Available online: October 28, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 336–346 (2020)
DOI: 10.31801/cfsuasmas.473923
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

LIGHTLIKE HYPERSURFACES WITH PLANAR NORMAL SECTIONS IN $\mathbb{R}_1^4$

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Abstract. In the present paper our aim is to investigate lightlike hypersurfaces of $\mathbb{R}_1^4$ having degenerate or non-degenerate planar normal sections. Firstly, we prove that lightlike hypersurfaces in $\mathbb{R}_1^4$ always have degenerate planar normal sections. Then we examine the conditions for lightlike hypersurfaces in $\mathbb{R}_1^4$ to have non-degenerate planar normal sections and obtain some characterizations for such lightlike hypersurfaces.

1. Introduction

In Euclidean spaces, B.Y. Chen [2] initiated the study of surfaces with planar normal sections. After this, an important literature has been created on such surfaces and submanifolds (for example, see [2], [6], [7], [9],[8]). The semi-Riemannian adaptation of such surfaces was done by Y. H. Kim [7]. Recently, the authors ([12], [11]) introduced lightlike surfaces with planar normal sections in Minkowski 3-space and halflightlike submanifolds of $\mathbb{R}_2^4$ having degenerate and non-degenerate planar normal sections (see also [13]).

By a similar manner in [12] and [11] we define the normal section of a lightlike hypersurface $\acute{N}$ in $\mathbb{R}_1^4$ and non-degenerate planar normal sections as follows:

For a point p in a lightlike hypersurface $\acute{N}$ of $\mathbb{R}_1^4$ and a lightlike vector ξ such that the radical space $Rad(T\acute{N}) = Span\{\xi\}$, the vector ξ and transversal space $tr(T\acute{N})$ to $\acute{N}$ at p determine a 2-dimensional subspace $E(p,\xi)$ in $\mathbb{R}_1^4$ through p. The intersection $\acute{N} \cap E(p,\xi)$ gives rise to a lightlike curve α in a neighborhood of p, which we call normal section of $\acute{N}$ at the point p in the direction of ξ. If each normal section α at p in the direction of ξ satisfies $\alpha' \wedge \alpha'' \wedge \alpha''' = 0$, for each $p \in \acute{N}$, then we say that $\acute{N}$ has degenerate pointwise planar normal sections.

Received by the editors: October 23, 2018; Accepted: October 07, 2019.
2010 *Mathematics Subject Classification.* Primary 53C42; Secondary 53C50.
Key words and phrases. Lightlike hypersurface, non-degenerate planar normal section, degenerate normal section.

On the other hand, let w be a non-degenerate vector tangent to $\acute{N}$ at p such that $w \in S(T\acute{N}) = Sp\{u, v\}$, where $S(T\acute{N})$ is the screen distribution of $\acute{N}$. Then the vector w and transversal space $tr(T\acute{N})$ to $\acute{N}$ at p determine a 2- dimensional subspace $E(p, w)$ in $\mathbb{R}^4_1$ through p. From the intersection of $\acute{N}$ and $E(p, w)$, we have a non-degenerate curve α in a neighborhood of p which is called the normal section of $\acute{N}$ at p in the direction of w. In this case, if $\alpha' \wedge \alpha'' \wedge \alpha''' = 0$ is satisfied, for each point p in $\acute{N}$, where α is a normal section of $\acute{N}$ at p in the direction of w, then $\acute{N}$ is said to have non-degenerate pointwise planar normal sections.

In this paper, we study lightlike hypersurfaces in $\mathbb{R}^4_1$ having degenerate and nondegenerate planar normal sections. We prove that every lightlike hypersurfaces of $\mathbb{R}^4_1$ has degenerate planar normal sections. Also we obtain some results for a lightlike hypersurface with non-degenerate planar normal sections. We prove that a lightlike hypersurface $\acute{N}$ in $\mathbb{R}^4_1$ has non-degenerate planar normal sections if and only if it is either screen conformal and totally umbilical or totally geodesic. We also obtain a characterization for non-umbilical screen conformal lightlike hypersurface with non-degenerate planar normal sections.

2. Preliminaries

Let $(\breve{N}, \breve{g})$ be an $(n+2)$-dimensional semi-Riemannian manifold with the indefinite metric $\breve{g}$ of index $q \in \{1, ..., n+1\}$ and $\acute{N}$ be a hypersurface of $\breve{N}$. We denote the tangent space at $x \in \acute{N}$ by $T_x\acute{N}$. Then

$$T_x\acute{N}^{\perp} = \{V_x \in T_x\breve{N} \mid \breve{g}_x(V_x, W_x) = 0, \forall W_x \in T_x\acute{N}\}$$

and

$$Rad\, T_x\acute{N} = T_x\acute{N} \cap T_x\acute{N}^{\perp}.$$

Then, $\acute{N}$ is called a lightlike hypersurface of $\breve{N}$ if $Rad\, T_x\acute{N} \neq \{0\}$, for any $x \in \acute{N}$. Thus $T\acute{N}^{\perp} = \bigcap_{x\in\acute{N}} T_x\acute{N}^{\perp}$ becomes a 1- dimensional distribution $Rad\, T\acute{N}$ on $\acute{N}$. Then there exists a vector field $\xi \neq 0$ on $\acute{N}$ such that

$$g(\xi, X) = 0, \quad \forall X \in \Gamma(T\acute{N}),$$

where g is the induced degenerate metric tensor on $\acute{N}$. We denote the algebra of differential functions on $\acute{N}$ by $F(\acute{N})$ and the $F(\acute{N})$-module of differentiable sections of a vector bundle E over $\acute{N}$ by $\Gamma(E)$.

A complementary vector bundle $S(T\acute{N})$ of $T\acute{N}^{\perp} = Rad\, T\acute{N}$ in $T\acute{N}$ defined by

$$T\acute{N} = Rad\, T\acute{N} \oplus_{orth} S(T\acute{N}), \tag{1}$$

is called a screen distribution on $\acute{N}$. It follows from the equation above that $S(T\acute{N})$ is a non-degenerate distribution. Moreover, since we assume that $\acute{N}$ is para-compact, there always exists a screen $S(T\acute{N})$. Thus, along $\acute{N}$ we have

$$T\breve{N}_{|\acute{N}} = S(T\acute{N}) \oplus_{orth} S(T\acute{N})^{\perp}, \quad S(T\acute{N}) \cap S(T\acute{N})^{\perp} \neq \{0\}, \tag{2}$$

that is, $S(T\acute{N})^{\perp}$ is the orthogonal complement to $S(T\acute{N})$ in $T\breve{N}\mid_{\acute{N}}$. Note that $S(T\acute{N})^{\perp}$ is also a non-degenerate vector bundle of rank 2. However, it includes $T\acute{N}^{\perp} = Rad\, T\acute{N}$ as its sub-bundle.

Let $(\acute{N}, g, S(T\acute{N}))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\breve{N}, \breve{g})$. Then there exists a unique vector bundle $tr(T\acute{N})$ of rank 1 over $\acute{N}$, such that for any non-zero section ξ of $T\acute{N}^{\perp}$ on a coordinate neighborhood $U \subset \acute{N}$, there exists a unique section N of $tr(T\acute{N})$ on U satisfying: $T\acute{N}^{\perp}$ in $S(T\acute{N})^{\perp}$ and take $V \in \Gamma\left(F\mid_U\right), V \neq 0$. Then $\breve{g}\left(\xi, V\right) \neq 0$ on U, otherwise $S(T\acute{N})^{\perp}$ would be degenerate at a point of U [5]. Define a vector field

$$N = \frac{1}{\breve{g}\left(V,\xi\right)}\left\{V - \frac{\breve{g}\left(V,V\right)}{2\breve{g}\left(V,\xi\right)}\xi\right\},$$

on U where $V \in \Gamma\left(F\mid_U\right)$ such that $\breve{g}\left(\xi, V\right) \neq 0$. Then we have

$$\breve{g}\left(N,\xi\right) = 1,\ \breve{g}\left(N,N\right) = 0,\ \breve{g}\left(N,W\right) = 0,\ \forall W \in \Gamma(S(T\acute{N})|_U). \tag{3}$$

Moreover, from (1) and (2) we have the following decomposition:

$$T\breve{N}\mid_{\acute{N}} = S(T\acute{N}) \oplus_{orth} (T\acute{N}^{\perp} \oplus tr(T\acute{N})) = T\acute{N} \oplus tr(T\acute{N}). \tag{4}$$

Locally, suppose $\{\xi, N\}$ is a pair of sections on $U \subset \acute{N}$ satisfying (3). Define a symmetric $F(U)$-bi-linear form B and a 1-form τ on U . Hence on U, for $X, Y \in \Gamma(T\acute{N}\mid_U)$, we write

$$\breve{\nabla}_X Y = \breve{\nabla}_X Y + B\left(X,Y\right)N, \tag{5}$$

$$\breve{\nabla}_X N = -A_N X + \tau\left(X\right)N, \tag{6}$$

which are called local Gauss and Weingarten formula, respectively. Since $\breve{\nabla}$ is a metric connection on $\breve{N}$, it is easy to see that

$$B\left(X,\xi\right) = 0, \forall X \in \Gamma(T\acute{N}\mid_U). \tag{7}$$

Consequently, the second fundamental form of $\acute{N}$ is degenerate [5]. Define a local 1-from η by

$$\eta\left(X\right) = \breve{g}\left(X,N\right), \forall \in \Gamma(T\acute{N}\mid_U). \tag{8}$$

Let P denote the projection morphism of $\Gamma(T\acute{N})$ on $\Gamma(S(T\acute{N}))$ with respect to the decomposition (1). We obtain

$$\breve{\nabla}_X PY = \breve{\nabla}^*_X PY + C\left(X,PY\right)\xi, \tag{9}$$

$$\breve{\nabla}_X \xi = -A^*_\xi X + \varepsilon\left(X\right)\xi = -A^*_\xi X - \tau\left(X\right)\xi, \tag{10}$$

where $\breve{\nabla}^*_X PY$ and $A^*_\xi X$ belong to $\Gamma(S(T\acute{N})), \breve{\nabla}$ and $\breve{\nabla}^*$ are linear connections on $\Gamma(S(T\acute{N}))$ and $T\acute{N}^{\perp}$, respectively, h^* is a $\Gamma(T\acute{N}^{\perp})$-valued $F(\acute{N})$-bi-linear form on $\Gamma(T\acute{N})\times\ \Gamma(S(T\acute{N}))$ and A^*_ξ is $\Gamma(S(T\acute{N}))$-valued $F(\acute{N})$-linear operator on $\Gamma(T\acute{N})$.

We call them the screen fundamental form and screen shape operator of $S(T\acute{N})$, respectively. Define

$$\begin{aligned} C\left(X, PY\right) &= \breve{g}\left(h^*(X, PY\right), N), & (11)\\ \varepsilon(X) &= \breve{g}(\breve{\nabla}^{*t}_X \xi, N), \forall X, Y \in \Gamma(T\acute{N}). & (12)\end{aligned}$$

One can easily show that $\varepsilon\left(X\right) = -\tau\left(X\right)$. Here, $C\left(X, PY\right)$ is called the local screen fundamental form of $S(T\acute{N})$. Precisely, the two local second fundamental forms of $\acute{N}$ and $S(T\acute{N})$ are related to their shape operators by

$$\begin{aligned} B\left(X, Y\right) &= \breve{g}\left(Y, A^*_\xi X\right), & (13)\\ A^*_\xi \xi &= 0, & (14)\\ \breve{g}\left(A^*_\xi PY, N\right) &= 0, & (15)\\ C\left(X, PY\right) &= \breve{g}\left(PY, A_N X\right), & (16)\\ \breve{g}\left(N, A_N X\right) &= 0. & (17)\end{aligned}$$

A lightlike hypersurface $(\acute{N}, g, S(T\acute{N}))$ of a semi-Riemannian manifold is called totally umbilical[5] if there is a smooth function ϱ, such that

$$B\left(X, Y\right) = \varrho g\left(X, Y\right), \forall X, Y \in \Gamma(T\acute{N}), \qquad (18)$$

where ϱ is non-vanishing smooth function on a neighborhood U in $\acute{N}$.

A lightlike hypersurface $(\acute{N}, g, S(T\acute{N}))$ of a semi-Riemannian manifold is called screen locally conformal if the shape operators A_N and A^*_ξ of $\acute{N}$ and $S(T\acute{N})$, respectively, are related by

$$A_N = \varphi A^*_\xi, \qquad (19)$$

where φ is non-vanishing smooth function on a neighborhood U in $\acute{N}$. Therefore, it follows that for any $X, Y \in \Gamma(S(T\acute{N}))$ and $\xi \in Rad\, T\acute{N}$ we have

$$C\left(X, \xi\right) = 0. \qquad (20)$$

For details about screen conformal lightlike hypersurfaces, we refer [1] and [5] .

3. Planar Normal Sections of Lightlike Hypersurfaces in $\mathbb{R}^4_1$

Let $\acute{N}$ be a lightlike hypersurface of $\mathbb{R}^4_1$. Now we shall investigate lightlike hypersurfaces with degenerate planar normal sections. If α is a null curve, for a point p in $\acute{N}$, we have

$$\begin{aligned} \alpha'\left(s\right) &= \xi, & (21)\\ \alpha''\left(s\right) &= \breve{\nabla}_\xi \xi = -\tau\left(\xi\right)\xi, & (22)\\ \alpha'''\left(s\right) &= -\left[\xi\left(\tau\left(\xi\right)\right) + \tau^2\left(\xi\right)\right]\xi. & (23)\end{aligned}$$

Then, α''' is a linear combination of α' and α''. Thus from (21), (22) and (23), we conclude $\alpha''' \wedge \alpha'' \wedge \alpha' = 0$.

Hence we give

Corollary 1. *Every lightlike hypersurface of $\mathbb{R}^4_1$ has degenerate planar normal sections.*

Let $\acute{N}$ be a lightlike hypersurface of $\mathbb{R}^4_1$. For a point p in $\acute{N}$ and a spacelike vector $w \in S(T\acute{N}) = Sp\{u, v\}$, where u, v are unit spacelike vectors tangent to $\acute{N}$ at p , the vector w and transversal space $tr(T\acute{N})$ to $\acute{N}$ at p determine a 2-dimensional subspace $E(p,w)$ in $\mathbb{R}^4_1$ through p. The intersection of $\acute{N}$ and $E(p,w)$ gives a spacelike curve α in a neighborhood of p, which is called the normal section of $\acute{N}$ at p in the direction of w.

Now, we shall research the conditions for a lightlike hypersurface of $\mathbb{R}^4_1$ to have non-degenerate planar normal sections.

Let $(\acute{N}, g, S(T\acute{N}))$ be a totally umbilical and screen conformal lightlike hypersurface of $\left(\mathbb{R}^4_1, \breve{g}\right)$. In this case $S(T\acute{N})$ is integrable [1]. We denote integral hypersurface of $S(T\acute{N})$ by $\acute{N}'$. Then, using (6), (11) and (19) we find

$$\begin{aligned} C(w,w)\xi + B(w,w)N &= \breve{g}(w,w)\{\rho\xi + \beta N\} \\ &= \lambda\{\rho\xi + \beta N\},\ \lambda = a^2 + b^2, \end{aligned} \tag{24}$$

where $\lambda, \rho, \beta \in \mathbb{R}$. In this case, we obtain

$$\alpha'(s) = w, \tag{25}$$

$$\alpha''(s) = \breve{\nabla}^*_w w + C(w,w)\xi + B(w,w)N, \tag{26}$$

$$\alpha''(s) = \breve{\nabla}^*_w w + \rho\xi + \beta N, \tag{27}$$

and

$$\begin{aligned} \alpha'''(s) &= \breve{\nabla}^*_w \breve{\nabla}^*_w w + C(w, \breve{\nabla}^*_w w)\xi + w(C(w,w))\xi \\ &\quad - C(w,w)A^*_\xi w + w(B(w,w))N \\ &\quad - B(w,w)A_N w + B(w, \breve{\nabla}^*_w w)N, \end{aligned} \tag{28}$$

which implies

$$\begin{aligned} \alpha'''(s) &= \breve{\nabla}^*_w \breve{\nabla}^*_w w + C(w, \breve{\nabla}^*_w w)\xi \\ &\quad + B(w, \breve{\nabla}^*_w w)N - \rho A^*_\xi w - \beta A_N w. \end{aligned} \tag{29}$$

Here $\breve{\nabla}^*$ and $\breve{\nabla}$ are linear connections on $S(T\acute{N})$ and $\Gamma(T\acute{N})$, respectively and $\alpha'(s) = w = au + bv$, $a, b \in \mathbb{R}$. Since $\acute{N}$ is a totally umbilical screen conformal lightlike hypersurface, we find

$$C(w, \breve{\nabla}^*_w w)\xi + B(w, \breve{\nabla}^*_w w)N = g(w, \breve{\nabla}^*_w w)\{\rho_1\xi + \beta_1 N\}, \tag{30}$$

where $\rho_1, \beta_1 \in \mathbb{R}$. On the other hand we write

$$\breve{\nabla}^*_w w = a^2\breve{\nabla}^*_u u + ab\breve{\nabla}^*_u v + ab\breve{\nabla}^*_v u + b^2\breve{\nabla}^*_v v \tag{31}$$

and

$$g(w, \breve{\nabla}^*_w w) = a^3 g(u, \breve{\nabla}^*_u u) + a^2 b g(u, \breve{\nabla}^*_u v) + a^2 b g(u, \breve{\nabla}^*_v u) + ab^2 g(u, \breve{\nabla}^*_v v)$$

$$+a^2bg(v,\breve{\nabla}^*_u v)+ab^2g(v,\breve{\nabla}^*_u v)+ab^2g(v,\breve{\nabla}^*_v u)+b^3g(v,\breve{\nabla}^*_v v).$$

Since $\breve{g}\,(u,u)=\breve{g}\,(v,v)=1$ and $\breve{g}\,(u,v)=0$, then by a direct computation, we obtain

$$\breve{\nabla}^*_u u=\lambda_1 v,\ \breve{\nabla}^*_v u=\lambda_2 v, \tag{32}$$

$$\lambda_1=-\lambda_3, \tag{33}$$

$$\lambda_2=-\lambda_4, \tag{34}$$

$$\breve{\nabla}^*_u v=\lambda_3 u,\ \breve{\nabla}^*_v v=\lambda_4 u, \tag{35}$$

where $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{R}$. Hence, from (32)-(35) we get

$$g(w,\breve{\nabla}^*_w w)=0$$

and

$$C(w,\breve{\nabla}^*_w w)\xi+B(w,\breve{\nabla}^*_w w)N=0.$$

Therefore, we obtain

$$\begin{aligned} C(w,\breve{\nabla}^*_w w) &= 0,\\ B(w,\breve{\nabla}^*_w w) &= 0. \end{aligned}$$

Since $\acute{N}$ is screen conformal, we find

$$\begin{aligned} \alpha'(s) &= w,\\ \alpha''(s) &= \lambda\left(\rho\xi+\beta N\right),\\ \alpha'''(s) &= -\lambda\rho A^*_\xi w-\lambda\beta A_N w, \end{aligned}$$

where $\rho,\beta\neq 0$. Then, we have

$$\alpha'''(s)=tA^*_\xi w,\ t=-2\lambda\rho.$$

Hence, we obtain

$$B\,(w,w)=g\left(A^*_\xi w,w\right)=\beta g\,(w,w)=g\,(\beta w,w)\,,$$

which implies $A^*_\xi w=\beta w$, that is, α' and α''' are linearly dependent and so $\acute{N}$ has non-degenerate planar normal sections.

Assume that $\acute{N}$ is a totally geodesic lightlike hypersurface of $\mathbb{R}^4_1$. Then, we have $B=0$, $A^*_\xi=0$. Hence, from (25)-(28), we write

$$\alpha'(s) = w, \tag{36}$$

$$\alpha''(s) = \breve{\nabla}^*_w w, \tag{37}$$

$$\alpha'''(s) = \breve{\nabla}^*_w\breve{\nabla}^*_w w. \tag{38}$$

Since $\alpha',\alpha'',\alpha'''\in\Gamma(S(T\acute{N}))$ and $\dim(S(T\acute{N}))=2$, we have $\alpha'''(s)\wedge\alpha''(s)\wedge\alpha'(s)=0$.

Conversely, we assume that $\acute{N}$ has non-degenerate planar normal sections. Then, from (25), (26) and (28) we obtain

$$w \wedge \begin{pmatrix} \breve{\nabla}^*_w w + C(w,w)\xi \\ +B(w,w)N \end{pmatrix} \wedge \begin{pmatrix} \breve{\nabla}^*_w \breve{\nabla}^*_w w + C(w, \breve{\nabla}^*_w w)\xi + w(C(w,w))\xi \\ -C(w,w)A^*_\xi w + w(B(w,w))N \\ -B(w,w)A_N w + B(w, \breve{\nabla}^*_w w)N \end{pmatrix} = 0.$$

Since $w = au + bv$, $a, b \in \mathbb{R}$, for the sake of simplicity, we choose $u = (0,1,0,0)$ and $v = (0,0,1,0)$, which give

$$\breve{\nabla}^*_w w = \left(0, ab\lambda_3 + b^2\lambda_4, a^2\lambda_1 + ab\lambda_2, 0\right). \tag{39}$$

If we take $a = b = 1$, from (32)-(34), we obtain

$$\breve{\nabla}^*_w w = (0, -(\lambda_1 + \lambda_2), \lambda_1 + \lambda_2, 0),$$

which yields that w and $\breve{\nabla}^*_w w$ are linearly dependent. Thus we find

$$w \wedge \breve{\nabla}^*_w w = 0 \tag{40}$$

for any $a, b \in \mathbb{R}$. Moreover, if we take $a, b \in \{-1, 1\}$, we have

$$\breve{\nabla}^*_w w = (0, b(a\lambda_1 + b\lambda_2), a(a\lambda_1 + b\lambda_2), 0),$$

namely, in any case w and $\breve{\nabla}^*_w w$ are linearly dependent.

From (31), we find

$$\begin{aligned} \breve{\nabla}^*_w \breve{\nabla}^*_w w &= a^3\lambda_1\lambda_3 u + a^2 b\lambda_1\lambda_3 v + a^2 b\lambda_2\lambda_3 u + ab^2\lambda_4\lambda_1 v \\ &\quad + a^2 b\lambda_1\lambda_4 u + a^2 b\lambda_2\lambda_3 v + ab^2\lambda_4\lambda_2 u + b^3\lambda_4\lambda_2 v. \end{aligned}$$

Here, for simplicity, if we take $a = b = 1$ then we obtain

$$\breve{\nabla}^*_w \breve{\nabla}^*_w w = \left(0, \lambda_1^2 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2^2, \lambda_1^2 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2^2, 0\right),$$

which yields

$$w \wedge \breve{\nabla}^*_w \breve{\nabla}^*_w w = 0. \tag{41}$$

Then we have

$$w \wedge (C(w,w)\xi + B(w,w)N) \wedge \left(\breve{\nabla}_w (C(w,w)\xi + B(w,w)N)\right) = 0. \tag{42}$$

Thus $C(w,w)\xi + B(w,w)N = 0$ or $\breve{\nabla}_w (C(w,w)\xi + B(w,w)N) = 0$. If $C(w,w)\xi + B(w,w)N = 0$, then $C = B = 0$, at $p \in \acute{N}$, which implies that $\acute{N}$ is totally geodesic and totally umbilical. If $\breve{\nabla}_w (C(w,w)\xi + B(w,w)N) = 0$, then we have

$$w(C(w,w))\xi + w(B(w,w))N - C(w,w)A^*_\xi w - B(w,w)A_N w = 0. \tag{43}$$

Hence $C(w,w)A^*_\xi w + B(w,w)A_N w = 0$, we find

$$A^*_\xi w = -\frac{B(w,w)}{C(w,w)} A_N w, \tag{44}$$

at $p \in \acute{N}$, which shows that $\acute{N}$ is a screen conformal lightlike hypersurface.

Consequently, we have the following.

Theorem 2. *Let $\acute{N}$ be a lightlike hypersurface of $\mathbb{R}^4_1$. Then $\acute{N}$ has non-degenerate planar normal sections if and only if either $\acute{N}$ is totally umbilical and screen conformal or $\acute{N}$ is totally geodesic.*

Proof. Assume that $\acute{N}$ is a totally umbilical and screen conformal lightlike hypersurface of $\mathbb{R}^4_1$. Then we have $A^*_\xi w = \beta w$, $\beta \in \mathbb{R}$. By using (25), (27) and (29), we obtain

$$\alpha''' (s) \wedge \alpha'' (s) \wedge \alpha' (s) = 0.$$

If we consider that $\acute{N}$ is totally geodesic, then, we have $C = B = 0$ and from (36)–(38), we see that w, $\breve{\nabla}^*_w w$ and $\breve{\nabla}^*_w \breve{\nabla}^*_w w$ belong to $S(T\acute{N})$. Since $\dim(S(T\acute{N})) = 2$, we conclude that $\alpha', \alpha'', \alpha'''$ are linearly dependent.

Conversely, we assume that $\acute{N}$ has non-degenerate planar normal sections. Then, from (42)–(44) we complete the proof. □

Theorem 3. *Let $(\acute{N}, g, S(T\acute{N}))$ be a screen conformal non-umbilical lightlike hypersurface of $\mathbb{R}^4_1$. Then, for $T (w, w) = C (w, w) \xi + B (w, w) N$, the following statements are equivalent:*

(1) $(\breve{\nabla}_w T) (w, w) = 0$, *for every spacelike vector* $w \in S(T\acute{N})$,
(2) $\breve{\nabla} T = 0$,
(3) *$\acute{N}$ has non-degenerate planar normal sections and each normal section at p has one of its vertices at p.*

Note that, by the vertex of curve $\alpha (s)$ we mean a point p on α such that its curvature κ satisfies $\frac{d\kappa^2(p)}{ds} = 0$, where $\kappa^2 = \langle \alpha'' (s) , \alpha'' (s)\rangle$.

Proof. From (25), (26), we have

$$(\breve{\nabla}_w T) (w, w) = \breve{\nabla}_w T (w, w) ,$$

which shows $(\breve{\nabla}_w T) (w, w) = 0$ if and only if $\breve{\nabla} T = 0$.

Assume that $\breve{\nabla} T = 0$. Then $\acute{N}$ is totally geodesic and Theorem 2 implies that $\acute{N}$ has (pointwise) planar normal sections. Let the $\alpha (s)$ be a normal section of $\acute{N}$ at p in a given direction $w \in S(T\acute{N})$. Then (25) shows that the curvature $\kappa (s)$ of $\alpha (s)$ satisfies

$$\begin{aligned} \kappa^2 (s) &= \langle \alpha'' (s) , \alpha'' (s)\rangle \\ &= 2C(w, w)B(w, w) \\ &= \langle T (w, w) , T (w, w)\rangle , \end{aligned} \tag{45}$$

where $w = \alpha' (s)$. Therefore we find

$$\frac{d\kappa^2 (p)}{ds} = \left\langle \breve{\nabla}_w T (w, w) , T (w, w) \right\rangle = \left\langle (\breve{\nabla}_w T) (w, w) , T (w, w) \right\rangle . \tag{46}$$

Since $\breve{\nabla}_w T(w,w) = 0$, this implies

$$\frac{d\kappa^2(0)}{ds} = 0,$$

at $p = \alpha(0)$. Thus p is a vertex of the normal section $\alpha(s)$.

If $\acute{N}$ has planar normal sections, then by using Theorem 2 we have

$$T(w,w) \wedge (\breve{\nabla}_w T)(w,w) = 0. \tag{47}$$

If p is a vertex of $\alpha(s)$, then we have

$$\frac{d\kappa^2(0)}{ds} = 0.$$

Thus, since $\acute{N}$ has planar normal sections, using (46) we find

$$\begin{aligned} \alpha'(s) \wedge \alpha''(s) \wedge \alpha'''(s) &= w \wedge (\breve{\nabla}^*_w w + T(w,w)) \\ &\quad \wedge (\breve{\nabla}^*_w \breve{\nabla}^*_w w + tT(w,w) + (\breve{\nabla}_w T)(w,w)) = 0, \end{aligned}$$

which yields

$$T(w,w) \wedge (\breve{\nabla}_w T)(w,w) = 0$$

and

$$\Big\langle (\breve{\nabla}_w T)(w,w), T(w,w) \Big\rangle = 0. \tag{48}$$

Combining (47) and (48) we obtain either $(\breve{\nabla}_w T)(w,w) = 0$ or $T(w,w) = 0$. Let us define $U = \Big\{ w \in S(T\acute{N}) \mid T(w,w) = 0 \Big\}$. If $int(U) \neq \varnothing$, we obtain $(\breve{\nabla}_w T)(w,w) = 0$ on $int(U)$. Thus, by continuity we have $\breve{\nabla} T = 0$. □

Considering those obtained results above with [12], we give the following example.

Example 4. *Let $\mathbb{R}^4_1$ be the space $\mathbb{R}^4$ endowed with the semi-Euclidean metric*

$$\breve{g}(x,y) = -u_0 v_0 + \sum_{a=1}^{3} u_a v_a, \qquad u = \sum_{a=0}^{3} u_a \frac{\partial}{\partial u_a}.$$

Consider the null cone of $\mathbb{R}^4_1$ given by

$$\wedge^3_0 = \left\{ (u_0, u_1, u_2, u_3) \mid -u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0, u_0,\ u_1,\ u_2,\ u_3 \in \mathbb{R} \right\}.$$

The radical bundle of null cone is

$$\xi = u_0 \frac{\partial}{\partial u_0} + u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_3}$$

and screen distribution is spanned by

$$w = -u_2 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2} - u_3 \frac{\partial}{\partial u_3}.$$

Then the lightlike transversal vector bundle is given by

$$Itr(T\wedge_0^3) = Span\left\{N = \frac{1}{2(u_0)^2}\left(-u_0\frac{\partial}{\partial u_0} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3}\right)\right\}.$$

Let $\wedge_0^3$ *be a lightlike hypersurfaces of* $\mathbb{R}_1^4$. *For a point* p *in* $\wedge_0^3$ *and a lightlike vector* ξ *which spans the radical distribution of a lightlike hypersurface, the vector* ξ *and transversal space* $tr(T\wedge_0^3)$ *to* $\wedge_0^3$ *at* p *determine a 2- dimensional subspace* $E(p,\xi)$ *in* $\mathbb{R}_1^4$ *through* p. *The intersection of* $\wedge_0^3$ *and* $E(p,\xi)$ *gives a lightlike curve* α *in a neighborhood of* p, *which is called the normal section of* $\wedge_0^3$ *at the point* p *in the direction of* ξ. *Therefore, we have*

$$\begin{aligned}\breve{\nabla}_\xi\xi &= u_0\frac{\partial}{\partial u_0} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3}\\ \breve{\nabla}_\xi\breve{\nabla}_\xi\xi &= u_0\frac{\partial}{\partial u_0} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3}.\end{aligned}$$

Then, we obtain

$$\alpha'''(s)\wedge\alpha''(s)\wedge\alpha'(s) = 0$$

which shows that null cone has degenerate planar normal sections.

On the other hand, by direct computations, we find

$$\breve{\nabla}_{\xi}w = \breve{\nabla}_\xi w = w$$

and

$$A_N w = \frac{1}{2(u_0)^2}A_\xi^* w.$$

Namely, $\wedge_0^3$ *is a screen conformal lightlike hypersurface of* $\mathbb{R}_1^4$ [5].

Now, for a point p *in* $\wedge_0^3$ *and a non-degenerate vector* w *tangent to* $\wedge_0^3$ *at* p $(w \in S(T\wedge_0^3))$, *the vector* w *and transversal space* $tr(T\wedge_0^3)$ *to* $\acute{N}$ *at* p *determine a 2- dimensional subspace* $E(p,w)$ *in* $\mathbb{R}_1^4$ *through* p. *The intersection of* $\wedge_0^3$ *and* $E(p,w)$ *gives a non-degenerate curve* α *in a neighborhood of* p. *Therefore, we have*

$$\begin{aligned}\alpha' &= w = -u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3},\\ \alpha'' &= \breve{\nabla}_w w + B(w,w)N\\ &= \frac{1}{2}u_0\frac{\partial}{\partial u_0} - \frac{3}{2}u_1\frac{\partial}{\partial u_1} - \frac{3}{2}u_2\frac{\partial}{\partial u_2} - \frac{3}{2}u_3\frac{\partial}{\partial u_3},\\ \alpha''' &= \breve{\nabla}_w\breve{\nabla}_w w + w(B(w,w))N + B(w,w)\breve{\nabla}_w N\\ &= \breve{\nabla}_w\breve{\nabla}_w w + B(w,\breve{\nabla}_w w)N + w(B(w,w))N - B(w,w)A_N w.\end{aligned}$$

Using $A_N w$ *in* α''' *we find*

$$\alpha''' = -\frac{1}{2}\left(-u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3}\right).$$

Therefore α''' and α' are linearly dependent at $p \in \wedge_0^3$ and we have

$$\alpha' \wedge \alpha'' \wedge \alpha''' = 0.$$

Namely, $\wedge_0^3$ has non-degenerate planar normal sections.

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