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SPECTRAL DISJOINTNESS AND INVARIANT SUBSPACES

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IN MEMORY OF RISTEARD TIMONEY

Abstract. Spectral disjointness confers a certain mutual independence on pairs of Banach algebra elements. Necessary and sufficient for full spectral disjointness of diagonal elements is that the structural idempotent is a holomorphic function of a block diagonal matrix, while a partial left-right spectral disjointness is sufficient for membership of the double commutant. For bounded linear Banach space operators with an invariant subspace, spectral disjointness for the restriction and quotient operators implies both hyperinvariance and reducing.

1. BLOK STRUCTURE

Our "spectral disjointness" applies to pairs of operators defined on different spaces, and we need a somewhat elaborate algebraic framework for them: accordingly, we look at matrices with block structure.

If G is a ring, with identity I, then [7] an idempotent

$$Q = Q^2 \in G$$

imposes a *block structure* on G:

$$G \cong \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

where A and B are rings with identity in their own right, while M and N are bimodules over A and B; there are also bilinear mappings

$$(m,n) \mapsto m \cdot n \ (M \times N \to A) \ ; \ (m,n) \mapsto n \cdot m \ (M \times N \to B)$$

The structure is laid bare by formal multiplication of 2×2 matrices. We can take

$$A = QGQ \ ; \ M = QG(I-Q) \ ; \ N = (I-Q)GQ \ ; \ B = (I-Q)G(I-Q)$$

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The identity I, the structural idempotent Q and a generic element $T \in G$ are now given by block matrices:

1.5 $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ; \ Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ ; \ T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} .$$

The commutant of the structural idempotent is the subring of *block diagonals*,

$$\mathrm{comm}(Q) = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \subseteq G$$

In the notation of (1.5),

1.7 $$QT = TQ \Longleftrightarrow T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ .$$

More generally [13] there are upper and lower *block triangles*:

$$QT = QTQ \Longleftrightarrow T = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \begin{pmatrix} A & M \\ O & B \end{pmatrix}$$

$$TQ = QTQ \Longleftrightarrow T = \begin{pmatrix} a & 0 \\ n & b \end{pmatrix} \in \begin{pmatrix} A & O \\ N & B \end{pmatrix}$$

2. INVERTIBILITY

An element $T \in G$ is said to be *invertible*, written $T \in G^{-1}$, if there is another element $T^{-1} \in G$, for which

$$T^{-1}T = I = TT^{-1}$$

More generally if

$$T'T = I$$

then we say that $T \in G^{-1}_{left}$ is *left invertible* and $T' \in G^{-1}_{right}$ is *right invertible*; we observe

$$G^{-1} = G^{-1}_{left} \cap G^{-1}_{right}$$

that the invertible group is the intersection of the left and right invertible semigroups. In general it is quite a complicated business to express the invertibility or otherwise of an element $T \in G$ in terms of the contributing elements $a \in A$, $m \in M$, $n \in N$ and $b \in B$ of (1.5); for the block diagonals of (1.7) it is however rather simple:

2.4 $$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G^{-1}_{left} \Longleftrightarrow a \in A^{-1}_{left} \ \& \ b \in B^{-1}_{left}$$

and

2.5 $$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G^{-1}_{right} \Longleftrightarrow a \in A^{-1}_{right} \ \& \ b \in B^{-1}_{right} \ ,$$

and hence

$$T \in G^{-1} \Longleftrightarrow a \in A^{-1} \ \& \ b \in B^{-1}$$

For upper block triangles [6] something more subtle obtains:

2.7 $$a \in A^{-1}_{left} \ \& \ b \in B^{-1}_{left} \Longrightarrow \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in G^{-1}_{left} \Longrightarrow a \in A^{-1}_{left}$$

2.8 $$a \in A^{-1}_{right} \ \& \ b \in B^{-1}_{right} \Longrightarrow \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in G^{-1}_{right} \Longrightarrow b \in B^{-1}_{right}$$

Also

2.9 $$T \in G^{-1}_{left} \ \& \ a \ \in A^{-1}_{right} \Longrightarrow b \in B^{-1}_{left}$$

and

2.10 $$T \in G^{-1}_{right} \ \& \ b \ \in B^{-1}_{left} \Longrightarrow a \in A^{-1}_{right} \ .$$

It follows, that of the three assertions

$$T \in G^{-1} \ ; \ a \in A^{-1} \ ; \ b \in B^{-1}$$

any two imply the third.

3. SPECTRUM

If the rings G, A and B are complex linear algebras, then invertibility breeds *spectrum*

3.1 $$\sigma_G^{left}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin G^{-1}_{left}\} \ ,$$

and

3.2 $$\sigma_G^{right}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin G^{-1}_{right}\} \ ,$$

and then

$$\sigma_G(T) = \sigma_G^{left}(T) \cup \sigma_G^{right}(T)$$

with corresponding notation for $\sigma_A(a)$ and $\sigma_B(b)$. Thus, for a block diagonal $T \in G$, we can rewrite (2.4) and (2.5) in the form

$$\sigma_G^{left}(T) = \sigma_A^{left}(a) \cup \sigma_B^{left}(b)$$

and

$$\sigma_G^{right}(T) = \sigma_A^{right}(a) \cup \sigma_B^{right}(b)$$

For upper block triangles $T \in G$, (2.7) and (2.8) take the form

$$\sigma_A^{left}(a) \subseteq \sigma_G^{left}(T) \subseteq \sigma_A^{left}(a) \cup \sigma_B^{left}(b)$$

and

$$\sigma_B^{right}(b) \subseteq \sigma_G^{right}(T) \subseteq \sigma_B^{right}(b) \cup \sigma_A^{right}(a)$$

Also (2.9) and (2.10) take the form

$$\sigma_B^{left}(b) \subseteq \sigma_G^{left}(T) \cup \sigma_A^{right}(a)$$

and

$$\sigma_A^{right}(a) \subseteq \sigma_G^{right}(T) \cup \sigma_B^{left}(b)$$

It follows that, of the three sets

$$\sigma_G(T) \ ; \ \sigma_A(a) \ ; \ \sigma_B(b)$$

each is a subset of the union of the other two:

$$\sigma_G(T) \subseteq \sigma_A(a) \cup \sigma_B(b) \cup \big(\sigma_A(a) \cap \sigma_B(b)\big)$$

We can improve on this: by (2.7)-(2.10) we have ([6] Theorem 3.1, Theorem 3.2)

$$\sigma_A(a) \cup \sigma_B(b) = \sigma_G(T) \cup \big(\sigma_A^{right}(a) \cap \sigma_B^{left}(b)\big)$$

4. SPECTRAL DISJOINTNESS

When the linear algebras G, A and B are complex Banach algebras, then the spectral theory begins to bite. When the structural idempotent $Q = Q^2 \in G$ is bounded, then it is necessary and sufficient, for spectral disjointness

4.1 $$\sigma_A(a) \cap \sigma_B(b) = \emptyset \ ,$$

that

4.2 $$Q \in \mathrm{Holo}(T) \ :$$

the structural idempotent is a holomorphic function of the generic $T \in G$ of (1.5). This of course means that there exists a holomorphic function $f : U \to \mathbf{C}$ defined on an open neighbourhood of the spectrum $\sigma_G(T) = \sigma_A(a) \cup \sigma_B(b)$ for which

$$Q = f(T) = \frac{1}{2\pi i} \oint_{\sigma(T)} f(z)(zI - T)^{-1} dz$$

is given by the Cauchy integral formula. Inspecting the contour integral, which winds $+1$ times round the spectrum of T, it is sufficient, and obviously necessary, that Q lies in the closed subalgebra generated by all rational functions of T: this is generated by the polynomials in T, together with all possible inverses $(\lambda I - T)^{-1}$. To see why the disjointness (4.1) gives (4.2), it is sufficient to take the *characteristic function*

$$f = \chi_K \ with \ K = \sigma_A(a)$$

Conversely if $Q = f(T)$ then $a = f(1)$ and $b = f(0)$ and hence, by the spectral mapping theorem,

$$\sigma_A(a) \cap \sigma_B(b) \subseteq f^{-1}(1) \cap \ f^{-1}(0) = \emptyset$$

Since the block diagonal T is in the commutant of the idempotent Q, it follows that generally the idempotent Q is also in the commutant of the block diagonal T. If however it turns out ([7] Theorem 1;[10]) that the idempotent Q is a holomorphic function of T, then it follows that the idempotent is in the *double commutant* of the block diagonal:

4.6 $$Q \in \mathrm{comm}^2(T) \ .$$

In finite dimensions, in particular for matrices, it turns out [14] that everything in the double commutant of T is a polynomial in T, and hence (4.6) and (4.2) are equivalent. In general Banach algebras, as we shall see, (4.2) is strictly stronger than (4.6). This whole argument extends [13] to upper and lower block triangles.

We might notice here another "spectral disjointness": if for example $f = p/q$ is a rational function, with "relatively prime" polynomials p and q,

$$f = \frac{p}{q} \in H = C(\Omega) \ with \ \Omega = D_f = \mathbf{C} \setminus q^{-1}(0)$$

then necessary and sufficent for $f(T)$ to exist is

$$\sigma_H(f) \cap \sigma_G(T) = \emptyset$$

none of the poles $q^{-1}(0)$ of f can be in the spectrum of T. For example

$$f = z^{-1} \Longrightarrow \sigma_H(f) = \{0\}$$

thus

$$0 \notin \sigma_G(T) \Longleftrightarrow T \in G^{-1}$$

5. PARTIAL SPECTRAL DISJOINTNESS

In Banach algebras we claim ([7] Theorem 2;[10]) that a weaker "left,right" spectral disjointness is sufficient for the double commutant property :

5.1 $$\sigma_A^{left}(a) \cap \sigma_B^{right}(b) = \emptyset$$

and

5.2 $$\sigma_A^{right}(a) \cap \sigma_B^{left}(b) = \emptyset$$

are together sufficient for (4.6). Specifically we claim that (5.1) implies

$$L_a - R_b \in B(M)^{-1}_{left}$$

the generalized inner derivation $L_a - R_b \in E = B(M)$ has a bounded left inverse. This is the spectral mapping theorem in two variables. With no need of tensor product theory

$$\sigma_E^{left}(L_a, R_b) \subseteq \sigma_E^{left}(L_a) \times \sigma_E^{left}(R_b) \subseteq \sigma_A^{left}(a) \times \sigma_B^{right}(b)$$

and then, since L_a and R_b commute, by the spectral mapping theorem

$$0 \in \sigma_E^{left}(L_a - R_b) \Longrightarrow 0 \in \sigma_A^{left}(a) - \sigma_B^{right}(b)$$

and the spectral disjointness (5.1) excludes 0 from the right hand side. If the inner derivation $L_a - R_b$ has a bounded left inverse then it is also "bounded below", and hence in particular one-to-one: if $m \in M$ there is implication

$$am = mb \Longrightarrow m = 0$$

This is one step on the way to the double commutivity (4.6). If instead (5.2) holds then instead the generalized derivation $L_b - R_a \in F = B(N)$ is left invertible and hence also one-one. Now for arbitrary $(c, u, v, d) \in A \times M \times N \times B$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & u \\ v & d \end{pmatrix} - \begin{pmatrix} c & u \\ v & d \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ac - ca & au - ub \\ bv - va & bd - db \end{pmatrix}$$

It follows that if $S = \begin{pmatrix} c & u \\ v & d \end{pmatrix}$ commutes with $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then c commutes with a and d commutes with b, while

$$(L_a - R_b)u = 0 \in M \textit{ and } (L_b - R_a)v = 0 \in N$$

The condition (5.1) therefore ensures that S is a lower block triangle, while (5.2) makes it an upper block triangle, and together they put it in the commutant of Q, givng the inclusion

$$\mathrm{comm}(T) \subseteq \mathrm{comm}(Q)$$

which is equivalent to (4.6).

The condition (5.2) also says that $L_a - R_b$ has a bounded right inverse in $E = B(M)$ and hence is onto:

$$M = aM + Mb$$

which confers a certain splitting "left,right exactness" [9] on the pair (a, b). Dually (5.1) says that also

$$N = bN + Na$$

Notice also [10] that left,right spectral disjointness makes block triangles "similar" to their block diagonals.

6. LINEAR OPERATORS

If the linear algebra $G = L(X)$ is all the linear operators on a linear vector space X, then an *invariant subspace* for $T \in G$ is a subspace $Y \subseteq X$ for which

6.1 $$T(Y) \subseteq Y \subseteq X \ .$$

In the purely linear environment, this will confer block structure on the algebra $L(X)$. For Banach algebra structure we need a Banach space, and to look at bounded operators $T \in B(X)$; evidently we will only be interested in invariant subspaces $Y \subseteq X$ which are norm closed. It is now not clear that this confers block structure on $G = B(X)$: it is necessary that the invariant subspace is also *complemented*. We can however still mount a similar discussion, courtesy of the *quotient*:

$$X/Y = \{[x]_Y \equiv x + Y : x \in X\}$$

the set of cosets $x + Y$, normed by the distance function:

$$\|[x]_Y\| = \mathrm{dist}(x, Y) = \inf\{\|x - y\| : y \in Y\}$$

Now if (6.1) holds then the operator $T \in G = L(X)$ has a *restriction*

$$T_Y \in L(Y)$$

and a *quotient*

$$T_{/Y} \in L(X/Y)$$

defined by setting, for each $y \in Y$ and each $x \in X$,

$$T_Y(y) = Ty \ ; \ T_{/Y}([x]_Y) = [Tx]_Y$$

When $T \in B(X)$ is bounded on a Banach space and $Y \subseteq X$ is closed, then both the restriction and the quotient are also bounded.

As in the block matrix situation the invertibility of $T \in G = L(X)$, $T_Y = a \in A = L(Y)$ and $T_{/Y} = b \in B = L(X/Y)$ are mutually constrained. In the purely linear environment, necessary and sufficient for two-sided invertibility is that an operator is both one-one and onto; for bounded operators on Banach space this continues to be the case, courtesy of the "Open Mapping Theorem". To see the mutual constraints observe [2] the implications

6.7 $$T_Y, T_{/Y} \textit{ one-one} \Longrightarrow T \textit{ one-one} \Longrightarrow T_Y \textit{ one-one} \ ;$$

6.8 $$T_Y, T_{/Y} \textit{ onto} \Longrightarrow T \textit{ onto} \Longrightarrow T_{/Y} \textit{ onto} \ ;$$

6.9 $$T \textit{ one-one} \ , \ T_Y \textit{ onto} \Longrightarrow T_{/Y} \textit{ one-one} \ ;$$

6.10 $$T \textit{ onto} \ , \ T_{/Y} \textit{ one-one} \Longrightarrow T_Y \textit{ onto} \ .$$

To verify these implications, express non singularity properties of T_Y and $T_{/Y}$ in terms of the whole space X:

$$T_Y \textit{ one-one} \Longleftrightarrow T^{-1}(0) \cap Y \subseteq O \equiv \{0\}$$

$$T_Y \textit{ onto} \Longleftrightarrow Y \subseteq T(Y)$$

$$T_{/Y} \textit{ one-one} \Longleftrightarrow T^{-1}(Y) \subseteq Y$$

$$T_{/Y} \textit{ onto} \Longleftrightarrow X \subseteq Y + T(X)$$

7. SPECTRAL THEORY

The spectrum of $T \in G$ is the same as always:

$$\sigma(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin G^{-1}\}$$

The *point spectrum* or *eigenvalues* of $T \in G$ is

$$\pi(T) = \{\lambda \in \mathbf{C} : (T - \lambda I)^{-1}(0) \neq \{0\}\} \subseteq \sigma^{left}(T)$$

The *defect spectrum* is in a sense dual to the point spectrum:

$$\pi'(T) = \{\lambda \in \mathbf{C} : (T - \lambda I)(X) \neq X\} \subseteq \sigma^{right}(T)$$

Evidently

$$\sigma(T) = \pi(T) \cup \pi'(T)$$

this is true both for $G = L(X)$ and for $G = B(X)$. From the implications (6.7)-(6.10) it follows that

7.5 $$\sigma(T) \subseteq \sigma(T_Y) \cup \sigma(T_{/Y}) \subseteq \sigma(T) \cup (\sigma(T_Y) \cap \sigma(T_{/Y})) \ .$$

It follows that disjointness

7.6 $$\sigma(T_Y) \cap \sigma(T_{/Y}) = \emptyset$$

implies equality

$$\sigma(T) = \sigma(T_Y) \cup \sigma(T_{/Y})$$

We see (7.6), in the Banach space situation, as a significant property of the invariant subspace $T(Y) \subseteq Y \subseteq X$: when it holds we shall describe the subspace Y as *spectrally invariant*.

Barnes ([1] Proposition 4) has an improvement (*cf* (3.11)) on the right hand side of (7.5): by (6.7)-(6.10)

$$\sigma(T_Y) \cup \sigma(T_{/Y}) = \sigma(T) \cup (\pi'(T_Y) \cap \pi(T_{/Y}))$$

8. PARTIALLY HYPERINVARIANT SUBSPACES

When $T \in G = B(X)$ is a bounded operator on a Banach space X then we describe a subspace $Y \subseteq X$ as an "invariant subspace" for T provided it is norm closed and satisfies the inclusion (6.1). We describe it as *hyperinvariant* provided

8.1 $$\mathrm{comm}(T)Y \subseteq Y \ :$$

this means that there is implication, for $S \in G$,

$$ST = TS \Longrightarrow S(Y) \subseteq Y \subseteq X$$

More generally we shall describe a subspace $Y \subseteq X$ as *comm-square invariant* for $T \in G$ provided

8.3 $$\mathrm{comm}^2(T)Y \subseteq Y \ .$$

More generally still we will say that Y is *holomorphically invariant* for T when

8.4 $$\mathrm{Holo}(T)Y \subseteq Y \ .$$

Evidently this is the same as *inverse invariant*, in the sense that if $\lambda \in \mathbf{C}$ there is implication

$$T - \lambda I \in G^{-1} \Longrightarrow (T - \lambda I)^{-1}Y \subseteq Y$$

There is obvious implication

$$(8.1) \Longrightarrow (8.3) \Longrightarrow (8.4) \Longrightarrow (6.1)$$

It turns out [2] that none of these three implications is reversible; the counterexamples can all be taken to be 2×2 matrices of familiar operators such as the forward and backward shift. It also turns out that a spectrally invariant subspace $Y \subseteq X$, in the sense of (7.6), is hyperinvariant, in the sense (8.1), and also *reducing*: this means that it has an invariant *complement*, in the sense of a closed subspace $Z \subseteq X$ for which

$$Y + Z = X \ , \ Y \cap Z = O \equiv \{0\} \ , \ T(Z) \subseteq Z$$

In general ([2] Example 5) neither of hyperinvariant and reducing implies the other; also ([2] Example 4) hyperinvariant and reducing do not together imply spectral invariance (7.6).

9. BLOCK STRUCTURE FOR OPERATORS

Associated with an invariant subspace $T(Y) \subseteq Y \subseteq X$ for a linear operator $T \in L(X)$ we have a family of block triangular matrices of operators

$$T_U = \begin{pmatrix} T_Y & U \\ 0 & T_{/Y} \end{pmatrix} : \begin{pmatrix} Y \\ X/Y \end{pmatrix} \to \begin{pmatrix} Y \\ X/Y \end{pmatrix}$$

with

9.2 $$U \in L(X/Y, Y) \ ;$$

in the bottom left hand corner we have (*cf* [2] (0.3))

9.3 $$K_Y T J_Y = T_{/Y} K_Y J_Y = K_Y J_Y T_Y = 0 \in L(Y, X/Y) \ .$$

If $f \in \mathrm{Holo}(\sigma(T_Y) \cup \sigma(T_{/Y}))$ then, with

9.4 $$T'_U = \begin{pmatrix} T_Y & T_Y U - U T_{/Y} \\ 0 & T_{/Y} \end{pmatrix}, Q_U = \begin{pmatrix} I_Y & U \\ 0 & 0_{/Y} \end{pmatrix},$$

we have 8

9.5 $$f(T'_U) = \begin{pmatrix} f(T_Y) & f(T_Y)U - U f(T_{/Y}) \\ 0 & f(T_{/Y}) \end{pmatrix},$$

and also ([13] Theorem 1) necessary and sufficient for spectral invariance (7.6) is that

9.6 $$Q_U \in \mathrm{Holo}(T'_U) \ .$$

As in the block diagonal case, the weaker left,right disjointness conditions (5.1) and (5.2) are ([13] Theorem 3) together sufficient for membership of the double commutant:

9.7 $$Q_U \in \mathrm{comm}^2(T'_U) \ .$$

This turns out ([2] Theorem 7) to be helpful towards a sort of converse [3] to Lomonosov's theorem.

10. PRIMES AND EUCLID

We observe [10] a curious analogy between the spectral theory of operators and the prime factorization of integers: if we write

$$n = p_1^{\nu_1(n)} p_2^{\nu_2(n)} \dots p_k^{\nu_k(n)}$$

for the *prime factorization* of $n \in \mathbf{N} \subseteq \mathbf{Z}$, with

$$p = (p_1, p_2, p_3, \dots) = (2, 3, 5, 7, 11, 13, \dots)$$

for the usual sequence of prime numbers, then it is tempting to interpret

$$\{p_j : j \in \mathbf{N}, \nu_j(n) \neq 0\} = \varpi(n)$$

as some kind of "spectrum" of $n \in \mathbf{N}$. For example

$$n = 1 \Longleftrightarrow \varpi(n) = \emptyset$$

$n \in 1 + \mathbf{N}$ is a prime power if and only if $\varpi(n)$ is a singleton,

$$\#\varpi(n) = 1$$

and is square free if and only if every prime factor occurs with multiplicity one:

$$j \in \mathbf{N} \Longrightarrow \nu_j(n) \leq 1$$

If $\{m, n\} \subseteq 1 + \mathbf{N}$ then ([16] Corollary 4.1.3, Theorem 7.2.2)

10.7 $$\varpi(mn) = \varpi(m) \cup \varpi(n) \ ,$$

and, by the *Euclidean algorithm*, spectral disjointness gives rise to a sort of "exactness":

$$\varpi(m) \cap \varpi(n) = \emptyset \Longrightarrow 1 \in \mathbf{Z}m + n\mathbf{Z}$$

The background motivation, stimulated by Rosenthal-cubed [16], would be to try and apply linear algebra intuitions to elementary number theory. In another direction, Read [15], using essentially (10.7) as the definition, shows that all Banach algebra primes "have closed range".

References

[1] S. V. Djordjevic, R. E. Harte and D. A. Larson, *Partially hyperinvariant subspaces*, Operators Matrices, **6** (2012) 97-106.
[2] R. E. Harte, *Commutivity and separation of spectra II*, Proc. Royal Irish Acad. **7**4A (1974) 239-244.
[3] R. E. Harte, *Cayley-Hamilton for eigenvalues*, Irish Math. Soc. Bull. **2**2 (1989) 66-68.
[4] R. E. Harte, *Block diagonalization in Banach algebras*, Proc. Amer. Math. Soc. **1**29 (2001) 181-190.
[5] R. E. Harte, *Spectral mapping theorems, a bluffer's guide*, Springer Briefs, (2014).
[6] R. E. Harte, *On non commutative Taylor invertibility*,Operators Matrices, **1**0 (2016) 1117-1131.
[7] R. E. Harte and C. M. Stack, *Invertibility of spectral triangles*, Operators Matrices, **1** (2007) 445-453.
[8] R. E. Harte and C. M. Stack, *Separation of spectra for block triangles*, Proc. Amer. Math. Soc. **1**36 (2008) 3159-3164.
[9] J.J. Koliha, *Block diagonalization*,Math. Bohemica, **1**26 (2001) 237-246.
[10] D. Rosenthal and P. Rosenthal, *A readable introduction to real mathematics*, Springer, (2014).
[11] C. J. Read, *All primes have closed range*, Bull. London Math. Soc.**3**3 (2001) 341-346.

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