variables t, x and u.

The Lie-B$\ddot{a}$cklund generator equivalent to (1.1) is

$$\bar{H}^\alpha = \phi^u\partial_u + \phi^{u_x}\partial u_x + \phi^{u_{xx}}\partial u_{xx} + \phi^{u_\alpha}\partial u_\alpha + \bar{\phi^u}\partial_{\bar{u}} \tag{3.23}$$

where

$$\phi^u = \phi - \xi u_x - \tau u_t, \qquad \bar{\phi^u} = \bar{\phi} - \bar{\xi}\bar{u_x} - \bar{\tau}\cdot\bar{u_t} \tag{3.24}$$

$$\phi^{u_x} = D_x(\phi - \xi u_x - \tau u_t), \qquad \phi^{u_{xx}} = D_x(\phi^{u_x}), \tag{3.25}$$

$$\phi^{u_\alpha} = D_t^\alpha(\phi - \xi u_x - \tau u_t). \tag{3.26}$$

Here, D_t, D_x and D_t^α are the total derivatives operators with respect to t, x and the fractional total derivative respectively.

Applying the Lie-B$\ddot{a}$cklund infinitesimal generator (3.23) in the equation (1.1) and using the invariance criteria leads to

$$\left(-\phi^u f_u + \phi^{u_x} - \phi^{u_{xx}} + \phi^{u_\alpha} - \bar{\phi}^u f_{\bar{u}}\right)\Big|_{u_\alpha = f - u_x + u_{xx}} = 0. \tag{3.27}$$

Substituting the prolongation (3.24)-(3.26) and replacing

$$D_t^{\alpha+1}(u) = f_u u_t + f_{\bar{u}}\bar{u_t} + u_{xxt} - u_{xt},\ D_t^\alpha(u_x) = f_u u_x + f_{\bar{u}}\bar{u_x} + u_{xxx} - u_{xx}, \tag{3.28}$$

in the determining equations (3.27), we obtain a simplified system of determining equations by equating the coefficients of various derivatives of u, i.e. u_x, u_{xx}, u_t, u_{xt},$\bar{u}_t$,$\bar{u}_x$... and $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$... for $n = 1, 2...$, to zero, as follows;

$$\xi_u = \tau_x = \tau_u = \phi_{uu} = \xi_t = 0, \tag{3.29}$$

$$-2\phi_{ux} - \xi_x + \xi_{xx} + \alpha\tau_t = 0, \tag{3.30}$$

$$2\xi_x - \alpha\tau_t = 0, \tag{3.31}$$

$$\bar{\xi} = \xi, \qquad \bar{\tau} = \tau, \tag{3.32}$$

$$\binom{\alpha}{n}\frac{\partial^n\phi_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0, \quad \forall n \in \mathbb{N}, \tag{3.33}$$

$$\frac{\partial^\alpha\phi}{\partial t^\alpha} + \phi_u f - \alpha f\tau_t - u\frac{\partial^\alpha\phi_u}{\partial t^\alpha} - \phi_{xx} + \phi_x - \phi f_u - f_{\bar{u}}\bar{\phi} = 0. \tag{3.34}$$

Differentiating equation (3.31) with respect to x, implies that ξ is linear in x i.e.,

$$\xi = c_1x + c_2. \tag{3.35}$$

And using (3.35), (3.31) we have

$$\tau = \frac{2c_1}{\alpha} t + c_3. \tag{3.36}$$

Finally, using equation (3.30), (3.33), (3.35) and (3.36) we have

$$\phi = \gamma(t,x) + \frac{c_1 u}{2} x + c_4 u. \tag{3.37}$$

Using the periodic properties of the infinitesimals τ, and ξ (3.32) i.e.,

$$\tau(t,x) = \tau(t-s,x), \qquad \xi(t,x) = \xi(t-s,x), \tag{3.38}$$

and from (3.30), (3.31), (3.35), (3.36) and (3.37), we have

$$\xi = c_2, \qquad \tau = c_3, \qquad \phi = \gamma(t,x) + c_4 u. \tag{3.39}$$

Since the lower limit of the integral in the definition of the fractional derivative (2.1) is fixed, it requires that the manifold $t = 0$ is invariant i.e.,

$$\tau(t,x)\Big|_{t=0} = 0.$$

Hence from (3.39) we have

$$\xi = c_2, \qquad \tau = 0, \qquad \phi = \gamma(t,x) + c_4 u. \tag{3.40}$$

Substituting (3.40) into (3.34) gives

$$c_4\left(uf_u + \bar{u}f_{\bar{u}} - f - \frac{ut^{-\alpha}}{\Gamma(1-\alpha)}\right) + \gamma f_u + \bar{\gamma} f_{\bar{u}} - \gamma_\alpha + \gamma_{xx} - \gamma_x = 0. \tag{3.41}$$

3.1. Classification

In this section, we analyze the classification equation (3.41) to find all possible functions $f(u,\bar{u})$ and γ that will satisfy it.

Differentiating (3.41) with respect to u and $\overline{u}$ respectively, we have the following system

$$\begin{cases} c_4\left(uf_{uu} + \bar{u}f_{\bar{u}u} - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right) + \gamma f_{uu} + \bar{\gamma} f_{\bar{u}u} = 0 \\ c_4(uf_{u\bar{u}} + \bar{u}f_{\bar{u}\bar{u}}) + \gamma f_{u\bar{u}} + \bar{\gamma} f_{\bar{u}\bar{u}} = 0. \end{cases} \tag{3.42}$$

The system (3.42) is an algebraic with respect to $\bar{\gamma}$ and γ, which have determinant of the matrix as

$$\Delta = f_{u\bar{u}}^2 - f_{uu} f_{\bar{u}\bar{u}}. \tag{3.43}$$

3.1.1. Case 1: $\Delta \neq 0$

Equation (3.42) is a classification equation and is it's assumed to be true for the solution of $f(u,\bar{u})$.

We have from the system (3.42)

$$\gamma = \frac{-c_4\left(f_{\bar{u}\bar{u}} t^{-\alpha} + u\Gamma(1-\alpha)\Delta\right)}{\Gamma(1-\alpha)\Delta}, \qquad \bar{\gamma} = \frac{c_4\left(f_{u\bar{u}} t^{-\alpha} - \bar{u}\Gamma(1-\alpha)\Delta\right)}{\Gamma(1-\alpha)\Delta}. \tag{3.44}$$

Since γ, $\bar{\gamma}$ are independent of u and $\overline{u}$, equation (3.44) implies that $\gamma = c_4 = 0$. Hence, we obtained the following infinitesimals

$$\xi = c_2, \qquad \tau = 0, \qquad \phi = 0. \tag{3.45}$$

Therefore, the minimal symmetry algebra for any arbitrary function $f(u,\overline{u})$ is one dimensional given as below;

$$H_1 = \partial_x. \tag{3.46}$$

To search for extra symmetry algebra, we have to consider the case when $\Delta = 0$ and solve for all possible function $f(u,\bar{u})$.

3.1.2. Case 2: $\Delta = 0$

Under this case, we analyse the solutions of $\Delta = 0$ to look for possibilities of larger extra symmetry algebra, solutions of this equation are discussed in details in [27, 24, 55, 15].

The following conditions are consider;

$f_{\bar{u}\bar{u}} \neq 0$. In this case the equation $\Delta = 0$ has the general solution

$$f_u = \psi(f_{\bar{u}}) \tag{3.47}$$

where ψ is an arbitrary function. Substituting (3.47) in the system (3.42) we have

$$\begin{cases} (\gamma\psi' + \bar{\gamma})\psi' f_{\bar{u}\bar{u}} = -c_4(u\psi' + \bar{u} - \Gamma(1-\alpha)t^{-\alpha})\psi' f_{\bar{u}\bar{u}} \\ (\gamma\psi' + \bar{\gamma})f_{\bar{u}\bar{u}} = -c_4(u\psi' + \bar{u})f_{\bar{u}\bar{u}}. \end{cases} \tag{3.48}$$

The system (3.48) reduces to

$$c_4(\Gamma(1-\alpha)t^{-\alpha})\psi' = 0. \tag{3.49}$$

To study equation (3.49), we consider the following cases

(*i*) $\psi' = 0$

From equation (3.49) and (3.48), we have $\bar{\gamma} = -c_4\bar{u}$ which implies $\phi = 0$ and so no extra symmetry algebra is possible in this case. Next, we consider the case $\psi' \neq 0$.

(*ii*) $\psi' \neq 0$

From equation (3.49) and (3.48), we have

$$c_4 = 0, \quad \text{and} \quad \gamma\psi' + \bar{\gamma} = 0. \tag{3.50}$$

Differentiating equation (3.50) with respect to $\bar{u}$ implies that

$$\gamma\psi'' = 0. \tag{3.51}$$

Similarly, differentiating (3.51) with respect to x and t respectively, we have

$$\psi''\gamma_x = 0, \qquad \psi''\gamma_t = 0. \tag{3.52}$$

If $\psi'' \neq 0$, it implies γ is constants, say $\gamma = c_5$ and we have

$$f(u,\bar{u}) = uG\left(\frac{\bar{u}}{u}\right) + c_6, \quad G_{u\bar{u}} \neq 0 \tag{3.53}$$

where $c_i, i = 1, 2, 3, ...$ are constants. Substituting (3.52) in (3.41) we have

$$c_4(c_6) = c_5\left(\frac{\bar{u}}{u}G_u - G - G_{\bar{u}}\right) \tag{3.54}$$

which implies that for an extra symmetry algebra to be possible, $c_6 = c_5 = 0$. Thus no extra algebra is possible.

We now consider a case $\psi'' = 0$, which this leads to

$$f_u = c_8 f_{\bar{u}} + c_7. \tag{3.55}$$

This can be solve to find

$$f(u,\bar{u}) = c_7 u + G(\bar{u} + uc_8), \quad \psi'' = 0, \quad G_{u\bar{u}} \neq 0. \tag{3.56}$$

Using (3.49) and (3.50), we have $c_4 = 0$ and $c_8\gamma = -\bar{\gamma}$ from which equation (3.41) reduced to

$$\gamma_\alpha - \gamma_{xx} + \gamma_x - c_7\gamma = 0 \tag{3.57}$$

where $\gamma_\alpha = \frac{\partial^\alpha \gamma}{\partial t^\alpha}$ and $c_8\gamma(t,x) = -\gamma(t-s,x)$. In this case the equation admits an infinite dimensional algebra for any solution of (3.57) i.e.,

$$H_\gamma = \gamma(t,x)\partial_u. \tag{3.58}$$

$f_{\bar{u}\bar{u}} = 0$. Since $f_{\bar{u}} \neq 0$, we have

$$f(u, \bar{u}) = c_9 \bar{u} + h(u). \tag{3.59}$$

From system (3.42) and (3.59) we have

$$(\gamma + c_4 u) f_{uu} - \frac{c_4}{\Gamma(1-\alpha)} t^{-\alpha} = 0, \tag{3.60}$$

from which we have;

(*i*) If $\gamma = c_4 = 0$ lead to a minimal symmetry, so no extra algebra.

(*ii*) If $f_{uu} = 0$, we have from (3.59) $h''(u) = 0$ i.e.,

$$f(u, \bar{u}) = c_9 \bar{u} + c_{10} u + c_{11}. \tag{3.61}$$

Substituting (3.61) in (3.41), we get

$$-\frac{c_4 u}{\Gamma(1-\alpha)} t^{-\alpha} - c_4 c_{11} + c_{10}\gamma + c_9 \bar{\gamma} - \gamma_\alpha + \gamma_{xx} - \gamma_x = 0. \tag{3.62}$$

Extra symmetry algebra is possible if $c_4 = 0$ and is given by

$$H_\gamma = \gamma \partial_u \tag{3.63}$$

where γ satisfies the fractional delay equation

$$c_{10}\gamma + c_9 \bar{\gamma} - \gamma_\alpha + \gamma_{xx} - \gamma_x = 0. \tag{3.64}$$

Theorem 1. *Minimal symmetry algebra of the time fractional convection-reaction-diffusion equation with a delay*

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = f(u, \bar{u}), \quad f_{\bar{u}} \neq 0$$

where $\bar{u} = u(t-s, x)$ is spanned by one dimensional infinitesimal generators

$$H_1 = \partial_x$$

for any arbitrary function $f(u, \bar{u})$. Extra symmetry algebra with corresponding functions are summarized in the Table 1.

Table 1: Extra Symmetry Generators

$f(u,\bar{u})$	Generators	Conditions	Dimensions
$c_7 u + G(\bar{u} - (\frac{\bar{\gamma}}{\gamma})u)$	H_1, $H_\gamma = \gamma\partial_u$	$G_{\bar{u}\bar{u}} \neq 0, \quad c_7\gamma - \gamma_\alpha - \gamma_{xx} + \gamma_x = 0$	Infinite
$c_9\bar{u} + c_{10}u + c_{11}$	H_1, H_γ	$c_{10}\gamma + c_9\bar{\gamma} - \gamma_\alpha + \gamma_{xx} - \gamma_x = 0$	Infinite

4. Invariant Solutions

In this section, we make the use of the admitted Lie symmetries of the time fractional equation (1.1) to obtain invariant solutions.

4.1. Sub-algebra $H_1 = \partial_x$ with $f = uG\left(\frac{\bar{u}}{u}\right)$

Similarly, the sub-algebra $H_1 = \partial_x$ has the following invariant solution

$$u(t,x) = \psi(t) \tag{4.1}$$

which reduces (1.1) to fractional delay differential equation

$$D_t^\alpha \psi(t) = \psi(t) G\left(\frac{\bar{\psi}}{\psi}\right). \tag{4.2}$$

4.2. Sub-algebra $H_1 + H_{\gamma=e^t} = e^t\partial_u + \partial_x$ with $f = u + G(\bar{u} - e^{-s}u)$

Solving characteristic equation corresponding to the sub-algebra, we have

$$u(t,x) = \psi(t) + xe^t \tag{4.3}$$

which simplifies the equation (1.1) reduces to

$$D_t^\alpha \psi(t) + e^t = \psi + G(\bar{\psi} - e^{-s}\psi). \tag{4.4}$$

4.3. $H_{\gamma=e^t} + H_1 = e^t\partial_u + \partial_x$ with $f = c_9\bar{u} + (1 - c_9 e^{-s})u + c_{11}$

The invariant solution in this case has the form;

$$u = \psi(t) + xe^t \tag{4.5}$$

which reduces, the equation (1.1) to

$$D_t^\alpha \psi(t) + e^t = c_9\psi(t-s) - c_9 e^{-s}\psi(t) + \psi(t). \tag{4.6}$$

If $c_9 = e^s$, equation (4.6) becomes

$$D_t^\alpha \psi(t) = e^s\psi(t-s) - e^t. \tag{4.7}$$

The equation (4.7) has the general solution [5, 13]

$$\psi(t) = (t-s)^{\alpha-1} E_{\alpha,\alpha}(e^s(t-s)^\alpha) - e^t. \tag{4.8}$$

Finally, the exact solution becomes

$$u(t,x) = (t-s)^{\alpha-1} E_{\alpha,\alpha}(e^s(t-s)^\alpha) + xe^t - e^t. \tag{4.9}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [5, 13].

4.4. $H_1 = \partial_x$ with $f = \bar{u}^2 + d$

The invariant solution corresponding to the infinitesimal generator H_2 is

$$u = \psi(t), \tag{4.10}$$

were $\psi(t)$ is the solution of

$$D_t^\alpha \psi(t) = \bar{u}^2 + d. \tag{4.11}$$

The fractional ODE has the following solutions [21, 22, 59], where d is an arbitrary constant

$$\psi(t) = \begin{cases} \sqrt{d}\tan((t-s)\sqrt{d}, \alpha) & \text{if } d > 0 \\ -\sqrt{d}\cot((t-s)\sqrt{d}, \alpha) & \text{if } d > 0 \\ -\sqrt{-d}\tanh((s-t)\sqrt{-d}, \alpha) & \text{if } d < 0 \\ -\sqrt{-d}\coth((s-t)\sqrt{-d}, \alpha) & \text{if } d < 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}(t-s)^{-\alpha} & \text{if } d = 0. \end{cases} \tag{4.12}$$

Remark 2. *Some of the exact solutions obtained are summarized in Table 2 below, where*

$$\bar{\psi} = \psi(t-s)$$

and $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [5, 13].

Table 2: Table of Solutions

Representing Equation	Generator	Invariant Solution	Reduced Equation	Exact Solution
$u_\alpha + u_x - u_{xx} = \overline{u}^2 + d$	∂_x	$u(t,x) = \psi(t)$	$D_t^\alpha \psi(t) = \psi^2(t-s) + d$	$\psi(t) = \begin{cases} \sqrt{d}\tan((t-s)\sqrt{d},\alpha) & \text{if } d>0 \\ -\sqrt{d}\cot((t-s)\sqrt{d},\alpha) & \text{if } d>0 \\ -\sqrt{-d}\tanh((s-t)\sqrt{-d},\alpha) & \text{if } d<0 \\ -\sqrt{-d}\coth((s-t)\sqrt{-d},\alpha) & \text{if } d<0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}(t-s)^{-\alpha} & \text{if } d=0. \end{cases}$
$u_\alpha + u_x - u_{xx} = c_9\bar{u} + u$	$e^t\partial_u + \partial_x$	$u(t,x) = \psi(t) + xe^t$	$D_t^\alpha \psi(t) = e^s\psi(t-s) - e^{-t}$	$u(t,x) = (t-s)^{\alpha-1}E_{\alpha,\alpha}(e^s(t-s)^\alpha) + xe^t - e^t$

5. Conclusion

A complete group classification of different kind of partial differential equation with delay are available in literature [27, 24, 55, 46, 56]. J. Zhang and Jun Zhang [61] discusses the symmetry of time-fractional convection-diffusion equation and prove that the equation can be reduced to fractional ordinary differential equations.

In this article, we extend the application of the Lie symmetry analysis theory to the study of time-fractional convection-diffusion equation with a delay i.e., an equation with both fractional and delay phenomena. We present a complete group classification of this model and prove that the minimal symmetry algebra for any arbitrary function $f(u,\bar{u})$ is one dimensional given by;

$$H_1 = \partial_x.$$

Furthermore, we demonstrate that for some special functions, there is a possibility of larger symmetry algebras which are infinity dimensional to be precise. We use these admitted Lie symmetries with the respective function $f(u,\bar{u})$ in each case to perform some similarity reductions of time-fractional convection-diffusion equation with a delay to obtain the corresponding invariant solutions. These solutions are then used to transform the initial equation into a fractional ODE. In the last two cases, the reduced fractional ODEs are solved to obtain exact solutions. These results are presented in *Table2*.

References

[1] H. Azad, M.T. Mustafa, M. Ziad, Group classification, optimal system and optimal symmetry reductions of a class of Klien Gorden equations, system and optimal symmetry reductions of a class of Klien Gorden equations, Communications in nonlinear science and numerical simulation, (2009).

[2] Y.A. Ahmad, On the Conservation Laws for a Certain Class of Nonlinear Wave Equation via a New Conservation Theorem, Communications in Nonlinear Science and Numerical Simulation, **17(4)**, (2012), 1566-1575.

[3] M.M. Alshamrani, H.A. Zedan, M. Abu-Nawas, Lie group method and fractional differential equations, JNSA, **10**, Issue 8(2017),4175-4180.

[4] G. Bluman, A. Cheviakov, S. Anco, Applications of Symmetry Methods to Partial Differential Equations, Applied Mathematical Sciences, Springer-Verlag New York, **168** (2010).

[5] C. Chen, J. Yao-Lin, Lie Group Analysis and Invariant Solutions for Nonlinear Time-Fractional Diffusion-Convection Equations, Commun. Theor. Phys. **68** (2017), 295.

[6] J. Cermak, J. Hornicek, T. Kisela, Stability regions for fractional differential systems with a time delay, Communications in Nonlinear Science and Numerical Simulation, **31(1-3)** (2016), 108-123.

[7] V. Kolmanovskii, A. Myshkis, Applied Theory of Functional Differential Equations, The Netherlands: Kluwer Academic Publishers, (1992).

[8] C. Abdallah, P. Dorato, J. Benitez-Read, R. Byrne, Delayed positive feedback can stabilize oscillatory system, ACC. San Francisco, 3106-3107 (1993).

[9] A.G. Diego, C.H. Galeano, J.M Mantilla, Computational examples of reaction?convection?diffusion equations solution under the influence of fluid flow: First example, Applied Mathematical Modelling, **36** (2012), 5029-5045.

[10] D. Deng, The study of a fourth-order multistep ADI method applied to nonlinear delay reaction?diffusion equations, Applied Numerical Mathematics 96(2015), 118-133.

[11] E. Fredericks, F.M. Mahomed, Formal Approach for Handling Lie Point Symmetries of Scalar First-Order Ito Stochastic Ordinary Differential Equations, Journal of Nonlinear Mathematical Physics **15** (2008), 44-59.

[12] R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk, Continuous transformation groups of fractional differential equations, Vestn. USATU **9**(2007), 125-35

[13] R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk, Symmetry properties of fractional diffusion equations, Phys. Scr. T136 (2009), 0140-161.

[14] M. Gaur, K. Singh, Symmetry analysis of time-fractional potential Burger's equation, Math. Commun. **22** (2017), 1-11.

[15] Y.N. Grigoriev, S.V. Meleshko, N.H. Ibragimov, V.F. Kovalev, Symmetries of Integro-Differential Equation, Springer: New York, (2010).

[16] J.K. Hale, L. Verduyn, M. Sjoerd, Introduction to Functional Differential Equations, Springer, New York, (1993).

[17] J. Hu, Y.J. Ye, S.F. Shen, J. Zhang, Lie symmetry analysis of the time fractional KdV-type equation, Appl.Math.Comput. **233**(2014), 439–444.

[18] N.H. Ibragimov, Elementary Lie group analysis and ordinary differential equations, John Wiley and Sons, Chichester (1999).

[19] E.H. El Kinani, A. Ouhadan, Lie symmetry analysis of some time fractional partial differential equations, Int. J. Mod. Phys. Conf. Ser. **38**(2015), 15600751-15600758.

[20] V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series. Longman Scientific and Technical, Longma Group, Harlow 1994.

[21] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: Jan V. Mill (Ed.), North-Holland Mathematics Studies, Elsevier, Amsterdam, The Netherlands, **204** (2006).

[22] Y. Laskri, N. Tatar, The critical exponent for an ordinary fractional differential problem, Computers and Mathematics with Applications **59** (2010), 1266-1270.

[23] F-S. Long, S.V. Meleshko, Symmetry analysis of the nonlinear two dimensional Klein-Gordon equation with a time-varying delay, Mathematical Methods in the Applied Sciences, **40** (2017), 4658-4673.

[24] F-S. Long, S. Meleshko, On the complete group classification of the one-dimensional nonlinear Klein-Gordon equation with a delay, Mathematical Methods in the Applied Sciences, **39(12)** (2016), 3255-3270.

[25] S.Y. Lukashchuk, A.V. Makunin, Group classification of nonlinear time-fractional diffusion equation with a source term, Appl.Math.Comput. **257**(2015), 335-343.

[26] J. Makungu, H. Haario, W.C. Mahera, A generalized 1-dimensional particle transport method for convection diffusion reaction model, Afr. Mat. (2012) 23:21-39.

[27] S.V. Meleshko, S. Moyo, On the complete group classification of the reaction diffusion equation with a delay, Journal of Mathematical Analysis and Applications, **2016** (2016), 963815.

[28] L. Morgadoa, N.J.Fordb, P.M. Limac, Analysis and numerical methods for fractional differential equations with delay, Journal of Computational and Applied Mathematics, **252** (2013), 159-168.

[29] A.M. Nass, Symmetry Analysis and Invariants Solutions of Laplace Equation on Surfaces of Revolution, Advances in Mathematics: Scientific Journal, **3(1)** (2014), 23-31.

[30] A.M. Nass, K. Mpungu, R. I. Nuruddeen, Group classification of the time fractional nonlinear Poisson equation, Math. Commun. 24(2019), 219-231.

[31] A.M. Nass, E. Fredericks, Lie Symmetry of Itô Stochastic Differential Equation Driven by Poisson Process, American Review of Mathematics and Statistics, **4(1)** (2016), 17-30.

[32] A.M. Nass, E. Fredericks, Symmetry of Jump-Diffusion Stochastic Differential Equations, Global and Stochastic Analysis, **3(1)** (2016), 11-23.

[33] A.M. Nass, E. Fredericks, W-symmetries of jump-diffusion It? stochastic differential equations, Nonlinear Dyn, **90(4)** (2017) , 2869-2877.

[34] L.V. Ovsiannikov, Group Analysis of Differential Equations, Nauka, Moscow (1978).

[35] Z. G. Ouyang, Existence and Uniqueness of the Solutions for a Class of Nonlinear Fractional Order Partial Differential Equations with delay, Comput. Math. Appl. 61 (2011), 860-870.

[36] V.G. Pimenov, A.S. Hendy, Numerical Studies for Fractional Functional Differential Equations with Delay Based on BDF-Type Shifted Chebyshev Approximations, Abstract and Applied Analysis, 2015.

[37] A.D. Polyanin, A.I. Zhurov, Exact separable solutions of delay reaction-diffusion equations and other nonlinear partial functional-differential equations, Communications in Nonlinear Science and Numerical Simulation, Vol. 19 (2014), No. 3, pp. 409-416.

[38] A.D. Polyanin, A.I. Zhurov, Functional constraints method for constructing exact solutions to delay reaction-diffusion equations and more complex nonlinear equations, Communications in Nonlinear Science and Numerical Simulation, Vol. 19, No. 3 (2014), pp. 417-430.

[39] A.D. Polyanin, A.I. Zhurov, New generalized and functional separable solutions to non-linear delay reaction-diffusion equations, International Journal of Non-Linear Mechanics, Vol. 59 (2014), p. 16-22.

[40] A.D. Polyanin, A.I. Zhurov, Generalized and functional separable solutions to nonlinear delay Klein-Gordon equations, Communications in Nonlinear Science and Numerical Simulation , Vol. 19 (2014), No. 8, pp. 2676-2689.

[41] A.D. Polyanin, A.I. Zhurov, Non-linear instability and exact solutions to some delay reaction-diffusion systems, Interna-

tional Journal of Non-Linear Mechanics, Vol. 62 (2014), pp. 33-40.
[42] A.D. Polyanin, A.I. Zhurov, Nonlinear delay reaction-diffusion equations with varying transfer coefficients: Exact methods and new solutions, Applied Mathematics Letters, Vol. 37 (2014), pp. 43-48.
[43] A.D. Polyanin, A.I. Zhurov, The functional constraints method: Application to non-linear delay reaction-diffusion equations with varying transfer coefficients, International Journal of Non-Linear Mechanics, Vol. 67 (2014), pp. 267-277.
[44] A.D. Polyanin, A.I. Zhurov, The generating equations method: Constructing exact solutions to delay reaction-diffusion systems and other non-linear coupled delay PDEs, International Journal of Non-Linear Mechanics, Vol. 71 (2015), pp. 104-115.
[45] A.D. Polyanin, V.G. Sorokin, Nonlinear delay reaction-diffusion equations: Traveling-wave solutions in elementary functions, Applied Mathematics Letters, Vol. 46 (2015), pp. 38-43.
[46] P. Pue-on, S.V. Meleshko, Group classification of second-order delay ordinary differential equations, Communications in Nonlinear Science and Numerical Simulation, **15** (2010), 1444-1453.
[47] K. Singla, R.K. Gupta, Generalized Lie symmetry approach for fractional order systems of differential equations III, Journal of Mathematical Physics **58** (2017), 061501.
[48] K. Singla, R.K. Gupta, Space-time fractional nonlinear partial differential equations: symmetry analysis and conservation laws, Nonlinear Dyn **89(1)** (2017), 321-331.
[49] K. Singla, R.K. Gupta, On invariant analysis of space-time fractional nonlinear systems of partial differential equations. II, Journal of Mathematical Physics **58** (2017), 051503.
[50] K. Singla, R.K. Gupta, On invariant analysis of some time fractional nonlinear systems of partial differential equations. I, Journal of Mathematical Physics **57** (2016), 101504 .
[51] K. Singla, R.K. Gupta, Comment on ?Lie symmetries and group classification of a class of time fractional evolution systems?, Journal of Mathematical Physics **57** (2017), 054101.
[52] J. W.H. So , W. Jianhong, Z. Xingfu, A reaction-diffusion model for a single species with age structure. I Travelling wavefronts on unbounded domains, Proceedings of the Royal Society of London. Series A Mathematical, Physical and Engineering Sciences, 457, 1841 (2001).
[53] P. Song, H. Zhao, X. Zhang, Dynamic analysis of a fractional order delayed predator-prey system with harvesting, Theory in Biosciences - Springer, **135** (2016), 59-72.
[54] V.E. Tarasov, On chain rule for fractional derivatives, Communications in Nonlinear Science and Numerical Simulation, **30(1-3)** (2016) , 1-4.
[55] J. Tanthanuch, Symmetry analysis of the nonhomogeneous inviscid burgers equation with delay, Communications in Nonlinear Science and Numerical Simulation **17** (2012), 4978-4987.
[56] J. Tanthanuch, S.V. Meleshko, On definition of an admitted Lie group for functional differential equations, Communications in Nonlinear Science and Numerical Simulation, **9** (2004), 117-125.
[57] G. Unal, Symmetries of Itô and Stratonovich Dynamical Systems and their Conserved Quantities, Nonlinear Dynamics, **32** (2003), 417-426.
[58] S.C. Wafo, F.M Mahomed, Integration of stochastic ordinary differential equations from a symmetry standpoint, Journal of Physics A: Mathematical and General; **34** (2001), 177-194.
[59] G.W. Wang, T.Z. Xu, T. Feng, Lie Symmetry Analysis and Explicit Solutions of the Time Fractional Fifth-Order KdV Equation, PLoS ONE9(2), (2014)
[60] Q. Zhang, M. Ran, D. Xu, Analysis of the compact difference scheme for the semilinear fractional partial differential equation with time delay, Applicable Analysis, **96(11)** (2017), 1867-1884.
[61] J. Zhang , J. Zhang, Symmetry analysis for time-fractional convection-diffusion equation, e-print arXiv:1512.01319.
[62] Q. Zhang, C. Zhang, A new linearized compact multisplitting scheme for the nonlinear convection-reaction-diffusion equations with delay, Commun Nonlinear Sci Numer Simulat 18 (2013), 3278-3288.
[63] B. Zhu, L. Liu, Y. Wu, Existence and Uniqueness of Global Mild Solutions for a Class of Nonlinear Fractional Reaction-diffusion Equations with delay, Comput. Math. Appl. (2016).