Maltepe Journal of Mathematics
ISSN:2667-7660, URL: https://dergipark.org.tr/tr/pub/mjm
Volume I Issue 2 (2019), Pages 66-88.

PPF DEPENDENT FIXED POINTS OF GENERALIZED WEAKLY CONTRACTION MAPS VIA C_G-SIMULATION FUNCTIONS

G. V. R. BABU* AND M. VINOD KUMAR**

*DEPARTMENT OF MATHEMATICS, ANDHRA UNIVERSITY, VISAKHAPATNAM-530 003, INDIA, ORCID: 0000-0002-6272-2645
**DEPARTMENT OF MATHEMATICS, ANDHRA UNIVERSITY, VISAKHAPATNAM-530 003, INDIA
PERMANENT ADDRESS : DEPARTMENT OF MATHEMATICS, ANITS, SANGIVALASA, VISKHAPATNAM -531 162, INDIA, ORCID: 0000-0001-6469-4855

Abstract. In this paper, we introduce the notion of generalized weakly $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$contraction maps with respect to the C_G-simulation function and prove the existence of PPF dependent fixed points of nonself maps in Banach spaces. For such maps, PPF dependent fixed points may not be unique. We provide an example to illustrate this phenomenon.

1. INTRODUCTION AND PRELIMINARIES

In fixed point theory, Banach contraction principle is one of the well known basic fundemental result and it gives an idea for the existence of fixed points with uniqueness in complete metric spaces. In 1997, Alber and Gurre-Delabriere [1] introduced weakly contractive maps which are extensions of contraction maps and obtained fixed point results in the setting of Hilbert spaces. Rhoades [9] extended this concept to metric spaces. Based on this idea, many authors generalized and extended the contraction maps and weakly contractive maps by introducing new functions like $\alpha-$admissible maps, $C-$class function, simulation function etc., for more details we refer [2, 10, 14, 18].

Throughout this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$, and $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of integers.

In 2011, Choudhury, Konar, Rhoades and Metiya [16] introduced the notion of generalized weakly contractive mapping as follows and proved the existence of fixed points of generalized weakly contractive mappings in complete metric spaces.

Definition 1.1. [16] *Let (X,d) be a metric space, T a self-mapping of X. We shall call T a generalized weakly contractive mapping if for any $x,y \in X$,*

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max\{d(x,y),d(y,Ty)\}),$$

2010 *Mathematics Subject Classification.* Primary: 47H10; Secondaries: 54H25.

Key words and phrases. $\alpha-$admissible mapping, $\mu-$subadmissible mapping, $C-$class function, Razumikhin class, PPF dependent fixed point, simulation function, C_G-simulation function.

where
(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous monotone increasing function with* $\psi(t) = 0 \iff t = 0$,
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function with* $\phi(t) = 0 \iff t = 0$,
(iii) $m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}$.

Theorem 1.1. [16] *Let* (X,d) *be a complete metric space,* T *a generalized weakly contractive self-mapping of* X*. Then* T *has a unique fixed point.*

In 2012, Samet, Vetro and Vetro [30] introduced the concept of $\alpha-$admissible mappings as follows.

Definition 1.2. [30] *Let* (X,d) *be a metric space. Let* $T : X \to X$ *and* $\alpha : X \times X \to \mathbb{R}^+$ *be two functions. Then* T *is said to be an* $\alpha-$*admissible mapping if*

$$\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1 \tag{1.1}$$

for all $x, y \in X$.

In 2013, Karapınar, Kumam and Salimi [23] introduced the notion of triangular $\alpha-$admissible mappings as follows.

Definition 1.3. [23] *Let* T *be a self-mapping of* X *and let* $\alpha : X \times X \to \mathbb{R}^+$ *be a function. Then* T *is said to be a triangular* $\alpha-$*admissible mapping if*

$$\begin{aligned} \alpha(x,y) \geq 1 &\implies \alpha(Tx,Ty) \geq 1 \text{ and} \\ \alpha(x,z) \geq 1,\ \alpha(z,y) \geq 1 &\implies \alpha(x,y) \geq 1 \end{aligned} \tag{1.2}$$

for all $x, y, z \in X$.

In 2014, Ansari [2] introduced the concept of $C-$class function as follows.

Definition 1.4. [2] *A mapping* $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is called a* $C-$*class function if it is continuous and for any* $s, t \in \mathbb{R}^+$*, the function* G *satisfies the following conditions:*
(i) $G(s,t) \leq s$ *and*
(ii) $G(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$.
The family of all $C-$*class functions is denoted by* Δ.

Example 1.1. [2] *The following functions belong to* Δ.
(i) $G(s,t) = s - t$ *for all* $s, t \in \mathbb{R}^+$.
(ii) $G(s,t) = ks$ *for all* $s, t \in \mathbb{R}^+$ *where* $0 < k < 1$.
(iii) $G(s,t) = \frac{s}{(1+t)^r}$ *for all* $s, t \in \mathbb{R}^+$ *where* $r \in \mathbb{R}^+$.
(iv) $G(s,t) = s\beta(s)$ *for all* $s, t \in \mathbb{R}^+$ *where* $\beta : \mathbb{R}^+ \to [0,1)$ *is continuous.*

In 2015, Khojasteh, Shukla and Radenović [24] introduced the notion of simulation function and proved the existence of fixed points of Z_H-contractions in complete metric spaces.

Definition 1.5. [24] *A function* $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is said to be a simulation function if it satisfies the following conditions:*
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ *for all* $t, s > 0$;
(ζ_3) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$*, then* $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation functions in the sense of Definition 1.5 by Z_H.

Example 1.2. [24, 22] *Let $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function with $\phi_i(t) = 0$ if and only if $t = 0$ for $i = 1, 2, 3$. Then the following functions $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ belong to Z_H.*
(i) $\zeta(t, s) = \frac{s}{s+1} - t$ for all $t, s \in \mathbb{R}^+$.
(ii) $\zeta(t, s) = \lambda s - t$ for all $t, s \in \mathbb{R}^+$ and $0 < \lambda < 1$.
(iii) $\zeta(t, s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in \mathbb{R}^+$, where $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$.

Definition 1.6. [24] *Let (X, d) be a metric space, $T : X \to X$ be a mapping and $\zeta \in Z_H$. Then T is called a Z_H-contraction with respect to ζ if*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \tag{1.3}$$

for all $x, y \in X$.

Theorem 1.2. [24] *Let (X, d) be a complete metric space and $T : X \to X$ be a Z_H-contraction with respect to ζ. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$ where $x_n = Tx_{n-1}$ for any $n \in \mathbb{N}$ converges to the fixed point of T.*

In 2015, Nastasi and Vetro [4] proved the existence of fixed points in complete metric spaces by using simulation functions and a lower semicontinuous function.

Theorem 1.3. [4] *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist a simulation function ζ and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that*

$$\zeta(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)) \geq 0 \tag{1.4}$$

for any $x, y \in X$. Then T has a unique fixed point u such that $\varphi(u) = 0$.

In 2018, Cho [14] introduced the notion of generalized weakly contractive mappings in metric spaces and proved the existence of its fixed points in complete metric spaces.

Definition 1.7. [14] *Let (X, d) be a metric space, T a self-mapping of X. Then T is called a generalized weakly contractive mapping if*

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi)) \tag{1.5}$$

for all $x, y \in X$, where
(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $\psi(t) = 0 \iff t = 0$,
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function and $\phi(t) = 0 \iff t = 0$,
(iii) $m(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)]\}$,
(iv) $l(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), \ d(y, Ty) + \varphi(y) + \varphi(Ty)\}$ and
(v) $\varphi : X \to \mathbb{R}^+$ is a lower semicontinuous function.

Theorem 1.4. [14] *Let (X, d) be a complete metric space. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

In 2018, Liu, Ansari, Chandok and Radenović [25] generalized the simulation function introduced by Khojasteh, Shukla and Radenović [24] by using $C-$class functions with C_G property.

Definition 1.8. [25] *A mapping $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ has the property C_G if there exists an $C_G \geq 0$ such that*
(i) for any $s, t \in \mathbb{R}^+$, $G(s,t) > C_G$ implies $s > t$, and
(ii) $G(t,t) \leq C_G$ for all $t \in \mathbb{R}^+$.

Example 1.3. [25] *The following functions are elements of Δ that have property C_G for all $t, s \in \mathbb{R}^+$:*
(i) $G(s,t) = s - t, C_G = r, r \in \mathbb{R}^+$,
(ii) $G(s,t) = s - \frac{(2+t)t}{1+t}, C_G = 0$,
(iii) $G(s,t) = \frac{s}{1+kt}, k \geq 1, C_G = \frac{r}{1+k}, r \geq 2$.

Definition 1.9. [25] *A function $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a C_G-simulation function if it satisfies the following conditions:*
(ζ_4) $\zeta(0,0) = 0$;
(ζ_5) $\zeta(t,s) < G(s,t)$ *for all $t, s > 0$ where $G \in \Delta$ has property C_G;*
(ζ_6) *if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_G$.*

We denote the set of all C_G-simulation functions by Z_G.

Example 1.4. [25] *The following functions ζ belong to Z_G.*
(i) Let $k \in \mathbb{R}$ be such that $k < 1$ and $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kG(s,t) - t$, here $C_G = 0$.
(ii) Let $k \in \mathbb{R}$ be such that $k < 1$ and let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kG(s,t)$, here $C_G = 1$.
(iii) We define $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $\zeta(t,s) = \lambda s - t$, where $\lambda \in (0,1)$ and $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $G(s,t) = s - t$ for any $s, t \in \mathbb{R}^+$.
Clearly $\zeta(0,0) = 0$ and $G \in \Delta$ with $C_G = 0$.
Clearly $\zeta(t,s) = \lambda s - t < s - t = G(s,t)$ and hence ζ satisfies (ζ_5).
If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = k > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then
$\limsup_{n\to\infty} \zeta(t_n, s_n) = \limsup_{n\to\infty} (\lambda s_n - t_n) = \lambda k - k = (\lambda - 1)k < 0.$
Therefore ζ satisfies (ζ_6) and hence $\zeta \in Z_G$.

In 1977, Bernfeld, Lakshmikantham and Reddy [12] introduced the concept of fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point, for more details we refer [6, 11, 17, 19, 21, 26].

Let $(E, ||.||_E)$ be a Banach space and we denote it simply by E. Let $I = [a,b] \subseteq \mathbb{R}$ and $E_0 = C(I, E)$, the set of all continuous functions on I equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup_{a \leq t \leq b} ||\phi(t)||_E$ for $\phi \in E_0$.

For a fixed $c \in I$, the Razumikhin class R_c of functions in E_0 is defined by $R_c = \left\{\phi \in E_0 / \, ||\phi||_{E_0} = ||\phi(c)||_E\right\}$. Clearly every constant function from I to E belongs to R_c so that R_c is a non-empty subset of E_0 .

Definition 1.10. [12] *Let R_c be the Razumikhin class of continuous functions in E_0. We say that* *(i) the class R_c is algebraically closed with respect to the difference if $\phi - \psi \in R_c$ whenever $\phi, \psi \in R_c$.*

(ii) the class R_c is topologically closed if it is closed with respect to the topology on E_0 by the norm $||.||_{E_0}$.

The Razumikhin class of functions R_c has the following properties.

Theorem 1.5. [5] *Let R_c be the Razumikhin class of functions in E_0. Then*

i) $E_0 = \underset{c\in[a,b]}{\cup} R_c$.

ii) *for any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$.*

iii) *the Razumikhin class R_c is topologically closed with respect to the norm defined on E_0.*

iv) $\underset{c\in[a,b]}{\cap} R_c = \{\phi \in E_0 \mid \phi : I \to E \text{ is constant}\}$.

Definition 1.11. [12] *Let $T : E_0 \to E$ be a mapping. A function $\phi \in E_0$ is said to be a PPF dependent fixed point of T if $T\phi = \phi(c)$ for some $c \in I$.*

Definition 1.12. [12] *Let $T : E_0 \to E$ be a mapping. Then T is called a Banach type contraction if there exists $k \in [0, 1)$ such that $||T\phi - T\psi||_E \le k\,||\phi - \psi||_{E_0}$ for all $\phi, \psi \in E_0$.*

Theorem 1.6. [12] *Let $T : E_0 \to E$ be a Banach type contraction. Let R_c be algebraically closed with respect to the difference and topologically closed. Then T has a unique PPF dependent fixed point in R_c.*

Definition 1.13. [28] *Let $c \in I$. Let $T : E_0 \to E$ and $\alpha : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be an α_c-admissible mapping if*

$$\alpha(\phi(c), \psi(c)) \ge 1 \implies \alpha(T\phi, T\psi) \ge 1 \tag{1.6}$$

for all $\phi, \psi \in E_0$.

In 2013, Hussain, Khaleghizadeh, Salimi and Akbar [21] introduced the concept of α_c-admissible mapping with respect to μ_c and proved theorems for the existence of PPF dependent fixed points and PPF dependent coincidence points for contractive mappings in Banach spaces.

Definition 1.14. [21] *Let $c \in I$ and $T : E_0 \to E$. Let $\alpha, \mu : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be an α_c-admissible mapping with respect to μ_c if*

$$\alpha(\phi(c), \psi(c)) \ge \mu(\phi(c), \psi(c)) \implies \alpha(T\phi, T\psi) \ge \mu(T\phi, T\psi) \tag{1.7}$$

for all $\phi, \psi \in E_0$.

Note that, if we take $\mu(x, y) = 1$ for all $x, y \in E$ then α_c-admissible mapping with respect to μ_c is an α_c-admissible mapping. If we take $\alpha(x, y) = 1$ for all $x, y \in E$ in (1.7) then we say that T is a μ_c-*subadmissible mapping*.

In 2014, Ćirić, Alsulami, Salimi and Vetro [13] introduced the concept of triangular α_c-admissible mapping with respect to μ_c as follows.

Definition 1.15. [13] *Let $c \in I$ and $T : E_0 \to E$. Let $\alpha, \mu : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be a triangular α_c-admissible mapping with respect* *to* μ_c *if*

$$\begin{cases} \text{(i) } \alpha(\phi(c),\psi(c)) \geq \mu(\phi(c),\psi(c)) \implies \alpha(T\phi,T\psi) \geq \mu(T\phi,T\psi) \\ \qquad\qquad\qquad\qquad \text{and} \\ \text{(ii) } \alpha(\phi(c),\psi(c)) \geq \mu(\phi(c),\psi(c)),\ \alpha(\psi(c),\varphi(c)) \geq \mu(\psi(c),\varphi(c)) \\ \qquad \implies \alpha(\phi(c),\varphi(c)) \geq \mu(\phi(c),\varphi(c)). \end{cases} \tag{1.8}$$

for all $\phi,\psi,\varphi \in E_0$.

Lemma 1.7. [13] *Let* T *be a triangular* α_c-*admissible mapping with respect to* μ_c. *We define the sequence* $\{\phi_n\}$ *by* $T\phi_n = \phi_{n+1}(c)$ *for all* $n \in \mathbb{N}\cup\{0\}$, *where* $\phi_0 \in R_c$ *is such that* $\alpha(\phi_0(c),T\phi_0) \geq \mu(\phi_0(c),T\phi_0)$. *Then* $\alpha(\phi_m(c),\phi_n(c)) \geq \mu(\phi_m(c),\phi_n(c))$ *for all* $m,n \in \mathbb{N}$ *with* $m < n$.

Remark. *If* $\mu(x,y) = 1$ *for any* $x,y \in E$ *in Lemma 1.7, we get the following lemma.*

Lemma 1.8. *Let* T *be a triangular* α_c-*admissible mapping. We define the sequence* $\{\phi_n\}$ *by* $T\phi_n = \phi_{n+1}(c)$ *for all* $n \in \mathbb{N}\cup\{0\}$, *where* $\phi_0 \in R_c$ *is such that* $\alpha(\phi_0(c),T\phi_0) \geq 1$. *Then* $\alpha(\phi_m(c),\phi_n(c)) \geq 1$ *for all* $m,n \in \mathbb{N}$ *with* $m < n$.

Remark. *If* $\alpha(x,y) = 1$ *for any* $x,y \in E$ *in Lemma 1.7, we get the following lemma.*

Lemma 1.9. *Let* T *be a triangular* μ_c-*subadmissible mapping. We define the sequence* $\{\phi_n\}$ *by* $T\phi_n = \phi_{n+1}(c)$ *for all* $n \in \mathbb{N}\cup\{0\}$, *where* $\phi_0 \in R_c$ *is such that* $\mu(\phi_0(c),T\phi_0) \leq 1$. *Then* $\mu(\phi_m(c),\phi_n(c)) \leq 1$ *for all* $m,n \in \mathbb{N}$ *with* $m < n$.

Lemma 1.10. [7] *Let* $\{\phi_n\}$ *be a sequence in* E_0 *such that* $||\phi_n - \phi_{n+1}||_{E_0} \to 0$ *as* $n \to \infty$. *If* $\{\phi_n\}$ *is not a Cauchy sequence, then there exists an* $\epsilon > 0$ *and two subsequences* $\{\phi_{m_k}\}$ *and* $\{\phi_{n_k}\}$ *of* $\{\phi_n\}$ *with* $m_k > n_k > k$ *such that* $||\phi_{n_k} - \phi_{m_k}||_{E_0} \geq \epsilon,\ ||\phi_{n_k} - \phi_{m_k-1}||_{E_0} < \epsilon$ *and*

i) $\lim_{k\to\infty} ||\phi_{n_k} - \phi_{m_k+1}||_{E_0} = \epsilon,$ ii) $\lim_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k}||_{E_0} = \epsilon,$

iii) $\lim_{k\to\infty} ||\phi_{n_k} - \phi_{m_k}||_{E_0} = \epsilon,$ iv) $\lim_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} = \epsilon.$

In Section 2, we introduce the notion of generalized weakly $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$contraction map with respect to a C_G-simulation function $\zeta \in Z_G$ and prove the existence of PPF dependent fixed points of these maps in Banach spaces(Theorem 2.1) which is the main result of this paper. For such maps, PPF dependent fixed points may not be unique. In Section 3, we draw some corollaries and an example is provided to illustrate our main result.

2. EXISTENCE OF PPF DEPENDENT FIXED POINTS

We denote

$\Psi = \{\xi \mid \xi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing and $\xi(t) = 0 \iff t = 0\}$

and

$\Phi = \{\eta \mid \eta : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\eta(t) = 0 \iff t = 0\}$.

Based on the results of [4, 14, 16] we introduce a notion of generalized weakly $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$contraction map with respect to $\zeta \in Z_G$ as follows.

Definition 2.1. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function and* $\zeta \in Z_G$. *If there exist* $\xi \in \Psi, \eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0,\infty)$, *and a lower semicontinuous* *function* $\varphi : E \to \mathbb{R}^+$ *such that*

$$\zeta(\alpha(\phi(c),\psi(c))\xi(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)), \mu(\phi(c),\psi(c))(\xi(M(\phi,\psi)) - \eta(N(\phi,\psi)))) \geq C_G \tag{2.1}$$

for all $\phi, \psi \in E_0$, *where* $\xi(t) > \eta(t)$ *for any* $t > 0$,

$$M(\phi,\psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi), \frac{1}{2}[||\phi(c) - T\psi||_E + \varphi(\phi(c)) + \varphi(T\psi) + ||\psi(c) - T\phi||_E + \varphi(\psi(c)) + \varphi(T\phi)]\}$$

and

$$N(\phi,\psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$

then we say that T *is a generalized weakly* $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$*contraction map with respect to* ζ.

Remark. *(i) If* $\varphi(x) = 0$ *for any* $x \in E$ *in the inequality (2.1) then* T *is called a generalized weakly* $Z_{G,\alpha,\mu,\xi,\eta}-$*contraction map with respect to* ζ.

(ii) If $\varphi(x) = 0, \mu(x,y) = 1 = \alpha(x,y)$ *for any* $x, y \in E$ *in the inequality (2.1) then* T *is called a generalized weakly* $Z_{G,\xi,\eta}-$*contraction map with respect to* ζ.

(iii) If $\varphi(x) = 0, \mu(x,y) = 1 = \alpha(x,y)$ *for any* $x, y \in E$ *and* $\xi(t) = t$ *for any* $t \in \mathbb{R}^+$ *in the inequality (2.1) then* T *is called a generalized weakly* $Z_{G,\eta}-$*contraction map with respect to* ζ.

Theorem 2.1. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) T *is a generalized weakly* $Z_{G,\alpha,\mu,\xi,\eta,\varphi}-$*contraction map with respect to* ζ,
(ii) T *is a triangular* α_c-*admissible mapping and triangular* μ_c-*subadmissible mapping,*
(iii) R_c *is algebraically closed with respect to the difference,*
(iv) *if* $\{\phi_n\}$ *is a sequence in* E_0 *such that* $\phi_n \to \phi$ *as* $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *then* $\alpha(\phi_n(c), \phi(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *and*
(v) *there exists* $\phi_0 \in R_c$ *such that* $\alpha(\phi_0(c), T\phi_0) \geq 1$ *and* $\mu(\phi_0(c), T\phi_0) \leq 1$.

Then T *has a PPF dependent fixed point* $\phi^* \in R_c$ *such that* $\varphi(\phi^*(c)) = 0$.

Proof. From (v) we have $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$. Let $\{\phi_n\}$ be a sequence in R_c defined by

$$T\phi_n = \phi_{n+1}(c) \tag{2.2}$$

for any $n = 0, 1, 2, 3...$.
Since R_c is algebraically closed with respect to the difference, we have

$$||\phi_{n+1} - \phi_n||_{E_0} = ||\phi_{n+1}(c) - \phi_n(c)||_E \tag{2.3}$$

for any $n = 0, 1, 2, 3...$.
Since T is triangular α_c-admissible and triangular μ_c-subadmissible mappings, by Lemma 1.8 and Lemma 1.9 we have

$$\begin{gathered} \alpha(\phi_m(c), \phi_n(c)) \geq 1 \\ \text{and} \\ \mu(\phi_m(c), \phi_n(c)) \leq 1 \end{gathered} \tag{2.4}$$

for any $m, n \in \mathbb{N}$ with $m < n$.
If there exists $n \in \mathbb{N} \cup \{0\}$ such that $\phi_n = \phi_{n+1}$ then $T\phi_n = \phi_{n+1}(c) = \phi_n(c)$ and hence $\phi_n \in R_c$ is a PPF dependent fixed point of T.

Suppose that $\phi_n \neq \phi_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$.
If either $M(\phi_n, \phi_{n+1}) = 0$ or $N(\phi_n, \phi_{n+1}) = 0$ then the result is trivial.
Suppose that $M(\phi_n, \phi_{n+1}) \neq 0$ and $N(\phi_n, \phi_{n+1}) \neq 0$.
We consider

$$\begin{aligned} M(\phi_n, \phi_{n+1}) = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\ &||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n),\\ &||\phi_{n+1}(c) - T\phi_{n+1}||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_{n+1}),\\ &\tfrac{1}{2}[||\phi_n(c) - T\phi_{n+1}||_E + \varphi(\phi_n(c)) + \varphi(T\phi_{n+1}) +\\ &\qquad ||\phi_{n+1}(c) - T\phi_n||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_n)]\}\\ = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\ &||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\ &||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)),\\ &\tfrac{1}{2}[||\phi_n - \phi_{n+2}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+2}(c)) +\\ &\qquad ||\phi_{n+1} - \phi_{n+1}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+1}(c))]\}\\ = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\ &||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))\} \end{aligned}$$

and

$$\begin{aligned} N(\phi_n, \phi_{n+1}) = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\ &||\phi_{n+1}(c) - T\phi_{n+1}||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_{n+1})\}\\ = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\ &||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))\}. \end{aligned}$$

Suppose that

$$\max\{||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))\}$$
$$= ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)).$$

Clearly $M(\phi_n, \phi_{n+1}) = N(\phi_n, \phi_{n+1}) = ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))$.
Since $\phi_{n+1} \neq \phi_{n+2}$, we have $||\phi_{n+1} - \phi_{n+2}||_{E_0} > 0$ and hence
$||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) > 0$ and which implies that
$\xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0$.
Therefore

$$\begin{aligned} &\alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1}))\\ &\qquad = \alpha(\phi_n(c), \phi_{n+1}(c))\xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0. \end{aligned}$$

Since $\xi(t) > \eta(t)$ for any $t > 0$ we have $\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1})) > 0$ and hence $\mu(\phi_n(c), \phi_{n+1}(c))(\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1}))) > 0$.
From (2.1), we have

$$\begin{aligned} C_G \leq \zeta(&\alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})),\\ &\mu(\phi_n(c), \phi_{n+1}(c))(\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1}))))\\ < G(&\mu(\phi_n(c), \phi_{n+1}(c))(\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1}))),\\ &\alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1}))). \end{aligned}$$

(by (ζ_5))
Now by the property C_G , we get

$$\begin{aligned} &\mu(\phi_n(c), \phi_{n+1}(c))(\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1})))\\ &\qquad > \alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})). \end{aligned}$$

Clearly

$$\begin{aligned} \xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) &= \xi(M(\phi_n, \phi_{n+1}))\\ &> \xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1})) \end{aligned}$$

$$\begin{aligned}&\geq \mu(\phi_n(c),\phi_{n+1}(c))(\xi(M(\phi_n,\phi_{n+1}))-\eta(N(\phi_n,\phi_{n+1})))\\ &> \alpha(\phi_n(c),\phi_{n+1}(c))\xi(||T\phi_n-T\phi_{n+1}||_E+\varphi(T\phi_n)+\varphi(T\phi_{n+1}))\\ &\geq \xi(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))),\end{aligned}$$

a contradiction.
Therefore
$||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c)) > ||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))$
and hence $M(\phi_n,\phi_{n+1}) = N(\phi_n,\phi_{n+1}) = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))$.
Let $d_n = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))$.
Then the sequence $\{d_n\}$ is a decreasing sequence and hence convergent.
Let $\lim_{n\to\infty} d_n = k$ (say). Suppose that $k > 0$.
Since $\phi_n \neq \phi_{n+1}$ we have $d_n = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c)) > 0$
and which implies that $\xi(d_n) = \xi(||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))) > 0$.
Similarly $\eta(d_n) > 0$. Clearly $M(\phi_n,\phi_{n+1}) = N(\phi_n,\phi_{n+1}) = d_n$ and hence
$\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)) > 0$.
Similarly $d_{n+1} > 0$ and which implies that $\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}) > 0$.
From (2.1), we have

$$\begin{aligned}C_G &\leq \zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\xi(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))),\\ &\qquad \mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)))\\ &= \zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}),\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)))\\ &< G(\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)),\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1})). \text{ (by}(\zeta_5))\end{aligned} \tag{2.5}$$

Now by the property C_G , we get that
$\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)) > \alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1})$.
Clearly

$$\begin{aligned}\xi(d_n) &> \xi(d_n)-\eta(d_n)\\ &\geq \mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n))\\ &> \alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1})\\ &\geq \xi(d_{n+1}).\end{aligned}$$

On applying limits as $n\to\infty$, we get that
$\lim_{n\to\infty}\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)) = \lim_{n\to\infty}\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}) = \xi(k) > 0$.
On applying limit superior to (2.5), we get that

$$\begin{aligned}C_G &\leq \limsup_{n\to\infty}\zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\xi(d_{n+1}),\mu(\phi_n(c),\phi_{n+1}(c))(\xi(d_n)-\eta(d_n)))\\ &< C_G, \text{ (by } (\zeta_6))\end{aligned}$$

a contradiction.
Therefore $k = 0$ and hence $\lim_{n\to\infty}[||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))] = 0$.
That is

$$\lim_{n\to\infty}||\phi_n-\phi_{n+1}||_{E_0} = 0 \text{ and } \lim_{n\to\infty}\varphi(\phi_n(c)) = 0. \tag{2.6}$$

We now show that the sequence $\{\phi_n\}$ is a Cauchy sequence in R_c.
Suppose that the sequence $\{\phi_n\}$ is not a Cauchy sequence.
Then there exists an $\epsilon > 0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{n_k}-\phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k}-\phi_{m_k-1}||_{E_0} < \epsilon$ and by Lemma 1.10 we have,

$$\begin{aligned}&\lim_{k\to\infty}||\phi_{n_k}-\phi_{m_k}||_{E_0} = \epsilon \text{ and}\\ &\lim_{k\to\infty}||\phi_{n_k}-\phi_{m_k+1}||_{E_0} = \epsilon = \lim_{k\to\infty}||\phi_{n_k+1}-\phi_{m_k}||_{E_0}\\ &\qquad = \lim_{k\to\infty}||\phi_{n_k+1}-\phi_{m_k+1}||_{E_0}.\end{aligned} \tag{2.7}$$

Let $d_{n_k m_k} = ||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c))$.
Then from (2.6) and (2.7) it follows that
$\lim\limits_{k\to\infty} d_{n_k m_k} = \epsilon = \lim\limits_{k\to\infty} d_{n_k+1 m_k+1}$.
Since ξ is continuous, we get that

$$\lim_{k\to\infty} \xi(d_{n_k+1 m_k+1}) = \xi(\epsilon) > 0. \tag{2.8}$$

We consider

$$\begin{aligned}
M(\phi_{n_k}, \phi_{m_k}) = \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)),\\
&||\phi_{n_k}(c) - T\phi_{n_k}||_E + \varphi(\phi_{n_k}(c)) + \varphi(T\phi_{n_k}),\\
&||\phi_{m_k}(c) - T\phi_{m_k}||_E + \varphi(\phi_{m_k}(c)) + \varphi(T\phi_{m_k}),\\
&\tfrac{1}{2}[||\phi_{n_k}(c) - T\phi_{m_k}||_E + \varphi(\phi_{n_k}(c)) + \varphi(T\phi_{m_k}) +\\
&\qquad ||\phi_{m_k}(c) - T\phi_{n_k}||_E + \varphi(\phi_{m_k}(c)) + \varphi(T\phi_{n_k})]\}\\
= \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)),\\
&||\phi_{n_k} - \phi_{n_k+1}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{n_k+1}(c)),\\
&||\phi_{m_k} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{m_k}(c)) + \varphi(\phi_{m_k+1}(c)),\\
&\tfrac{1}{2}[||\phi_{n_k} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k+1}(c)) +\\
&\qquad ||\phi_{m_k} - \phi_{n_k+1}||_{E_0} + \varphi(\phi_{m_k}(c)) + \varphi(\phi_{n_k+1}(c))]\}\\
= \max\{&d_{n_k m_k}, d_{n_k n_k+1}, d_{m_k m_k+1}, \tfrac{1}{2}[d_{n_k m_k+1} + d_{m_k n_k+1}]\}.
\end{aligned}$$

On applying limits as $k \to \infty$, we get that $\lim\limits_{k\to\infty} M(\phi_{n_k}, \phi_{m_k}) = \epsilon$.
We consider

$$\begin{aligned}
N(\phi_{n_k}, \phi_{m_k}) = \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)),\\
&\quad ||\phi_{m_k}(c) - T\phi_{m_k}||_E + \varphi(\phi_{m_k}(c)) + \varphi(T\phi_{m_k})\}\\
= \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)),\\
&\quad ||\phi_{m_k} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{m_k}(c)) + \varphi(\phi_{m_k+1}(c))\}\\
= \max\{&d_{n_k m_k}, d_{m_k m_k+1}\}.
\end{aligned}$$

On applying limits as $k \to \infty$, we get that $\lim\limits_{k\to\infty} N(\phi_{n_k}, \phi_{m_k}) = \epsilon$.
Since ξ, η are continuous, we have
$\lim\limits_{k\to\infty} \xi(M(\phi_{n_k}, \phi_{m_k})) = \xi(\epsilon) > 0$ and $\lim\limits_{k\to\infty} \eta(N(\phi_{n_k}, \phi_{m_k})) = \eta(\epsilon) > 0$.
Therefore

$$\lim_{k\to\infty} (\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) = \xi(\epsilon) - \eta(\epsilon) > 0. \tag{2.9}$$

(since $\xi(t) > \eta(t)$ for $t > 0$)
From (2.8) and (2.9), there exists $k_1 \in \mathbb{N}$ such that

$$\begin{aligned}
&\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})) > \tfrac{\xi(\epsilon)-\eta(\epsilon)}{2} > 0\\
&\text{and}\\
&\xi(d_{n_k+1 m_k+1}) > \tfrac{\eta(\epsilon)}{2} > 0
\end{aligned} \tag{2.10}$$

for any $k \geq k_1$.
From (2.4), we have

$$\begin{aligned}
&\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1 m_k+1}) \geq \xi(d_{n_k+1 m_k+1}) > 0 \text{ and}\\
&\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) > 0.
\end{aligned} \tag{2.11}$$

for any $k \geq k_1$.
For any $k \geq k_1$, from (2.1) we have

$$\begin{aligned}
C_G \leq \zeta(&\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(||T\phi_{n_k} - T\phi_{m_k}||_E + \varphi(T\phi_{n_k}) + \varphi(T\phi_{m_k})),\\
&\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))))
\end{aligned}$$

$$
\begin{aligned}
&= \zeta(\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{n_k+1}(c)) + \varphi(\phi_{m_k+1}(c))), \\
&\qquad \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))))\\
&= \zeta(\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1m_k+1}), \\
&\qquad \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})))) \qquad (2.12)\\
&< G(\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))), \\
&\qquad \alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1m_k+1})). \\
&\qquad \text{(by (2.11) and } (\zeta_5))
\end{aligned}
$$

Now by the property C_G, we have

$$
\begin{aligned}
&\mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) \\
&\qquad > \alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1m_k+1}). \qquad (2.13)
\end{aligned}
$$

Clearly

$$
\begin{aligned}
\xi(M(\phi_{n_k}, \phi_{m_k})) &> \xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})) \\
&\geq \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) \\
&> \alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1m_k+1}) \text{ (by(2.13))} \\
&\geq \xi(d_{n_k+1m_k+1}).
\end{aligned}
$$

On applying limits as $k \to \infty$, we get that

$$
\begin{aligned}
&\lim_{k\to\infty} \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) \\
&\qquad = \lim_{k\to\infty} \alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1m_k+1}) = \xi(\epsilon) > 0. \qquad (2.14)
\end{aligned}
$$

On applying limit superior as $k \to \infty$ to (2.12), by (2.13) ,(2.14) and (ζ_6) we get

$$
\begin{aligned}
C_G &\leq \limsup_{k\to\infty} \zeta(\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\xi(d_{n_k+1m_k+1}), \\
&\qquad \mu(\phi_{n_k}(c), \phi_{m_k}(c))(\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})))) \\
&< C_G,
\end{aligned}
$$

a contradiction.
Therefore the sequence $\{\phi_n\}$ is a Cauchy sequence in R_c.
Since E_0 is complete, there exists $\phi^* \in E_0$ such that $\phi_n \to \phi^*$.
Since R_c is topologically closed, we have $\phi^* \in R_c$.
Clearly $||\phi^*||_{E_0} = ||\phi^*(c)||_E$. (since $\phi^* \in R_c$)
Since φ is lower semicontinuous function, we have
$\varphi(\phi^*(c)) \leq \liminf_{n\to\infty} \varphi(\phi_n(c)) = 0$ and hence $\varphi(\phi^*(c)) = 0$.
We now show that $T\phi^* = \phi^*(c)$. Suppose that $T\phi^* \neq \phi^*(c)$.
From (2.4) we have $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$.
From (iv) we get that $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ and $\mu(\phi_n(c), \phi^*(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$.
We consider

$$
\begin{aligned}
M(\phi_n, \phi^*) = \max\{&||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)), \\
&||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n), \\
&||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*), \\
&\tfrac{1}{2}[||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) + \\
&\qquad ||\phi^*(c) - T\phi_n||_E + \varphi(\phi^*(c)) + \varphi(T\phi_n)]\} \\
= \max\{&||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)), \\
&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \\
&||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*), \\
&\tfrac{1}{2}[||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) +
\end{aligned}
$$

$||\phi^* - \phi_{n+1}||_{E_0} + \varphi(\phi^*(c)) + \varphi(\phi_{n+1}(c))]\}$

and

$N(\phi_n, \phi^*) = \max\{||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)),$
$\quad ||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*)\}.$

If either $M(\phi_n, \phi^*) = 0$ or $N(\phi_n, \phi^*) = 0$ then $T\phi^* = \phi^*(c)$, a contradiction.

Therefore $M(\phi_n, \phi^*) > 0$ and $N(\phi_n, \phi^*) > 0$.

Clearly $M(\phi_n, \phi^*) \geq N(\phi_n, \phi^*)$.

Since $\xi(t) > \eta(t)$ for $t > 0$ we have $\xi(M(\phi_n, \phi^*)) \geq \xi(N(\phi_n, \phi^*)) > \eta(N(\phi_n, \phi^*))$ and hence $\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)) > 0$.

Clearly

$$\mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) > 0. \tag{2.15}$$

If $||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*) = 0$ then $\phi_{n+1}(c) = T\phi_n = T\phi^*$.

On applying limits as $n \to \infty$, we get $\phi^*(c) = T\phi^*$, a contradiction.

Therefore $||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*) > 0$ and hence $\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) > 0$.

Clearly

$$\alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) > 0. \tag{2.16}$$

From (2.1) we have

$$\begin{aligned} C_G &\leq \zeta(\alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \\ &\qquad \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)))) \\ &< G(\mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))), \\ &\qquad \alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*))). \end{aligned}$$

Now by the property C_G, we get that

$$\begin{aligned} &\mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) \\ &\quad > \alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)). \end{aligned} \tag{2.17}$$

On applying limits as $n \to \infty$ to $M(\phi_n, \phi^*)$ and $N(\phi_n, \phi^*)$, we get that $\lim_{n\to\infty} M(\phi_n, \phi^*) = ||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*) = \lim_{n\to\infty} N(\phi_n, \phi^*)$.

Since ξ is continuous, we get that

$\lim_{n\to\infty} \xi(M(\phi_n, \phi^*)) = \xi(||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*)) > 0$. (since $T\phi^* \neq \phi^*(c)$)

Clearly

$$\begin{aligned} \xi(M(\phi_n, \phi^*)) &> \xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)) \\ &\geq \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) \\ &> \alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) \\ &\geq \xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) \\ &= \xi(||\phi_{n+1}(c) - T\phi^*||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi^*)). \end{aligned}$$

On applying limits as $n \to \infty$, we get

$$\begin{aligned} &\lim_{n\to\infty} \alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) \\ &= \lim_{n\to\infty} \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) \\ &= \xi(||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*)) > 0. \end{aligned}$$

From (2.1) we have

$$\begin{aligned} C_G &\leq \zeta(\alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \\ &\qquad \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)))). \end{aligned}$$

On applying limit superior as $n \to \infty$, by (ζ_6) we get that

$$\begin{aligned} C_G \leq \limsup_{n\to\infty} \zeta(\alpha(\phi_n(c), \phi^*(c))\xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \\ \mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)))) \\ < C_G, \end{aligned}$$

a contradiction.

Therefore $T\phi^* = \phi^*(c)$ and hence $\phi^* \in R_c$ is a PPF dependent fixed point of T such that $\varphi(\phi^*(c)) = 0$. □

3. COROLLARIES AND EXAMPLES

Corollary 3.1. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *T is a generalized weakly $Z_{G,\alpha,\mu,\xi,\eta}-$contraction map with respect to ζ,*

(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ then $\alpha(\phi_n(c), \phi(c)) \geq 1$ and $\mu(\phi_n(c), \phi(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ and*

(v) *there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$.*

Then T has a PPF dependent fixed point in R_c.

Proof. By taking $\varphi(x) = 0$ for any $x \in E$ in Theorem 2.1 we obtain the desired result. □

By choosing $\alpha(x, y) = 1 = \mu(x, y)$ for any $x, y \in E$ in Corollary 3.1 we get the following corollary.

Corollary 3.2. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *T is a generalized weakly $Z_{G,\xi,\eta}-$contraction map with respect to ζ and*

(ii) *R_c is algebraically closed with respect to the difference.*

Then T has a PPF dependent fixed point in R_c.

By choosing $\xi(t) = t$ for any $t \in \mathbb{R}^+$ in Corollary 3.2 we get the following corollary.

Corollary 3.3. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *T is a generalized weakly $Z_{G,\eta}-$contraction map with respect to ζ and*

(ii) *R_c is algebraically closed with respect to the difference.*

Then T has a PPF dependent fixed point in R_c.

By choosing $\alpha(x, y) = 1 = \mu(x, y)$ for any $x, y \in E,\ \xi(t) = t$ for any $t \in \mathbb{R}^+$ and $C_G = 0$ in Theorem 2.1 we get the following corollary.

Corollary 3.4. *Let $c \in I$ and $\zeta \in Z_G$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *if there exist $\eta \in \Phi$ and a lower semicontinuous function $\varphi : E \to \mathbb{R}^+$ such that*

$$\zeta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi), M(\phi, \psi) - \eta(N(\phi, \psi))) \geq 0$$

for any $\phi, \psi \in E_0$, where $\eta(t) < t$ for any $t > 0$,

$M(\phi, \psi) = \max\{||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)), ||\phi(c)-T\phi||_E+\varphi(\phi(c))+\varphi(T\phi),$

$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi),$$
$$\tfrac{1}{2}[||\phi(c) - T\psi||_E + \varphi(\phi(c)) + \varphi(T\psi) + ||\psi(c) - T\phi||_E + \varphi(\psi(c)) + \varphi(T\phi)]\},$$
$$N(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
and

(ii) R_c *is algebraically closed with respect to the difference.*

Then T *has a PPF dependent fixed point* $\phi^* \in R_c$ *such that* $\varphi(\phi^*(c)) = 0$.

By choosing $\varphi(x) = 0$ for any $x \in E$ in Corollary 3.4 we get the following corollary.

Corollary 3.5. *Let* $c \in I$ *and* $\zeta \in Z_G$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exists* $\eta \in \Phi$ *such that*
$$\zeta(||T\phi - T\psi||_E, M(\phi, \psi) - \eta(N(\phi, \psi))) \geq 0$$
for any $\phi, \psi \in E_0$, *where* $\eta(t) < t$ *for any* $t > 0$,
$$M(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E, \tfrac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\},$$
$$N(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\psi(c) - T\psi||_E\}$$
and

(ii) R_c *is algebraically closed with respect to the difference.*

Then T *has a PPF dependent fixed point in* R_c.

By choosing $\zeta(t, s) = \lambda s - t, G(s, t) = s - t$ for any $s, t \in \mathbb{R}^+, C_G = 0$ and $\lambda \in (0, 1)$ in Theorem 2.1 we get the following corollary.

Corollary 3.6. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exist* $\xi \in \Psi, \eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0, \infty), \lambda \in (0, 1)$ *and a lower semicontinuous function* $\varphi : E \to \mathbb{R}^+$ *such that*
$$\alpha(\phi(c), \psi(c))\xi(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda\mu(\phi(c), \psi(c))(\xi(M(\phi, \psi)) - \eta(N(\phi, \psi))) \tag{3.1}$$
for any $\phi, \psi \in E_0$, *where* $\xi(t) > \eta(t)$ *for any* $t > 0$,
$M(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi),$$
$$\tfrac{1}{2}[||\phi(c) - T\psi||_E + \varphi(\phi(c)) + \varphi(T\psi) + ||\psi(c) - T\phi||_E + \varphi(\psi(c)) + \varphi(T\phi)]\},$$
$$N(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\},$$

(ii) T *is a triangular* α_c-*admissible mapping and triangular* μ_c-*subadmissible mapping,*

(iii) R_c *is algebraically closed with respect to the difference,*

(iv) *if* $\{\phi_n\}$ *is a sequence in* E_0 *such that* $\phi_n \to \phi$ *as* $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *then* $\alpha(\phi_n(c), \phi(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *and*

(v) *there exists* $\phi_0 \in R_c$ *such that* $\alpha(\phi_0(c), T\phi_0) \geq 1$ *and* $\mu(\phi_0(c), T\phi_0) \leq 1$.

Then T *has a PPF dependent fixed point* $\phi^* \in R_c$ *such that* $\varphi(\phi^*(c)) = 0$.

By choosing $\xi(t) = t,\ t \in \mathbb{R}^+$ in Corollary 3.6 we get the following corollary.

Corollary 3.7. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exist* $\eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0,\infty), \lambda \in (0,1)$ *and a lower semicontinuous function* $\varphi : E \to \mathbb{R}^+$ *such that*

$$\begin{aligned}\alpha(\phi(c),\psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi))\\ \leq \lambda\mu(\phi(c),\psi(c))(M(\phi,\psi) - \eta(N(\phi,\psi))\end{aligned} \tag{3.2}$$

for any $\phi,\psi \in E_0$, *where* $\eta(t) < t$ *for any* $t > 0$,

$M(\phi,\psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),$

$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi),$

$\frac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\}$,

$N(\phi,\psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)),$

$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$,

(ii) T *is a triangular* α_c-*admissible mapping and triangular* μ_c-*subadmissible mapping,*

(iii) R_c *is algebraically closed with respect to the difference,*

(iv) *if* $\{\phi_n\}$ *is a sequence in* E_0 *such that* $\phi_n \to \phi$ *as* $n \to \infty, \alpha(\phi_n(c),\phi_{n+1}(c)) \geq 1$ *and* $\mu(\phi_n(c),\phi_{n+1}(c)) \leq 1$ *for any* $n \in \mathbb{N}\cup\{0\}$ *then* $\alpha(\phi_n(c),\phi(c)) \geq 1$ *and* $\mu(\phi_n(c),\phi(c)) \leq 1$ *for any* $n \in \mathbb{N}\cup\{0\}$ *and*

(v) *there exists* $\phi_0 \in R_c$ *such that* $\alpha(\phi_0(c),T\phi_0) \geq 1$ *and* $\mu(\phi_0(c),T\phi_0) \leq 1$.

Then T *has a PPF dependent fixed point* $\phi^* \in R_c$ *such that* $\varphi(\phi^*(c)) = 0$.

By choosing If $\varphi(x) = 0$ for any $x \in E$ in Corollay 3.7 we get the following corollary.

Corollary 3.8. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exist* $\eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0,\infty)$ *and* $\lambda \in (0,1)$ *such that*

$$\alpha(\phi(c),\psi(c))||T\phi - T\psi||_E \leq \lambda\mu(\phi(c),\psi(c))(M(\phi,\psi) - \eta(N(\phi,\psi))) \tag{3.3}$$

for any $\phi,\psi \in E_0$, *where* $\eta(t) < t$ *for any* $t > 0$,

$M(\phi,\psi) = \max\{||\phi - \psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E,$

$\frac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\}$,

$N(\phi,\psi) = \max\{||\phi - \psi||_{E_0}, ||\psi(c) - T\psi||_E\}$,

(ii) T *is a triangular* α_c-*admissible mapping and triangular* μ_c-*subadmissible mapping,*

(iii) R_c *is algebraically closed with respect to the difference,*

(iv) *if* $\{\phi_n\}$ *is a sequence in* E_0 *such that* $\phi_n \to \phi$ *as* $n \to \infty, \alpha(\phi_n(c),\phi_{n+1}(c)) \geq 1$ *and* $\mu(\phi_n(c),\phi_{n+1}(c)) \leq 1$ *for any* $n \in \mathbb{N}\cup\{0\}$ *then* $\alpha(\phi_n(c),\phi(c)) \geq 1$ *and* $\mu(\phi_n(c),\phi(c)) \leq 1$ *for any* $n \in \mathbb{N}\cup\{0\}$ *and*

(v) *there exists* $\phi_0 \in R_c$ *such that* $\alpha(\phi_0(c),T\phi_0) \geq 1$ *and* $\mu(\phi_0(c),T\phi_0) \leq 1$.

Then T *has a PPF dependent fixed point in* R_c.

By choosing $\alpha(x,y) = 1 = \mu(x,y)$ for any $x,y \in E$ in Corollay 3.6 we get the following corollary.

Corollary 3.9. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exist* $\xi \in \Psi, \eta \in \Phi, \lambda \in (0,1)$ *and a lower semicontinuous function* $\varphi : E \to \mathbb{R}^+$ *such that*

$$\xi(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda(\xi(M(\phi,\psi)) - \eta(N(\phi,\psi))) \tag{3.4}$$

for any $\phi, \psi \in E_0$*, where* $\xi(t) > \eta(t)$ *for any* $t > 0$,
$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),$
$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi),$
$\frac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\},$
$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)),$
$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\},$
(ii) R_c *is algebraically closed with respect to the difference.*
Then T *has a PPF dependent fixed point* $\phi^* \in R_c$ *such that* $\varphi(\phi^*(c)) = 0$.

By choosing $\varphi(x) = 0$ for any $x \in E$ in Corollay 3.9 we get the following corollary.

Corollary 3.10. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*
(i) *if there exist* $\xi \in \Psi, \eta \in \Phi$ *and* $\lambda \in (0,1)$ *such that*

$$\xi(||T\phi - T\psi||_E) \leq \lambda(\xi(M(\phi,\psi)) - \eta(N(\phi,\psi))) \tag{3.5}$$

for any $\phi, \psi \in E_0$*, where* $\xi(t) > \eta(t)$ *for any* $t > 0$,
$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E,$
$\frac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\},$
$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\psi(c) - T\psi||_E\},$
(ii) R_c *is algebraically closed with respect to the difference.*
Then T *has a PPF dependent fixed point in* R_c.

By choosing $\xi(t) = t$ for any $t \in \mathbb{R}^+$ in Corollary 3.10 we get the following corollary.

Corollary 3.11. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*
(i) *if there exist* $\eta \in \Phi$ *and* $\lambda \in (0,1)$ *such that*
$$||T\phi - T\psi||_E \leq \lambda(M(\phi,\psi) - \eta(N(\phi,\psi)))$$
for any $\phi, \psi \in E_0$*, where* $\eta(t) < t$ *for any* $t > 0$,
$M(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E,$
$\frac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\},$
$N(\phi,\psi) = \max\{||\phi-\psi||_{E_0}, ||\psi(c) - T\psi||_E\}$
and
(ii) R_c *is algebraically closed with respect to the difference.*
Then T *has a PPF dependent fixed point in* R_c.

We present the following example in support of Theorem 2.1, which suggests that under the hypotheses of Theorem 2.1, T may have more than one fixed point.

Example 3.1. *Let* $E = \mathbb{R},\ c = 1 \in I = [\frac{1}{2}, 2] \subseteq \mathbb{R},\ E_0 = C(I,E)$.
We define $T : E_0 \to E, \alpha : E \times E \to \mathbb{R}^+, \mu : E \times E \to (0,\infty)$ *by*

$$T\phi = \begin{cases} -2 & \text{if } \phi(c) \leq 0 \\ \frac{3\phi(c)-4}{2} & \textit{if } 0 \leq \phi(c) < \frac{1}{2} \\ -\frac{1}{2} & \text{if } \phi(c) \geq \frac{1}{2}, \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1 & \textit{if } x \geq y \\ 0 & \text{if } x < y, \end{cases}$$

and

$$\mu(x,y) = \begin{cases} \frac{1}{\sqrt{2}} & \textit{if } x \geq y \\ 2 & \text{if } x < y. \end{cases}$$

We first prove that T is an α_c−admissible mapping.
For any $\phi, \psi \in E_0$, we suppose that $\alpha(\phi(c), \psi(c)) \geq 1$.
From the definition of α, we get $\phi(c) \geq \psi(c)$.
Case (i): Suppose that $0 \leq \phi(c), \psi(c) < \frac{1}{2}$.
Clearly $3\phi(c) - 4 \geq 3\psi(c) - 4$ and which implies that $\frac{3\phi(c)-4}{2} \geq \frac{3\psi(c)-4}{2}$.
Therefore $T\phi \geq T\psi$ and hence $\alpha(T\phi, T\psi) \geq 1$.
Case (ii): Suppose that $\phi(c), \psi(c) \geq \frac{1}{2}$.
Clearly $T\phi = -\frac{1}{2} = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.
Case (iii): Suppose that $\phi(c), \psi(c) \leq 0$.
Clearly $T\phi = -2 = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.
Case (iv): Suppose that $0 \leq \phi(c) < \frac{1}{2}$ and $\psi(c) \leq 0$.
Since $\phi(c) \geq 0$ we have $T\phi = \frac{3\phi(c)-4}{2} \geq -2 = T\psi$
and which implies that $\alpha(T\phi, T\psi) \geq 1$.
Case (v): Suppose that $\phi(c) \geq \frac{1}{2}$ and $\psi(c) \leq 0$.
Clearly $T\phi = -\frac{1}{2} > -2 = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.
Case (vi): Suppose that $\phi(c) \geq \frac{1}{2}$ and $0 \leq \psi(c) < \frac{1}{2}$.
Since $\psi(c) \leq 1$ we have $T\phi = -\frac{1}{2} \geq \frac{3\psi(c)-4}{2} = T\psi$ and
which implies that $\alpha(T\phi, T\psi) \geq 1$.
From the above cases, we get that T is an α_c−admissible mapping.
For any $\phi, \psi, \gamma \in E_0$, we suppose that $\alpha(\phi(c), \psi(c)) \geq 1$ and $\alpha(\psi(c), \gamma(c)) \geq 1$.
From the definition of α, we get $\phi(c) \geq \psi(c) \geq \gamma(c)$.
Therefore $\phi(c) \geq \gamma(c)$ and hence $\alpha(\phi(c), \gamma(c)) \geq 1$.
Therefore T is a traingular α_c−admissible mapping.
Similarly, we can prove that T is a triangular μ_c−subadmissible mapping.
Let $\lambda = \frac{1}{\sqrt{2}}$. Then $\lambda \in (0, 1)$.
We define $\varphi : E \to \mathbb{R}^+$ by

$$\varphi(x) = \begin{cases} 0 & \textit{if } \; x \leq 0 \\ x & \text{if } \; 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \; x \geq \frac{1}{2}. \end{cases}$$

Clearly φ is a lower semicontinuous function.
We define $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ by $\eta(t) = \frac{t}{2}$ for any $t \in \mathbb{R}^+$. Clearly $\eta \in \Phi$.
Let $\phi, \psi \in E_0$.
If $\phi(c) < \psi(c)$ then from the definition of α, the inequality (3.2) trivially holds.
Without loss of generality, we assume that $\phi(c) \geq \psi(c)$.
From the definition of α, we get $T\phi \geq T\psi$.
We consider
$||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi) \leq T\phi - T\psi + T\phi + T\psi = 2\ T\phi$.
Therefore

$$\alpha(\phi(c), \psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq 2\ T\phi. \tag{3.6}$$

Also we have

$$\begin{aligned} M(\phi, \psi) = \max\{&||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi), \\ &||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi), \\ &\tfrac{1}{2}[||\phi(c) - T\psi||_E + \varphi(\phi(c)) + \varphi(T\psi) + ||\psi(c) - T\phi||_E + \varphi(\psi(c)) + \varphi(T\phi)]\} \\ \geq \max\{&||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\} \end{aligned}$$

which implies that

$$
\begin{aligned}
M(\phi,\psi)-\eta(N(\phi,\psi)) &\geq \tfrac{1}{2}\max\{||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)),\\
&\qquad\qquad ||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}\\
&\geq \tfrac{1}{2}\max\{||\phi(c)-\psi(c)||_E+\varphi(\phi(c))+\varphi(\psi(c)),\\
&\qquad\qquad ||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}\\
&= \tfrac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),\\
&\qquad\qquad ||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}.\\
&\qquad\qquad\qquad\qquad \textit{(since } \phi(c)\geq\psi(c)\textit{)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
M(\phi,\psi)-\eta(N(\phi,\psi)) \geq \tfrac{1}{2}\max\{&\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),\\
&||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}.
\end{aligned}
\tag{3.7}
$$

Case (i): Suppose that $T\psi=\psi(c)$.
If $\psi\in R_c$ then ψ is a PPF dependent fixed point of T and hence the result holds.
Let us suppose $\psi\notin R_c$.
We define $\psi_1: I\to E$ by $\psi_1(x)=\psi(c),\ x\in I$. Clearly $\psi_1\in R_c$.
From the definition of T, we have

$$
T\psi_1=\begin{cases} -2 & \text{if } \psi_1(c)\leq 0\\ \frac{3\psi_1(c)-4}{2} & \textit{if } 0\leq\psi_1(c)<\frac{1}{2}\\ -\frac{1}{2} & \text{if } \psi_1(c)\geq\frac{1}{2}.\end{cases}
$$

That is

$$
T\psi_1=\begin{cases} -2 & \text{if } \psi(c)\leq 0\\ \frac{3\psi(c)-4}{2} & \textit{if } 0\leq\psi(c)<\frac{1}{2}\\ -\frac{1}{2} & \text{if } \psi(c)\geq\frac{1}{2}.\end{cases}
$$

Therefore $T\psi_1=T\psi=\psi(c)=\psi_1(c)$.
Hence ψ_1 is a PPF dependent fixed point of T in R_c and the result follows.
Case (ii): Suppose that $\psi(c)<T\psi$.
From the definition of T we have $\psi(c)<-2$ and hence $T\psi=-2$.
Since $\phi(c)\geq\psi(c)$ we have $\phi(c)\leq 0$ or $0\leq\phi(c)<\frac{1}{2}$ or $\phi(c)\geq\frac{1}{2}$.
Suppose that $\phi(c)\leq 0$. Clearly $T\phi=-2$.
From (3.7) we have

$$
\begin{aligned}
M(\phi,\psi)-\eta(N(\phi,\psi)) &\geq \tfrac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),\\
&\qquad\qquad ||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}\\
&= \tfrac{1}{2}\max\{\phi(c)-\psi(c),T\psi-\psi(c)\}\\
&\qquad\qquad \textit{(since } \varphi(\phi(c))=\varphi(\psi(c))=\varphi(T\psi)=0\textit{)}\\
&\geq \tfrac{1}{2}\max\{0,T\psi-\psi(c)\}\geq\tfrac{1}{2}\max\{0,T\psi-\phi(c)\}.\\
&\qquad\qquad \textit{(since } \phi(c)\geq\psi(c)\implies-\psi(c)\geq-\phi(c)\textit{)}
\end{aligned}
$$

If $\phi(c)<T\psi$ then $T\psi-\phi(c)>0$ and hence
$M(\phi,\psi)-\eta(N(\phi,\psi))\geq\frac{1}{2}(T\psi-\phi(c))=-1-\frac{\phi(c)}{2}$.
Clearly
$\lambda\mu(\phi(c),\psi(c))(M(\phi,\psi)-\eta(N(\phi,\psi)))\geq-\frac{1}{2}-\frac{\phi(c)}{4}\geq 2\ T\phi$.
(since $-\frac{1}{2}-\frac{\phi(c)}{4}\geq-4\iff\phi(c)\leq 14$)
If $\phi(c)>T\psi$ then $T\psi-\phi(c)<0$ and hence
$M(\phi,\psi)-\eta(N(\phi,\psi))\geq 0>-4=2\ (-2)=2\ T\phi$.
Suppose that $0\leq\phi(c)<\frac{1}{2}$. Clearly $T\phi=\frac{3\phi(c)-4}{2}$.
From (3.7) we have
$M(\phi,\psi)-\eta(N(\phi,\psi))\geq\frac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),$

$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \frac{1}{2}\max\{\phi(c) - \psi(c) + \phi(c), T\psi - \psi(c)\}$$
(since $\varphi(\psi(c)) = \varphi(T\psi) = 0$)
$$= \frac{1}{2}\max\{2\phi(c) - \psi(c), T\psi - \psi(c)\}$$
$$\geq \frac{1}{2}\max\{2\psi(c) - \psi(c), T\psi - \psi(c)\}$$
$$= \frac{1}{2}\max\{\psi(c), T\psi - \psi(c)\}$$
$$= \frac{1}{2}(T\psi - \psi(c)) = -1 - \frac{\psi(c)}{2} \geq -1 - \frac{\phi(c)}{2}.$$
(since $\psi(c) < -2$ and $T\psi - \psi(c) > 0$)

Clearly
$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq -\frac{1}{2} - \frac{\phi(c)}{4} \geq 2\ T\phi.$
(since $-\frac{1}{2} - \frac{\phi(c)}{4} \geq 3\phi(c) - 4 \iff \phi(c) \leq \frac{14}{13}$)
Suppose that $\phi(c) \geq \frac{1}{2}$. Clearly $T\phi = -\frac{1}{2}$.
From (3.7) we have
$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \frac{1}{2}\max\{\phi(c) - \psi(c), T\psi - \psi(c)\}$$
(since $\varphi(\phi(c)) = \varphi(\psi(c)) = \varphi(T\psi) = 0$)
$$= \frac{1}{2}(\phi(c) - \psi(c))$$
(since $\phi(c) > T\psi$ we have $\phi(c) - \psi(c) > T\psi - \psi(c) > 0$)
$$> 0.$$

Clearly
$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) > 0 > -1 = 2(-\frac{1}{2}) = 2\ T\phi.$
Case (iii): Suppose that $\psi(c) > T\psi$.
From the definition of T we have $0 \leq \psi(c) < \frac{1}{2}$ or $-2 < \psi(c) \leq 0$ or $\psi(c) \geq \frac{1}{2}$.
Sub-case (i): Suppose that $0 \leq \psi(c) < \frac{1}{2}$. Clearly $T\psi = \frac{3\psi(c)-4}{2} < 0$.
Since $\phi(c) \geq \psi(c)$ we have either $0 \leq \phi(c) < \frac{1}{2}$ or $\phi(c) \geq \frac{1}{2}$.
Suppose that $0 \leq \phi(c) < \frac{1}{2}$. Clearly $T\phi = \frac{3\phi(c)-4}{2}$
From (3.7) we have
$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \frac{1}{2}\max\{\phi(c) - \psi(c) + \phi(c) + \psi(c), \psi(c) - T\psi + \psi(c)\}$$
(since $T\psi < 0$ we have $\varphi(T\psi) = 0$)
$$= \frac{1}{2}\max\{2\phi(c), 2\psi(c) - T\psi\} \geq \frac{1}{2}max\{2\psi(c), 2\psi(c) - T\psi\}.$$
(since $\phi(c) \geq \psi(c)$)
$$= \psi(c) - \frac{T\psi}{2}.\ \textit{(since } T\psi < 0\textit{)}$$

Clearly
$\lambda\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c)}{2} - \frac{T\psi}{4} = \frac{\psi(c)}{2} - \frac{3\psi(c)-4}{8} = \frac{\psi(c)+4}{8} \geq 2T\phi.$
(since $\phi(c) \geq \psi(c)$ and $\frac{\psi(c)+4}{8} \geq 3\phi(c) - 4 \iff \psi(c) \leq \frac{36}{23}$)
Suppose that $\phi(c) \geq \frac{1}{2}$. Clearly $T\phi = -\frac{1}{2}$.
From (3.7) we have
$$M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \frac{1}{2}\max\{\phi(c) - \psi(c) + \psi(c), \psi(c) - T\psi + \psi(c)\}$$

$$(since\ \phi(c) \geq \tfrac{1}{2}\ and\ T\psi < 0\ we\ have\ \varphi(\psi(c)) = \varphi(T\psi) = 0)$$
$$= \tfrac{1}{2}\max\{\phi(c), 2\psi(c) - T\psi\} \geq \tfrac{1}{2}max\{\psi(c), 2\psi(c) - T\psi\}.$$
(since $\phi(c) \geq \psi(c)$*)*
$$= \psi(c) - \tfrac{T\psi}{2}.\ (since\ T\psi < 0)$$

Clearly
$$\lambda\mu(\phi(c), \psi(c))(M(\phi,\psi) - \eta(N(\phi,\psi))) \geq \tfrac{\psi(c)}{2} - \tfrac{T\psi}{4} = \tfrac{\psi(c)}{2} - \tfrac{3\psi(c)-4}{8}$$
$$= \tfrac{\psi(c)+4}{8} \geq 2(-\tfrac{1}{2}) = 2\ T\phi.$$
(since $\frac{\psi(c)+4}{8} \geq -1 \iff \psi(c) \geq -12$)

<u>*Sub-case (ii):*</u> *Suppose that* $-2 < \psi(c) \leq 0$. *Clearly* $T\psi = -2$.
Since $\phi(c) \geq \psi(c)$ *we have either* $-2 < \phi(c) \leq 0$ *or* $0 \leq \phi(c) < \frac{1}{2}$ *or* $\phi(c) \geq \frac{1}{2}$.
Suppose that $-2 < \phi(c) \leq 0$. *Clearly* $T\phi = -2$.
From (3.7) we have
$$M(\phi,\psi) - \eta(N(\phi,\psi)) \geq \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \tfrac{1}{2}\max\{\phi(c) - \psi(c), \psi(c) - T\psi\}$$
$$(since\ \phi(c), \psi(c), T\psi < 0\ we\ have\ \varphi(T\psi) = \varphi(\phi(c)) = \varphi(\psi(c)) = 0)$$
$$\geq \tfrac{1}{2}\max\{0, \psi(c) + 2\} = \tfrac{\psi(c)+2}{2}.\ (since\ \psi(c) + 2 > 0)$$

Clearly
$$\lambda\mu(\phi(c), \psi(c))(M(\phi,\psi) - \eta(N(\phi,\psi))) \geq \tfrac{\psi(c)+2}{4} \geq -4 = 2\ T\phi.$$
(since $\frac{\psi(c)+2}{4} \geq -4 \iff \psi(c) \geq -18$*)*

Suppose that $0 \leq \phi(c) < \frac{1}{2}$. *Clearly* $T\phi = \frac{3\phi(c)-4}{2}$.
From (3.7) we have
$$M(\phi,\psi) - \eta(N(\phi,\psi)) \geq \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \phi(c), \psi(c) - T\psi\}$$
(since $\psi(c), T\psi \leq 0$ *we have* $\varphi(T\psi) = \varphi(\psi(c)) = 0$*)*
$$\geq \tfrac{1}{2}\max\{\phi(c), \psi(c) + 2\} \geq \tfrac{1}{2}max\{\psi(c), \psi(c) + 2\}$$
(since $\psi(c) + 2 > 0$*)*
$$= \tfrac{\psi(c)+2}{2}.$$

Clearly
$$\lambda\mu(\phi(c), \psi(c))(M(\phi,\psi) - \eta(N(\phi,\psi))) \geq \tfrac{\psi(c)+2}{4} \geq 2\ T\phi.$$
(since $\phi(c) \geq \psi(c)$ *and* $\frac{\psi(c)+2}{4} \geq 3\phi(c) - 4 \iff \psi(c) \leq \frac{18}{11}$)

Suppose that $\phi(c) \geq \frac{1}{2}$. *Clearly* $T\phi = -\frac{1}{2}$.
From (3.7) we have
$$M(\phi,\psi) - \eta(N(\phi,\psi)) \geq \tfrac{1}{2}\max\{\phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)),$$
$$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}$$
$$= \tfrac{1}{2}\max\{\phi(c) - \psi(c), \psi(c) - T\psi\}$$
$$(since\ \psi(c), T\psi \leq 0\ and\ \phi(c) \geq \tfrac{1}{2}\ we\ have\ \varphi(T\psi) = \varphi(\phi(c)) = \varphi(\psi(c)) = 0)$$
$$\geq \tfrac{1}{2}\max\{0, \psi(c) + 2\} = \tfrac{\psi(c)+2}{2}.\ (since\ \psi(c) + 2 > 0)$$

Clearly

$\lambda\mu(\phi(c),\psi(c))(M(\phi,\psi)-\eta(N(\phi,\psi))) \geq \frac{\psi(c)+2}{4} \geq 2\ T\phi.$

(since $\frac{\psi(c)+2}{4} \geq -1 \iff \psi(c) \geq -6$)

Sub-case (iii): Suppose that $\psi(c) \geq \frac{1}{2}$. *Clearly* $T\psi = -\frac{1}{2}$.
Since $\phi(c) \geq \psi(c)$ *we have* $\phi(c) \geq \frac{1}{2}$. *Clearly* $T\phi = -\frac{1}{2}$.
From (3.7) we have

$$\begin{aligned} M(\phi,\psi)-\eta(N(\phi,\psi)) &\geq \tfrac{1}{2}\max\{\phi(c)-\psi(c)+\varphi(\phi(c))+\varphi(\psi(c)),\\ &\qquad ||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi)\}\\ &= \tfrac{1}{2}\max\{\phi(c)-\psi(c),\psi(c)-T\psi\} \end{aligned}$$

(since $T\psi \leq 0$ *and* $\psi(c),\phi(c) \geq \frac{1}{2}$ *we have* $\varphi(T\psi)=\varphi(\phi(c))=\varphi(\psi(c))=0$*)*

$$\geq \tfrac{1}{2}\max\{0,\psi(c)+\tfrac{1}{2}\} = \tfrac{\psi(c)}{2}+\tfrac{1}{4}.$$ *(since* $\psi(c)+\frac{1}{2}>0$*)*

Clearly
$\lambda\mu(\phi(c),\psi(c))(M(\phi,\psi)-\eta(N(\phi,\psi))) \geq \frac{\psi(c)}{4}+\frac{1}{8} \geq 2\ T\phi.$

(since $\frac{\psi(c)}{4}+\frac{1}{8} \geq -1 \iff \psi(c) \geq -\frac{9}{2}$)

From all the above cases, we get

$$\lambda\mu(\phi(c),\psi(c))(M(\phi,\psi)-\eta(N(\phi,\psi))) \geq \alpha(\phi(c),\psi(c))(||T\phi-T\psi||_E+\varphi(T\phi)+\varphi(T\psi)).$$

Therefore the inequality (3.2) *is holds.*
Let $\{\phi_n\}$ *be a sequence in* E_0 *such that* $\alpha(\phi_n(c),\phi_{n+1}(c)) \geq 1$ *and* $\mu(\phi_n(c),\phi_{n+1}(c)) \leq 1$ *for any* $n \in \mathbb{N}\cup\{0\}$.
Then from the definition of α, *we have* $\phi_n(c) \geq \phi_{n+1}(c)$ *for any* $n \in \mathbb{N}\cup\{0\}$ *and hence convergent. Since* $\mathbb{R}$ *is complete, there exists* $r \in \mathbb{R}$ *such that* $\phi_n(c) \to r$ *as* $n \to \infty$.
We define $\gamma: I \to E$ *by* $\gamma(x)=r, x \in I$. *Then* $\gamma \in R_c$ *and* $\gamma(c)=r$.
Therefore $\phi_n(c) \to \gamma(c)$ *as* $n \to \infty$. *Clearly* $\phi_n(c) \geq \gamma(c)$ *for any* $n \in \mathbb{N}\cup\{0\}$.
From the definition of α *and* μ, *we get* $\alpha(\phi_n(c),\gamma(c)) \geq 1$ *and* $\mu(\phi_n(c),\gamma(c)) \leq 1$ *for any* $n \in \mathbb{N}\cup\{0\}$. *Therefore the condition (iv) is satisfied.*
For any $n \in \mathbb{R}$, *we define* $\phi_n: I \to E$ *by*

$$\phi_n(x) = \begin{cases} nx^2 & \textit{if } x \in [\frac{1}{2},1] \\ \frac{n}{x^2} & \text{if } x \in [1,2]. \end{cases}$$

Clearly $\phi_n \in E_0, ||\phi_n||_{E_0} = ||\phi_n(c)||_E$ *and hence* $\phi_n \in R_c$ *for any* $n \in \mathbb{R}$.
Let $\mathrm{F}_0 = \{\phi_n \mid n \in \mathbb{R}\}$. *Then* $\mathrm{F}_0 \subseteq R_c$ *and* F_0 *is algebraically closed with respect to the difference.*
Clearly $\phi_2(c) \geq T\phi_2$ *and hence* $\alpha(\phi_2(c),T\phi_2) \geq 1$ *and* $\mu(\phi_2(c),T\phi_2) \leq 1$.
Therefore the condition (v) is satisfied.

Therefore T *satisfies all the hypotheses of Corollary 3.7 which in turn* T *satisfies all the hypotheses of Theorem 2.1 with* $\zeta(t,s)=\lambda s-t, G(s,t)=s-t, \xi(t)=t$ *for any* $s,t \in \mathbb{R}^+, C_G=0$ *and* $\lambda=\frac{1}{\sqrt{2}} \in (0,1)$ *and hence* $\phi_{-2} \in R_c$ *is a PPF dependent fixed point of* T *such that* $\varphi(\phi_{-2}(c))=0$.
We define $\gamma_1: I \to E$ *by*

$$\gamma_1(x) = \begin{cases} -2x & \textit{if } x \in [\frac{1}{2},1] \\ 2x-4 & \text{if } x \in [1,2]. \end{cases}$$

Clearly $||\gamma_1||_{E_0} = 2 = ||\gamma_1(c)||_E$ *and hence* $\gamma_1 \in R_c$.
We observe that $T\gamma_1 = \gamma_1(c)$. *(since* $\gamma_1(c) = -2 < 0$, *we have* $T\gamma_1 = -2 = \gamma_1(c)$*)*
Therefore $\gamma_1 \in R_c$ *is another PPF dependent fixed point of* T *such that* $\varphi(\gamma_1(c)) = 0$.

References

[1] Ya. I. Alber, S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces New results in Operator theory*, Adv. Appl., Vol.98 , Birkhauser Verlag, (1997), 7-22 .

[2] A. H. Ansari, *Note on $\phi - \psi -$ contractive type mappings and related fixed point*, The 2nd Regional Conference on Mathematics and Applications, Payame Noor University Tehran, (2014), 377-380.

[3] A. H. Ansari, J. Kaewcharoen, *$C-$ class functions and fixed point theorems for generalized $\alpha - \eta - \psi - \phi - F-$contraction type mappings in $\alpha - \eta$ complete metric spaces*, J. Nonlinear Sci. Appl., **9(6)**(2016), 4177-4190.

[4] Antonella Nastasi and P. Vetro, *Fixed point results on metric and partial metric spaces via simuation functions*, J. Nonlinear Sci. Appl., **8**(2015), 1059-1069.

[5] G.V.R. Babu, G. Satyanarayana and M. Vinod Kumar, *Properties of Razumikhin class of functions and PPF dependent fixed points of Weakly contractive type mappings*, Bull. Int. Math. Virtual Institute, **9(1)**(2019), 65-72.

[6] G.V.R. Babu and M. Vinod Kumar, *PPF dependent coupled fixed points via $C-$class functions*, J. Fixed Point Theory, **2019**(2019), Article ID 7.

[7] G.V.R. Babu and M. Vinod Kumar, *PPF dependent fixed points of generalized Suzuki type contractions via simulation functions*, Advances in the Theory of Nonlinear Anal. and its Appl., **3(3)**(2019), 121-140.

[8] G.V.R. Babu and M. Vinod Kumar, *PPF dependent fixed points of generalized contractions via C_G-simulation functions*, Communications in Nonlinear Anal., **7(1)**(2019), 1-16.

[9] B. E. Rhoades, *Some theorems on weakly contractive mappings*, Nonlinear Anal. 47 (2001) 2683-2693.

[10] Banach S.: *Sur les operations dans les ensembles abstraits et leur appliacation aux equations integrales*, Fund. math., **3**(1922), 133-181.

[11] Bapurao C. Dhage, *On some common fixed point theorems with PPF dependence in Banach spaces*, J. Nonlinear Sci. Appl., **5**(2012), 220-232.

[12] S. R. Bernfeld, V. Lakshmikantham, and Y. M. Reddy, *Fixed point theorems of operators with PPF dependence in Banach spaces*, Appl. Anal., **6(4)**(1977), 271-280.

[13] L. Ćirić, S. M. Alsulami, P. Salimi and P. Vetro, *PPF dependent fixed point results for triangular α_c-admissible mappings*, Hindawi Publishing corporation, (2014), Article ID 673647, 10 pages.

[14] S. Cho, *Fixed point theorems for generalized weakly contractive mappings in metric spaces with application*, Fixed point theory and Appl., **2018**(2018).

[15] S. H. Cho, *A fixed point theorem for weakly $\alpha-$contractive mappings with application*, Appl. Mathematical Sciences, **7**(2013), No. 60, 2953-2965.

[16] B. S. Choudhury, P. Konar, B. E. Rhoades and N. Metiya, *Fixed point theorems for generalized weakly contractive mappings*, Nonlinear Anal., **74**(2011), 2116-2126.

[17] Z. Dirci, F. A. McRae and J. Vasundharadevi, *Fixed point theorems in partially ordered metric spaces for operators with PPF dependence*, Nonlinear Anal., **67**(2007), 641-647.

[18] D. Doric, *Common fixed point for generalized $(\psi, \phi)-$weak contractions*, Appl. Mathematics Letters, **22**(2009), 1896-1900.

[19] A. Farajzadeh, A.Kaewcharoen and S.Plubtieng, *PPF dependent fixed point theorems for multivalued mappings in Banach spaces*, Bull. Iranian Math.Soc., **42(6)**(2016), 1583-1595.

[20] Haitham Quwagneh, Mohd Salmi MD Noorani, Wasfi Shatanawi and Habes Alsamir, *Common fixed points for pairs of triangular $\alpha-$admissible mappings*, J. Nonlinear Sci. Appl., **10**(2017), 6192 - 6204.

[21] N. Hussain, S. Khaleghizadeh, P. Salimi and F. Akbar, *New Fixed Point Results with PPF dependence in Banach Spaces Endowed with a Graph*, Abstr. Appl. Anal., (2013), Article ID 827205.

[22] E. Karapınar, *Fixed points results via simulation functions*, Filomat, **30(8)**(2016), 2343 - 2350.

[23] E. Karapınar, P. Kumam and P. Salimi, *On a $\alpha - \psi-$Meir-Keeler contractive mappings*, Fixed point theory Appl., (2013), 2013:94,

[24] F. Khojasteh, Satish Shukla and S. Radenovic, *A new approach to the study of fixed point theory for simulation function*, Filomat, **29(6)**(2015), 1189-1194.

[25] X. L. Liu, A. H. Ansari, S. Chandok and S. Radenović, *On some results in metric spaces using auxiliary simulation functions via new functions*, J. Comput. Anal. Appl., **24(6)**(2018).

[26] Marwan Amin Kutbi and Wutiphol Sintunavarat, *On sufficient coniditons for the existence of Past-Present-Future dependent fixed point in Razumikhin class and application*, Abstr. Appl Anal., (2014), Article ID 342684.

[27] S. Radenović, F. Vetro and J. Vujaković, *An alternative and easy approach to fixed point results via simulation functions*, Demonstr. Math., **50(1)**(2017).

[28] Ravi P. Agarwal, P. Kumam and Wutiphol Sintunavarat, *PPF dependent fixed point theorems for an α_c-admissible non-self mapping in the Razumikhin class*, Fixed Point Theory Appl., 2013(1)(2013), 280.

[29] A. R. Roldán-Lopez-de-Hierro, E. Karapınar, C. Roldán-Lopez-de-Hierro, J. Martines-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math., **275**(2015), 345-355.

[30] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha - \psi-$contractive type mappings*, Nonlinear Anal., **75(4)**(2012), 2154-2165.

G. V. R. Babu,
Department of Mathematics, Andhra University, Visakhapatnam-530 003, India
E-mail address: gvr_babu@hotmail.com

M. Vinod Kumar,
Department of Mathematics, Andhra University, Visakhapatnam-530 003, India, Permanent Address : Department of Mathematics, ANITS, Sangivalasa, Viskhapatnam -531 162, India
E-mail address: dravinodvivek@gmail.com