PPF DEPENDENT FIXED POINTS OF GENERALIZED WEAKLY CONTRACTION MAPS VIA \( C_G \)-SIMULATION FUNCTIONS

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Abstract. In this paper, we introduce the notion of generalized weakly \( Z_{G,\alpha,\mu,\xi,\eta,\phi} \)-contraction maps with respect to the \( C_G \)-simulation function and prove the existence of PPF dependent fixed points of nonself maps in Banach spaces. For such maps, PPF dependent fixed points may not be unique. We provide an example to illustrate this phenomenon.

1. INTRODUCTION AND PRELIMINARIES

In fixed point theory, Banach contraction principle is one of the well known basic fundamental result and it gives an idea for the existence of fixed points with uniqueness in complete metric spaces. In 1997, Alber and Guerre-Delabriere [1] introduced weakly contractive maps which are extensions of contraction maps and obtained fixed point results in the setting of Hilbert spaces. Rhoades [9] extended this concept to metric spaces. Based on this idea, many authors generalized and extended the contraction maps and weakly contractive maps by introducing new functions like \( \alpha \)-admissible maps, \( C \)-class function, simulation function etc., for more details we refer [2, 10, 14, 18].

Throughout this paper, we denote the real line by \( \mathbb{R}, \mathbb{R}^+ = [0, \infty) \), and \( \mathbb{N} \) is the set of all natural numbers, \( \mathbb{Z} \) is the set of integers.

In 2011, Choudhury, Konar, Rhoades and Metiya [16] introduced the notion of generalized weakly contractive mapping as follows and proved the existence of fixed points of generalized weakly contractive mappings in complete metric spaces.

**Definition 1.1.** [16] Let \((X, d)\) be a metric space, \(T\) a self-mapping of \(X\). We shall call \(T\) a generalized weakly contractive mapping if for any \(x, y \in X\),

\[
\psi(d(Tx, Ty)) \leq \psi(m(x, y)) - \phi(\max\{d(x, y), d(y, Ty)\}),
\]

2010 Mathematics Subject Classification. Primary: 47H10; Secondaries: 54H25.

Key words and phrases. \( \alpha \)-admissible mapping, \( \mu \)-subadmissible mapping, \( C \)-class function, Razumikhin class, PPF dependent fixed point, simulation function, \( C_G \)-simulation function.

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where
\( i \) \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous monotone increasing function with
\( \psi(t) = 0 \iff t = 0, \)
\( ii \) \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function with \( \phi(t) = 0 \iff t = 0, \)
\( iii \) \( m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}. \)

**Theorem 1.1.** Let \((X, d)\) be a complete metric space, \(T\) a generalized weakly contractive self-mapping of \(X\). Then \(T\) has a unique fixed point.

In 2012, Samet, Vetro and Vetro \cite{30} introduced the concept of \(\alpha\)-admissible mappings as follows.

**Definition 1.2.** Let \((X, d)\) be a metric space. Let \(T : X \to X\) and \(\alpha : X \times X \to \mathbb{R}^+\) be two functions. Then \(T\) is said to be an \(\alpha\)-admissible mapping if
\[
\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1
\]
for all \(x, y \in X\).

In 2013, Karapınar, Kumam and Salimi \cite{23} introduced the notion of triangular \(\alpha\)-admissible mappings as follows.

**Definition 1.3.** Let \(T\) be a self-mapping of \(X\) and let \(\alpha : X \times X \to \mathbb{R}^+\) be a function. Then \(T\) is said to be a triangular \(\alpha\)-admissible mapping if
\[
\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1
\]
and
\[
\alpha(x, z) \geq 1, \ \alpha(z, y) \geq 1 \implies \alpha(x, y) \geq 1
\]
for all \(x, y, z \in X\).

In 2014, Ansari \cite{2} introduced the concept of \(C\)-class function as follows.

**Definition 1.4.** A mapping \(G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}\) is called a \(C\)-class function if it is continuous and for any \(s, t \in \mathbb{R}^+\), the function \(G\) satisfies the following conditions:
\(i\) \(G(s, t) \leq s\) and
\(ii\) \(G(s, t) = s\) implies that either \(s = 0\) or \(t = 0\).
The family of all \(C\)-class functions is denoted by \(\Delta\).

**Example 1.1.** The following functions belong to \(\Delta\).
\(i\) \(G(s, t) = s - t\) for all \(s, t \in \mathbb{R}^+\).
\(ii\) \(G(s, t) = ks\) for all \(s, t \in \mathbb{R}^+\) where \(0 < k < 1\).
\(iii\) \(G(s, t) = \frac{s}{(s + t)}\) for all \(s, t \in \mathbb{R}^+\) where \(r \in \mathbb{R}^+\).
\(iv\) \(G(s, t) = s\beta(s)\) for all \(s, t \in \mathbb{R}^+\) where \(\beta : \mathbb{R}^+ \to [0, 1)\) is continuous.

In 2015, Khojasteh, Shukla and Radenović \cite{24} introduced the notion of simulation function and proved the existence of fixed points of \(Z_H\)-contractions in complete metric spaces.

**Definition 1.5.** A function \(\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}\) is said to be a simulation function if it satisfies the following conditions:
\((c_1)\) \(\zeta(0, 0) = 0;\)
\((c_2)\) \(\zeta(t, s) < s - t\) for all \(t, s > 0;\)
\((c_3)\) if \(\{t_n\}, \{s_n\}\) are sequences in \((0, \infty)\) such that \(\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0\), then
\[
\lim_{n \to \infty} \sup_{n \to \infty} \zeta(t_n, s_n) < 0.
\]
We denote the set of all simulation functions in the sense of Definition 1.5 by $Z_H$.

**Example 1.2.** [24, 22] Let $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function with $\phi_i(t) = 0$ if and only if $t = 0$ for $i = 1, 2, 3$. Then the following functions $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ belong to $Z_H$.

(i) $\zeta(t, s) = \frac{t}{s+1} - t$ for all $t, s \in \mathbb{R}^+$.
(ii) $\zeta(t, s) = \lambda s - t$ for all $t, s \in \mathbb{R}^+$ and $0 < \lambda < 1$.
(iii) $\zeta(t, s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in \mathbb{R}^+$, where $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$.

**Definition 1.6.** [24] Let $(X, d)$ be a metric space, $T : X \to X$ be a mapping and $\zeta \in Z_H$. Then $T$ is called a $Z_H$–contraction with respect to $\zeta$ if

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad (1.3)$$

for all $x, y \in X$.

**Theorem 1.2.** [24] Let $(X, d)$ be a complete metric space and $T : X \to X$ be a $Z_H$–contraction with respect to $\zeta$. Then $T$ has a unique fixed point $u$ in $X$ and for every $x_0 \in X$ the Picard sequence $\{x_n\}$ where $x_n = Tx_{n-1}$ for any $n \in \mathbb{N}$ converges to the fixed point of $T$.

In 2015, Nastasi and Vetro [4] proved the existence of fixed points in complete metric spaces by using simulation functions and a lower semicontinuous function.

**Theorem 1.3.** [4] Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist a simulation function $\zeta$ and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that

$$\zeta(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)) \geq 0 \quad (1.4)$$

for any $x, y \in X$. Then $T$ has a unique fixed point $u$ such that $\varphi(u) = 0$.

In 2018, Cho [14] introduced the notion of generalized weakly contractive mappings in metric spaces and proved the existence of its fixed points in complete metric spaces.

**Definition 1.7.** [14] Let $(X, d)$ be a metric space, $T$ a self-mapping of $X$. Then $T$ is called a generalized weakly contractive mapping if

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi)) \quad (1.5)$$

for all $x, y \in X$, where

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $\psi(t) = 0 \iff t = 0$,
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function and $\phi(t) = 0 \iff t = 0$,
(iii) $m(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(Tx, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)]\}$,
(iv) $l(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(Tx, Ty) + \varphi(y) + \varphi(Ty)\}$ and
(v) $\varphi : X \to \mathbb{R}^+$ is a lower semicontinuous function.

**Theorem 1.4.** [14] Let $(X, d)$ be a complete metric space. If $T$ is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$. 


**Definition 1.8.** [25] A mapping \( G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) has the property \( C_G \) if there exists an \( C_G \geq 0 \) such that

(i) for any \( s, t \in \mathbb{R}^+ \), \( G(s, t) > C_G \) implies \( s > t \), and

(ii) \( G(t, t) \leq C_G \) for all \( t \in \mathbb{R}^+ \).

**Example 1.3.** [25] The following functions are elements of \( \Delta \) that have property \( C_G \) for all \( t, s \in \mathbb{R}^+ \):

(i) \( G(s, t) = s - t, C_G = r, r \in \mathbb{R}^+ \),

(ii) \( G(s, t) = s - \frac{(2 + t)k}{1 + k}, C_G = 0 \),

(iii) \( G(s, t) = \frac{s}{1 + k}, k \geq 1, C_G = \frac{r}{1 + k}, r \geq 2 \).

**Definition 1.9.** [25] A function \( \zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) is said to be a \( C_G \)-simulation function if it satisfies the following conditions:

\[
\begin{align*}
(\zeta_1) & \quad \zeta(0, 0) = 0; \\
(\zeta_\alpha) & \quad \zeta(t, s) < G(s, t) \text{ for all } t, s > 0 \text{ where } G \in \Delta \text{ has property } C_G; \\
(\zeta_\beta) & \quad \text{if } \{t_n\}, \{s_n\} \text{ are sequences in } (0, \infty) \text{ such that } \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0 \text{ and } \\
& \quad t_n < s_n \text{ then } \limsup_{n \to \infty} \zeta(t_n, s_n) < C_G.
\end{align*}
\]

We denote the set of all \( C_G \)-simulation functions by \( Z_G \).

**Example 1.4.** [24] The following functions \( \zeta \) belong to \( Z_G \):

(i) Let \( k \in \mathbb{R} \) be such that \( k < 1 \) and \( \zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) be the function defined by \( \zeta(t, s) = kG(s, t) - t \), here \( C_G = 0 \).

(ii) Let \( k \in \mathbb{R} \) be such that \( k < 1 \) and let \( \zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) be the function defined by \( \zeta(t, s) = kG(s, t) \), here \( C_G = 1 \).

(iii) We define \( \zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) by \( \zeta(t, s) = \lambda s - t \), where \( \lambda \in (0, 1) \) and \( G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) by \( G(s, t) = s - t \) for any \( s, t \in \mathbb{R}^+ \).

Clearly \( \zeta(0, 0) = 0 \) and \( G \in \Delta \) with \( C_G = 0 \).

Clearly \( \zeta(t, s) = \lambda s - t < s - t = G(s, t) \) and hence \( \zeta \) satisfies \( (\zeta_\alpha) \).

If \( \{t_n\}, \{s_n\} \) are sequences in \( (0, \infty) \) such that \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = k > 0 \) and \( t_n < s_n \) for all \( n \in \mathbb{N} \), then

\[ \limsup_{n \to \infty} \zeta(t_n, s_n) = \limsup_{n \to \infty} (\lambda s_n - t_n) = \lambda k - k = (\lambda - 1)k < 0. \]

Therefore \( \zeta \) satisfies \( (\zeta_\beta) \) and hence \( \zeta \in Z_G \).

In 1977, Bernfeld, Lakshmikantham and Reddy [12] introduced the concept of fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point, for more details we refer to [6, 11, 17, 19, 21, 26].

Let \((E, \| \cdot \|_E)\) be a Banach space and we denote it simply by \( E \). Let \( I = [a, b] \subseteq \mathbb{R} \) and \( E_0 = C(I, E) \), the set of all continuous functions on \( I \) equipped with the supremum norm \( \| \cdot \|_{E_0} \) and we define it by \( \| \phi \|_{E_0} = \sup_{a \leq t \leq b} \| \phi(t) \|_E \) for \( \phi \in E_0 \).

For a fixed \( c \in I \), the Razumikhin class \( R_c \) of functions in \( E_0 \) is defined by

\[ R_c = \{ \phi \in E_0 / \| \phi \|_{E_0} = \| \phi(c) \|_E \}. \]

Clearly every constant function from \( I \) to \( E \) belongs to \( R_c \) so that \( R_c \) is a non-empty subset of \( E_0 \).

**Definition 1.10.** [12] Let \( R_c \) be the Razumikhin class of continuous functions in \( E_0 \). We say that
Definition 1.11. Let \( R_c \) be the Razumikhin class of functions in \( E_0 \). Then
\[
R_c = \bigcup_{c \in [a, b]} R_c.
\]
(i) the class \( R_c \) is algebraically closed with respect to the difference if \( \phi - \psi \in R_c \) whenever \( \phi, \psi \in R_c \).
(ii) the class \( R_c \) is topologically closed if it is closed with respect to the topology on \( E_0 \) by the norm \( \| \cdot \|_{E_0} \).

The Razumikhin class of functions \( R_c \) has the following properties.

Theorem 1.5. [5] Let \( R_c \) be the Razumikhin class of functions in \( E_0 \). Then
\[
i) E_0 = \bigcup_{c \in [a, b]} R_c.
\]

Definition 1.12. \[12\] Let \( T : E_0 \rightarrow E \) be a mapping. A function \( \phi \in E_0 \) is said to be a PPF dependent fixed point of \( T \) if \( T\phi = \phi(c) \) for some \( c \in I \).

Definition 1.13. \[12\] Let \( T : E_0 \rightarrow E \) be a mapping. Then \( T \) is called a Banach type contraction if there exists \( k \in [0, 1) \) such that \( \| T\phi - T\psi \|_E \leq k \| \phi - \psi \|_{E_0} \) for all \( \phi, \psi \in E_0 \).

Theorem 1.6. \[12\] Let \( T : E_0 \rightarrow E \) be a Banach type contraction. Let \( R_c \) be algebraically closed with respect to the difference and topologically closed. Then \( T \) has a unique PPF dependent fixed point in \( R_c \).

Definition 1.14. \[25\] Let \( c \in I \). Let \( T : E_0 \rightarrow E \) and \( \alpha : E \times E \rightarrow \mathbb{R}^+ \) be two functions. Then \( T \) is said to be an \( \alpha_c \)-admissible mapping if
\[
\alpha(\phi(c), \psi(c)) \geq 1 \implies \alpha(T\phi, T\psi) \geq 1 \tag{1.6}
\]
for all \( \phi, \psi \in E_0 \).

In 2013, Hussain, Khaleghizadeh, Salimi and Akbar \[21\] introduced the concept of \( \alpha_c \)-admissible mapping with respect to \( \mu_c \) and proved theorems for the existence of PPF dependent fixed points and PPF dependent coincidence points for contractive mappings in Banach spaces.

Definition 1.15. \[21\] Let \( c \in I \) and \( T : E_0 \rightarrow E \). Let \( \alpha, \mu : E \times E \rightarrow \mathbb{R}^+ \) be two functions. Then \( T \) is said to be an \( \alpha_c \)-admissible mapping with respect to \( \mu_c \) if
\[
\alpha(\phi(c), \psi(c)) \geq \mu(\phi(c), \psi(c)) \implies \alpha(T\phi, T\psi) \geq \mu(T\phi, T\psi) \tag{1.7}
\]
for all \( \phi, \psi \in E_0 \).

Note that, if we take \( \mu(x, y) = 1 \) for all \( x, y \in E \) then \( \alpha_c \)-admissible mapping with respect to \( \mu_c \) is an \( \alpha_c \)-admissible mapping. If we take \( \alpha(x, y) = 1 \) for all \( x, y \in E \) in (1.7) then we say that \( T \) is a \( \mu_c \)-subadmissible mapping.

In 2014, Ćirić, Alsulami, Salimi and Vetro \[13\] introduced the concept of triangular \( \alpha_c \)-admissible mapping with respect to \( \mu_c \) as follows.

Definition 1.16. \[13\] Let \( c \in I \) and \( T : E_0 \rightarrow E \). Let \( \alpha, \mu : E \times E \rightarrow \mathbb{R}^+ \) be two functions. Then \( T \) is said to be a triangular \( \alpha_c \)-admissible mapping with respect
Lemma 1.7. [13] Let $T$ be a triangular $\alpha_c$–admissible mapping with respect to $\mu_c$. We define the sequence $\{\phi_n\}$ by $T\phi_n = \phi_{n+1}(c)$ for all $n \in \mathbb{N} \cup \{0\}$, where $\phi_0 \in \mathcal{E}_c$ is such that $\alpha(\phi_0(c), T\phi_0) \geq \mu(\phi_0(c), T\phi_0)$. Then $\alpha(\phi_m(c), \phi_n(c)) \geq \mu(\phi_m(c), \phi_n(c))$ for all $m, n \in \mathbb{N}$ with $m < n$.

Remark. If $\mu(x, y) = 1$ for any $x, y \in E$ in Lemma 1.7, we get the following lemma.

Lemma 1.8. Let $T$ be a triangular $\alpha_c$–admissible mapping. We define the sequence $\{\phi_n\}$ by $T\phi_n = \phi_{n+1}(c)$ for all $n \in \mathbb{N} \cup \{0\}$, where $\phi_0 \in \mathcal{E}_c$ is such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Then $\alpha(\phi_m(c), \phi_n(c)) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.

Remark. If $\alpha(x, y) = 1$ for any $x, y \in E$ in Lemma 1.7, we get the following lemma.

Lemma 1.9. Let $T$ be a triangular $\mu_c$–subadmissible mapping. We define the sequence $\{\phi_n\}$ by $T\phi_n = \phi_{n+1}(c)$ for all $n \in \mathbb{N} \cup \{0\}$, where $\phi_0 \in \mathcal{E}_c$ is such that $\mu(\phi_0(c), T\phi_0) \leq 1$. Then $\mu(\phi_m(c), \phi_n(c)) \leq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.

Lemma 1.10. [7] Let $\{\phi_n\}$ be a sequence in $E_0$ such that $||\phi_n - \phi_{n+1}||_{E_0} \to 0$ as $n \to \infty$. If $\{\phi_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{m_k} - \phi_{m_{k+1}}||_{E_0} \geq \epsilon$, $||\phi_{n_k} - \phi_{m_{k+1}}||_{E_0} \leq \epsilon$ and

- i) $\lim_{k \to \infty} ||\phi_{m_k} - \phi_{m_{k+1}}||_{E_0} = \epsilon$,
- ii) $\lim_{k \to \infty} ||\phi_{n_k} - \phi_{m_{k+1}}||_{E_0} = \epsilon$,
- iii) $\lim_{k \to \infty} ||\phi_{n_k} - \phi_{m_k}||_{E_0} = \epsilon$, and
- iv) $\lim_{k \to \infty} ||\phi_{m_k} - \phi_{m_{k+1}}||_{E_0} = \epsilon$.

In Section 2, we introduce the notion of generalized weakly $Z_{G, \alpha, \mu, \xi, \eta, \varphi}$–contraction map with respect to a $C_G$–simulation function $\xi \in Z_G$ and prove the existence of PPF dependent fixed points of these maps in Banach spaces (Theorem 2.1), which is the main result of this paper. For such maps, PPF dependent fixed points may not be unique. In Section 3, we draw some corollaries and an example is provided to illustrate our main result.

2. EXISTENCE OF PPF DEPENDENT FIXED POINTS

We denote

$\Psi = \{\xi \mid \xi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous, nondecreasing and } \xi(t) = 0 \iff t = 0\}$

and

$\Phi = \{\eta \mid \eta : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous and } \eta(t) = 0 \iff t = 0\}$.

Based on the results of [4] [14] [10] we introduce a notion of generalized weakly $Z_{G, \alpha, \mu, \xi, \eta, \varphi}$–contraction map with respect to $\xi \in Z_G$ as follows.

Definition 2.1. Let $c \in I$. Let $T : E_0 \to E$ be a function and $\xi \in Z_G$. If there exist $\xi \in \Psi, \eta \in \Phi, \alpha : E \times E \to \mathbb{R}^+, \mu : E \times E \to (0, \infty)$, and a lower semicontinuous
function \( \varphi : E \rightarrow \mathbb{R}^+ \) such that
\[
\zeta(\alpha(\phi(c), \psi(c))\xi(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)), \mu(\phi(c), \psi(c))\xi(M(\phi, \psi)) - \eta(N(\phi, \psi))) \geq C_G
\]
for all \( \phi, \psi \in E_0 \), where \( \xi(t) > \eta(t) \) for any \( t > 0 \),
\[
M(\phi, \psi) = \max\{||\phi - \psi||_E, \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi), \frac{1}{2}||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi) + ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}
\]
and
\[
N(\phi, \psi) = \max\{||\phi - \psi||_E, \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\}
\]
then we say that \( T \) is a generalized weakly \( Z_{G, \alpha, \mu, \xi, \eta, \varphi} \)-contraction map with respect to \( \zeta \).

**Remark.** (i) If \( \varphi(x) = 0 \) for any \( x \in E \) in the inequality (2.1) then \( T \) is called a generalized weakly \( Z_{G, \alpha, \mu, \xi, \eta, \varphi} \)-contraction map with respect to \( \zeta \).

(ii) If \( \varphi(x) = 0, \mu(x, y) = 1 = \alpha(x, y) \) for any \( x, y \in E \) in the inequality (2.1) then \( T \) is called a generalized weakly \( Z_{G, \xi, \eta} \)-contraction map with respect to \( \zeta \).

(iii) If \( \varphi(x) = 0, \mu(x, y) = 1 = \alpha(x, y) \) for any \( x, y \in E \) and \( \xi(t) = t \) for any \( t \in \mathbb{R}^+ \) in the inequality (2.1) then \( T \) is called a generalized weakly \( Z_{G, \xi, \eta} \)-contraction map with respect to \( \zeta \).

**Theorem 2.1.** Let \( c \in I \). Let \( T : E_0 \rightarrow E \) be a function satisfying the following conditions:

(i) \( T \) is a generalized weakly \( Z_{G, \alpha, \mu, \xi, \eta, \varphi} \)-contraction map with respect to \( \zeta \),

(ii) \( T \) is a triangular \( \alpha, \mu \)-admissible mapping and triangular \( \mu, \alpha \)-subadmissible mapping,

(iii) \( R_c \) is algebraically closed with respect to the difference,

(iv) if \( \{\phi_n\} \) is a sequence in \( E_0 \) such that \( \phi_n \rightarrow \phi \) as \( n \rightarrow \infty \), \( \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1 \) and \( \mu(\phi_n(c), \phi_{n+1}(c)) \leq 1 \) for any \( n \in \mathbb{N} \cup \{0\} \) then \( \alpha(\phi_n(c), \phi(c)) \geq 1 \) and \( \mu(\phi_n(c), \phi(c)) \leq 1 \) for any \( n \in \mathbb{N} \cup \{0\} \) and

(v) there exists \( \phi_0 \in R_c \) such that \( \alpha(\phi_0(c), T\phi_0) \geq 1 \) and \( \mu(\phi_0(c), T\phi_0) \leq 1 \).

Then \( T \) has a PPF dependent fixed point \( \phi^* \in R_c \) such that \( \varphi(\phi^*(c)) = 0 \).

**Proof.** From (v) we have \( \phi_0 \in R_c \) such that \( \alpha(\phi_0(c), T\phi_0) \geq 1 \) and \( \mu(\phi_0(c), T\phi_0) \leq 1 \). Let \( \{\phi_n\} \) be a sequence in \( R_c \) defined by
\[
T\phi_n = \phi_{n+1}(c)
\]
for any \( n = 0, 1, 2, 3, \ldots \).

Since \( R_c \) is algebraically closed with respect to the difference, we have
\[
||\phi_{n+1} - \phi_n||_E = ||\phi_{n+1}(c) - \phi_n(c)||_E
\]
for any \( n = 0, 1, 2, 3, \ldots \).

Since \( T \) is triangular \( \alpha, \mu \)-admissible and triangular \( \mu, \alpha \)-subadmissible mappings, by Lemma 1.8 and Lemma 1.9 we have
\[
\alpha(\phi_n(c), \phi_n(c)) \geq 1
\]
and
\[
\mu(\phi_n(c), \phi_n(c)) \leq 1
\]
for any \( m, n \in \mathbb{N} \) with \( m < n \).

If there exists \( n \in \mathbb{N} \cup \{0\} \) such that \( \phi_n = \phi_{n+1} \) then \( T\phi_n = \phi_{n+1}(c) = \phi_n(c) \) and hence \( \phi_n \in R_c \) is a PPF dependent fixed point of \( T \).
Suppose that $\phi_n \neq \phi_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$.

If either $M(\phi_n, \phi_{n+1}) = 0$ or $N(\phi_n, \phi_{n+1}) = 0$ then the result is trivial.

Suppose that $M(\phi_n, \phi_{n+1}) \neq 0$ and $N(\phi_n, \phi_{n+1}) \neq 0$.

We consider

\[ M(\phi_n, \phi_{n+1}) = \max\{||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \]
\[ ||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n), \]
\[ ||\phi_{n+1}(c) - T\phi_{n+1}||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_{n+1}), \]
\[ \frac{1}{2}||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n) + \]
\[ ||\phi_{n+1}(c) - T\phi_{n+1}||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_{n+1}) \} \]

\[ = \max\{||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \]
\[ ||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \]
\[ ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)), \]
\[ \frac{1}{2}||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+2}(c)) + \]
\[ \frac{1}{2}||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) \} \]

and

\[ N(\phi_n, \phi_{n+1}) = \max\{||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \]
\[ ||\phi_{n+1}(c) - T\phi_{n+1}||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_{n+1}) \} \]

\[ = \max\{||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \]
\[ ||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \]
\[ ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) \} \]

Suppose that

\[ \max\{||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), ||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) \} \]

\[ = ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) \].

Clearly $M(\phi_n, \phi_{n+1}) = N(\phi_n, \phi_{n+1}) = ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))$.

Since $\phi_{n+1} \neq \phi_{n+2}$, we have $||\phi_{n+1} - \phi_{n+2}||_{E_0} > 0$ and hence $||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) > 0$ and which implies that $\xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0$.

Therefore

\[ \alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})) \]
\[ = \alpha(\phi_n(c), \phi_{n+1}(c))\xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0. \]

Since $\xi(t) > \eta(t)$ for any $t > 0$ we have $\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1})) > 0$ and hence $\mu(\phi_n(c), \phi_{n+1}(c))\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1})) > 0$.

From (2.4), we have

\[ C_G \leq \xi(\alpha(\phi_n(c), \phi_{n+1}(c)))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})) \]
\[ = \xi(\alpha(\phi_n(c), \phi_{n+1}(c)))\xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0. \]

(by $\xi$)

Now by the property $C_G$, we get

\[ \mu(\phi_n(c), \phi_{n+1}(c))\xi(M(\phi_n, \phi_{n+1})) - \eta(N(\phi_n, \phi_{n+1})) \]
\[ > \alpha(\phi_n(c), \phi_{n+1}(c))\xi(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})). \]

Clearly

\[ \xi(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) = \xi(M(\phi_n, \phi_{n+1})) > \xi(\alpha(\phi_n(c), \phi_{n+1}(c))) - \eta(N(\phi_n, \phi_{n+1})) \]
\[ \geq \mu(\phi_n(c), \phi_{n+1}(c)) \xi(\|M(\phi_n, \phi_{n+1}) - \eta(N(\phi_n, \phi_{n+1}))\|) \]
\[ > \alpha(\phi_n(c), \phi_{n+1}(c)) \xi(||T\phi_n - T\phi_{n+1}\|_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})) \]
\[ \geq \xi(||\phi_{n+1} - \phi_{n+2}\|_E_0 + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) , \]

a contradiction.

Therefore
\[ ||\phi_n - \phi_{n+1}\|_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)) > ||\phi_{n+1} - \phi_{n+2}\|_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) \]

and hence
\[ M(\phi_n, \phi_{n+1}) = N(\phi_n, \phi_{n+1}) = ||\phi_n - \phi_{n+1}\|_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)). \]

Let \( d_n = ||\phi_n - \phi_{n+1}\|_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)). \)

Then the sequence \( \{d_n\} \) is a decreasing sequence and hence convergent.

Let \( \lim_{n \to \infty} d_n = k \) (say). Suppose that \( k > 0. \)

Since \( \phi_n \neq \phi_{n+1} \) we have \( d_n = ||\phi_n - \phi_{n+1}\|_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)) > 0 \)

and which implies that \( \xi(d_n) = \xi(||\phi_n - \phi_{n+1}\|_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c))) > 0. \)

Similarly \( \eta(d_n) > 0. \) Clearly \( M(\phi_n, \phi_{n+1}) = N(\phi_n, \phi_{n+1}) = d_n \) and hence
\[ \mu(\phi_n(c), \phi_{n+1}(c))(\xi(d_n) - \eta(d_n)) > 0. \]

Similarly \( d_{n+1} > 0 \) and which implies that \( \alpha(\phi_n(c), \phi_{n+1}(c))\xi(d_{n+1}) > 0. \)

From \( \text{[2.1]} \), we have
\[ C_G \leq \xi(\alpha(\phi_n(c), \phi_{n+1}(c))\xi(||\phi_n - \phi_{n+1}\|_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))), \]
\[ \mu(\phi_n(c), \phi_{n+1}(c))(\xi(d_n) - \eta(d_n))) \]
\[ = \xi(\alpha(\phi_n(c), \phi_{n+1}(c))\xi(d_{n+1}), \mu(\phi_n(c), \phi_{n+1}(c))(\xi(d_n) - \eta(d_n))) \]
\[ < G(\mu(\phi_n(c), \phi_{n+1}(c))(\xi(d_n) - \eta(d_n)), \alpha(\phi_n(c), \phi_{n+1}(c))(\xi(d_{n+1}))) \) (by \( \xi_5 \))

Now by the property \( C_G \), we get that
\[ \mu(\phi_n(c), \phi_{n+1}(c))(\xi(d_n) - \eta(d_n)) > \alpha(\phi_n(c), \phi_{n+1}(c))\xi(d_{n+1}). \]

Clearly
\[ \xi(d_n) > \xi(d_n) - \eta(d_n) \]
\[ \geq \mu(\phi_n(c), \phi_{n+1}(c))(\xi(d_n) - \eta(d_n)) \]
\[ > \alpha(\phi_n(c), \phi_{n+1}(c))\xi(d_{n+1}) \]
\[ \geq \xi(d_{n+1}). \]

On applying limits as \( n \to \infty \), we get that
\[ \lim_{n \to \infty} \mu(\phi_n(c), \phi_{n+1}(c))(\xi(d_n) - \eta(d_n)) = \lim_{n \to \infty} \alpha(\phi_n(c), \phi_{n+1}(c))\xi(d_{n+1}) = \xi(k) > 0. \]

On applying limit superior to \( \text{[2.3]} \), we get that
\[ C_G \leq \limsup_{n \to \infty} \xi(\alpha(\phi_n(c), \phi_{n+1}(c))\xi(d_{n+1}), \mu(\phi_n(c), \phi_{n+1}(c))(\xi(d_n) - \eta(d_n))) \]
\[ < C_G, \) (by \( \xi_6 \))

a contradiction.

Therefore \( k = 0 \) and hence \( \lim_{n \to \infty} ||\phi_n - \phi_{n+1}\|_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)) = 0. \)

That is
\[ \lim_{n \to \infty} ||\phi_n - \phi_{n+1}\|_{E_0} = 0 \) and \( \lim_{n \to \infty} \varphi(\phi_n(c)) = 0. \) \( \text{[2.6]} \)

We now show that the sequence \( \{\phi_n\} \) is a Cauchy sequence in \( R_c. \)

Suppose that the sequence \( \{\phi_n\} \) is not a Cauchy sequence.

Then there exists an \( \epsilon > 0 \) and two subsequences \( \{\phi_{m_k}\} \) and \( \{\phi_{n_k}\} \) of \( \{\phi_n\} \) with \( m_k > n_k > k \) such that
\[ ||\phi_{n_k} - \phi_{m_k}\|_{E_0} \geq \epsilon, \]
\[ ||\phi_{n_k} - \phi_{m_k-1}\|_{E_0} < \epsilon \] and by Lemma \( \text{[1.10]} \), we have,
\[ \lim_{k \to \infty} ||\phi_{n_k} - \phi_{m_k}\|_{E_0} = \epsilon \) and
\[ \lim_{k \to \infty} ||\phi_{n_k} - \phi_{m_k+1}\|_{E_0} = \epsilon = \lim_{k \to \infty} ||\phi_{n_k+1} - \phi_{m_k}\|_{E_0} \]
\[ = \lim_{k \to \infty} ||\phi_{n_k+1} - \phi_{m_k+1}\|_{E_0}. \] \( \text{[2.7]} \)
Let \( d_{nk} = \|\phi_{nk} - \phi_{mk}\|_{E_0} + \varphi(\phi_{nk}(c)) + \varphi(\phi_{mk}(c)). \)

Then from (2.6) and (2.7) it follows that

\[
\lim_{k \to \infty} d_{nk} = \epsilon = \lim_{k \to \infty} d_{nk+1m_k+1}.
\]

Since \( \xi \) is continuous, we get that

\[
\lim_{k \to \infty} \xi(d_{nk+1m_k+1}) = \xi(\epsilon) > 0. \tag{2.8}
\]

We consider

\[
M(\phi_{nk}, \phi_{mk}) = \max\{\|\phi_{nk} - \phi_{mk}\|_{E_0} + \varphi(\phi_{nk}(c)) + \varphi(\phi_{mk}(c)),
\|\phi_{nk}(c) - T\phi_{nk}\|_{E} + \varphi(\phi_{nk}(c)) + \varphi(T\phi_{nk}),
\|\phi_{mk}(c) - T\phi_{mk}\|_{E} + \varphi(\phi_{mk}(c)) + \varphi(T\phi_{mk}),
\frac{1}{2}[\|\phi_{nk}(c) - T\phi_{mk}\|_{E} + \varphi(\phi_{nk}(c)) + \varphi(T\phi_{nk}) + \varphi(T\phi_{mk})]\}
= \max\{\|\phi_{nk} - \phi_{mk}\|_{E_0} + \varphi(\phi_{nk}(c)) + \varphi(\phi_{mk}(c)),
\|\phi_{nk} - \phi_{nk+1}\|_{E_0} + \varphi(\phi_{nk}(c)) + \varphi(\phi_{nk+1}(c)),
\|\phi_{mk} - \phi_{mk+1}\|_{E_0} + \varphi(\phi_{mk}(c)) + \varphi(\phi_{mk+1}(c)),
\frac{1}{2}[\|\phi_{nk} - \phi_{mk+1}\|_{E_0} + \varphi(\phi_{nk}(c)) + \varphi(\phi_{mk+1}(c)) + \varphi(T\phi_{mk})]\}
= \max\{d_{nk}, d_{nk+1}, d_{mk}, d_{mk+1}, \frac{1}{2}[d_{nk} + d_{mk}]\}.
\]

On applying limits as \( k \to \infty \), we get that \( \lim_{k \to \infty} M(\phi_{nk}, \phi_{mk}) = \epsilon \).

We consider

\[
N(\phi_{nk}, \phi_{mk}) = \max\{\|\phi_{nk} - \phi_{mk}\|_{E_0} + \varphi(\phi_{nk}(c)) + \varphi(\phi_{mk}(c)),
\|\phi_{mk}(c) - T\phi_{mk}\|_{E} + \varphi(\phi_{mk}(c)) + \varphi(T\phi_{mk})\}
= \max\{\|\phi_{nk} - \phi_{mk}\|_{E_0} + \varphi(\phi_{nk}(c)) + \varphi(\phi_{mk}(c)),
\|\phi_{mk} - \phi_{mk+1}\|_{E_0} + \varphi(\phi_{mk}(c)) + \varphi(\phi_{mk+1}(c))\}
= \max\{d_{nk}, d_{mk}\}.
\]

On applying limits as \( k \to \infty \), we get that \( \lim_{k \to \infty} N(\phi_{nk}, \phi_{mk}) = \epsilon \).

Since \( \xi, \eta \) are continuous, we have

\[
\lim_{k \to \infty} \xi(M(\phi_{nk}, \phi_{mk})) = \xi(\epsilon) > 0 \quad \text{and} \quad \lim_{k \to \infty} \eta(N(\phi_{nk}, \phi_{mk})) = \eta(\epsilon) > 0.
\]

Therefore

\[
\lim_{k \to \infty} (\xi(M(\phi_{nk}, \phi_{mk})) - \eta(N(\phi_{nk}, \phi_{mk}))) = \xi(\epsilon) - \eta(\epsilon) > 0. \tag{2.9}
\]

(since \( \xi(t) > \eta(t) \))

for \( t > 0 \).

From (2.8) and (2.9), there exists \( k_1 \in \mathbb{N} \) such that

\[
\xi(M(\phi_{nk}, \phi_{mk})) - \eta(N(\phi_{nk}, \phi_{mk})) > \frac{\xi(\epsilon) - \eta(\epsilon)}{2} > 0 \tag{2.10}
\]

and

\[
\xi(d_{nk+1m_k+1}) > \frac{\xi(\epsilon)}{2} > 0
\]

for any \( k \geq k_1 \).

From (2.4), we have

\[
\alpha(\phi_{nk}(c), \phi_{mk}(c))\xi(d_{nk+1m_k+1}) \geq \xi(d_{nk+1m_k+1}) \geq 0 \quad \text{and} \quad \mu(\phi_{nk}(c), \phi_{mk}(c))\left(\xi(M(\phi_{nk}, \phi_{mk})) - \eta(N(\phi_{nk}, \phi_{mk}))\right) > 0. \tag{2.11}
\]

for any \( k \geq k_1 \).

For any \( k \geq k_1 \), from (2.11) we have

\[
C_G \leq \xi(\alpha(\phi_{nk}(c), \phi_{mk}(c))\xi(|T\phi_{nk} - T\phi_{mk}|_{E} + \varphi(T\phi_{nk}) + \varphi(T\phi_{mk})),
\mu(\phi_{nk}(c), \phi_{mk}(c))\left(\xi(M(\phi_{nk}, \phi_{mk})) - \eta(N(\phi_{nk}, \phi_{mk}))\right))).
\]
Now by the property \( C_G \), we have
\[
\mu(\phi_{n_k}(c), \phi_{m_k}(c)) (\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) > \alpha(\phi_{n_k}(c), \phi_{m_k}(c)) \xi(d_{n_k + 1m_k + 1}).
\]
(2.13)

Clearly
\[
\xi(M(\phi_{n_k}, \phi_{m_k})) > \xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k})) \\
\geq \mu(\phi_{n_k}(c), \phi_{m_k}(c)) (\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) \\
> \alpha(\phi_{n_k}(c), \phi_{m_k}(c)) \xi(d_{n_k + 1m_k + 1}) \text{ (by (2.13))} \\
\geq \xi(d_{n_k + 1m_k + 1}).
\]

On applying limits as \( k \to \infty \), we get that
\[
\lim_{k \to \infty} \mu(\phi_{n_k}(c), \phi_{m_k}(c)) (\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) = \lim_{k \to \infty} \alpha(\phi_{n_k}(c), \phi_{m_k}(c)) \xi(d_{n_k + 1m_k + 1}) = \xi > 0.
\]
(2.14)

On applying limit superior as \( k \to \infty \) to (2.12), by (2.13), (2.14) and \( (\zeta_6) \) we get
\[
C_G \leq \limsup_{k \to \infty} \xi(\alpha(\phi_{n_k}(c), \phi_{m_k}(c))) (\xi(M(\phi_{n_k}, \phi_{m_k})) - \eta(N(\phi_{n_k}, \phi_{m_k}))) \\
< C_G,
\]
a contradiction.

Therefore the sequence \( \{\phi_n\} \) is a Cauchy sequence in \( R_c \).

Since \( E_0 \) is complete, there exists \( \phi^* \in E_0 \) such that \( \phi_n \to \phi^* \).

Since \( R_c \) is topologically closed, we have \( \phi^* \in R_c \).

Clearly \( ||\phi^*||_{E_0} = ||\phi^*(c)||_E \). (since \( \phi^* \in R_c \))

Since \( \varphi \) is lower semicontinuous function, we have
\[
\varphi(\phi^*(c)) \leq \liminf_{n \to \infty} \varphi(\phi_n(c)) = 0 \text{ and hence } \varphi(\phi^*(c)) = 0.
\]

We now show that \( T\phi^* = \phi^*(c) \). Suppose that \( T\phi^* \neq \phi^*(c) \).

From (2.4) we have \( \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1 \) and \( \mu(\phi_n(c), \phi_{n+1}(c)) \leq 1 \)
for any \( n \in N \cup \{0\} \).

From (iv) we get that \( \alpha(\phi_n(c), \phi^*(c)) \geq 1 \) and \( \mu(\phi_n(c), \phi^*(c)) \leq 1 \)
for any \( n \in N \cup \{0\} \).

We consider
\[
M(\phi_n, \phi^*) = \max\{||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)), \\
||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n), \\
||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*), \\
\frac{1}{2}||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) + \varphi(T\phi_n), \\
\frac{1}{2}||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) + \varphi(T\phi_n)\}
\]

= \max\{||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)), \\
||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \\
||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*), \\
\frac{1}{2}||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) + \varphi(T\phi_n), \\
\frac{1}{2}||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) + \varphi(T\phi_n)\}$
On applying limits as $n \to \infty$ we get
$$\phi^* = \lim_{n \to \infty} \phi_n = \phi^*.$$

Next we have
$$\xi(\phi_n, \phi^*) = \xi(\phi, \phi^*) \leq \xi(\phi_n, \phi^*).$$

Therefore we get
$$\xi(\phi_n, \phi^*) > 0 \quad \text{and} \quad \eta(\phi_n, \phi^*) > 0.$$

Clearly $M(\phi_n, \phi^*) \geq N(\phi_n, \phi^*)$. Since $\xi(t) > \eta(t)$ for $t > 0$ we have $\xi(M(\phi_n, \phi^*)) \geq \xi(N(\phi_n, \phi^*)) > \eta(N(\phi_n, \phi^*))$ and hence $\xi(M(\phi_n, \phi^*)) < \eta(N(\phi_n, \phi^*)) > 0.$

Clearly
$$\mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) > 0. \quad \text{(2.15)}$$

On applying limits as $n \to \infty$, we get $\phi^*(c) = T\phi^*$, a contradiction.

Therefore $M(\phi_n, \phi^*) > 0$ and $N(\phi_n, \phi^*) > 0.$

Clearly
$$\mu(\phi_n(c), \phi^*(c))(\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*))) > 0. \quad \text{(2.15)}$$

From (2.1) we have
$$\alpha(\phi_n(c), \phi^*(c)) \leq \xi(\phi_n, \phi^*).$$

On applying limits as $n \to \infty$ to $M(\phi_n, \phi^*)$ and $N(\phi_n, \phi^*)$, we get that
$$\lim_{n \to \infty} M(\phi_n, \phi^*) = \lim_{n \to \infty} N(\phi_n, \phi^*).$$

Since $\xi$ is continuous, we get that
$$\lim_{n \to \infty} \xi(M(\phi_n, \phi^*)) = \xi(\phi^*(c) - T\phi^*||E + \phi(T\phi^*)) > 0. \quad \text{(2.16)}$$

From (2.1) we have
$$C_G \leq \xi(\alpha(\phi_n(c), \phi^*(c))) \xi(|T\phi_n - T\phi^*||E + \phi(T\phi^*) + \phi(\phi^*) - \eta(N(\phi_n, \phi^*) - \eta(N(\phi_n, \phi^*)))).$$

Now by the property $C_G$, we get that
$$\mu(\phi_n(c), \phi^*(c)) \xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)) > 0. \quad \text{(2.17)}$$

On applying limits as $n \to \infty$, we get
$$\lim_{n \to \infty} M(\phi_n, \phi^*) = \lim_{n \to \infty} N(\phi_n, \phi^*).$$

Since $\xi$ is continuous, we get that
$$\lim_{n \to \infty} \xi(M(\phi_n, \phi^*)) = \xi(\phi^*(c) - T\phi^*||E + \phi(T\phi^*)) > 0. \quad \text{(2.16)}$$

Clearly
$$\xi(M(\phi_n, \phi^*)) > \xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)) \geq \mu(\phi_n(c), \phi^*(c))\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)) > 0.$$

On applying limits as $n \to \infty$, we get
$$\lim_{n \to \infty} \mu(\phi_n(c), \phi^*(c))\xi(|T\phi_n - T\phi^*||E + \phi(T\phi_n) + \phi(T\phi^*)) = \lim_{n \to \infty} \mu(\phi_n(c), \phi^*(c))\xi(M(\phi_n, \phi^*)) - \eta(N(\phi_n, \phi^*)) > 0.$$

From (2.1) we have
$$C_G \leq \xi(\alpha(\phi_n(c), \phi^*(c))) \xi(|T\phi_n - T\phi^*||E + \phi(T\phi_n) + \phi(T\phi^*) + \phi(\phi^*) - \eta(N(\phi_n, \phi^*) - \eta(N(\phi_n, \phi^*)))).$$

On applying limit superior as $n \to \infty$, by (2.6) we get that
Let \( \phi_n \) be a sequence in \( E_0 \) such that
\[
\limsup_{n \to \infty} \phi_n = 0.
\]
Then \( T \) has a PPF dependent fixed point in \( E_0 \).

**Proof.** By taking \( \varphi(x) = 0 \) for any \( x \in E \) in Theorem 2.1 we obtain the desired result.

By choosing \( \alpha(x, y) = 1 = \mu(x, y) \) for any \( x, y \in E \) in Corollary 3.1 we get the following corollary.

**Corollary 3.2.** Let \( c \in I \). Let \( T : E_0 \to E \) be a function satisfying the following conditions:
(i) \( T \) is a generalized weakly \( Z_{G, \alpha, \mu, \xi, \eta} \)-contraction map with respect to \( \zeta \),
(ii) \( T \) is triangular \( \alpha_c \)-admissible mapping and triangular \( \mu_c \)-subadmissible mapping,
(iii) \( R_c \) is algebraically closed with respect to the difference,
(iv) \( \{ \phi_n \} \) is a sequence in \( E_0 \) such that \( \phi_n \to \phi \) as \( n \to \infty \), \( \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1 \) and \( \mu(\phi_n(c), \phi_{n+1}(c)) \leq 1 \) for any \( n \in \mathbb{N} \cup \{ 0 \} \) then \( \alpha(\phi_n(c), \phi(c)) \geq 1 \) and \( \mu(\phi_n(c), \phi(c)) \leq 1 \) for any \( n \in \mathbb{N} \cup \{ 0 \} \) and
(v) there exists \( \phi_0 \in R_c \) such that \( \alpha(\phi_0(c), T\phi_0) \geq 1 \) and \( \mu(\phi_0(c), T\phi_0) \leq 1 \).
Then \( T \) has a PPF dependent fixed point in \( R_c \).

By choosing \( \xi(t) = t \) for any \( t \in \mathbb{R}^+ \) in Corollary 3.2 we get the following corollary.

**Corollary 3.3.** Let \( c \in I \). Let \( T : E_0 \to E \) be a function satisfying the following conditions:
(i) \( T \) is a generalized weakly \( Z_{G, \eta} \)-contraction map with respect to \( \zeta \) and
(ii) \( R_c \) is algebraically closed with respect to the difference.
Then \( T \) has a PPF dependent fixed point in \( R_c \).

By choosing \( \alpha(x, y) = 1 = \mu(x, y) \) for any \( x, y \in E \), \( \xi(t) = t \) for any \( t \in \mathbb{R}^+ \) and \( C_G = 0 \) in Theorem 2.1 we get the following corollary.

**Corollary 3.4.** Let \( c \in I \) and \( \zeta \in Z_G \). Let \( T : E_0 \to E \) be a function satisfying the following conditions:
(i) there exist \( \eta \in \Phi \) and a lower semicontinuous function \( \varphi : E \to \mathbb{R}^+ \) such that
\[
\zeta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi), M(\phi, \psi) - \eta(N(\phi, \psi))) \geq 0
\]
for any \( \phi, \psi \in E_0 \), where \( \eta(t) < t \) for any \( t > 0 \),
\[
M(\phi, \psi) = \max(||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),
\]
\[
for any \( \phi, \psi \in E_0 \)
\]
\[
\eta(t) < t \]
\[
C_G \leq \limsup_{n \to \infty} \zeta(\alpha(\phi_n(c), \phi^*(c)) \xi(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*))
\[
\mu(\phi_n(c), \phi^*(c)) \xi(M(\phi_n, \phi^*) - \eta(N(\phi_n, \phi^*)))
\]
\[
\]
Corollary 3.6. Let \( \psi \) be a function satisfying the following conditions:

\[
\frac{1}{2} \left[ ||\phi(c) - T\phi||_{E} + ||\phi(c) + \varphi(T\psi) + \varphi(T\phi)||_{E} \right],
\]

\[
N(\phi, \psi) = \max \left\{ ||\phi - \psi||_{E} + ||\varphi(\phi(c)) + \varphi(\psi(c))||_{E} \right\},
\]

and

(ii) \( R_{c} \) is algebraically closed with respect to the difference.

Then \( T \) has a PPF dependent fixed point \( \phi^{*} \in R_{c} \) such that \( \varphi(\phi^{*}(c)) = 0 \).

By choosing \( \varphi(x) = 0 \) for any \( x \in E \) in Corollary 3.4 we get the following corollary.

Corollary 3.5. Let \( c \in I \) and \( \zeta \in Z_{G} \). Let \( T : E_{0} \to E \) be a function satisfying the following conditions:

(i) if there exists \( \eta \in \Phi \) such that

\[
\zeta(||T\phi - T\psi||_{E}, M(\phi, \psi) - \eta(N(\phi, \psi))) \geq 0
\]

for any \( \phi, \psi \in E_{0} \), where \( \eta(t) < t \) for any \( t > 0 \),

\[
M(\phi, \psi) = \max \left\{ \frac{1}{2} \left[ ||\phi(c) - T\phi||_{E} + ||\phi(c) + \varphi(T\psi) + \varphi(T\phi)||_{E} \right] \right\},
\]

\[
N(\phi, \psi) = \max \left\{ ||\phi - \psi||_{E_{0}} + ||\varphi(\phi(c)) + \varphi(\psi(c))||_{E} \right\}
\]

and

(ii) \( R_{c} \) is algebraically closed with respect to the difference.

Then \( T \) has a PPF dependent fixed point in \( R_{c} \).

By choosing \( \zeta(t, s) = \lambda s - t, G(s, t) = s - t \) for any \( s, t \in \mathbb{R}^{+}, C_{G} = 0 \) and \( \lambda \in (0, 1) \) in Theorem 2.1 we get the following corollary.

Corollary 3.6. Let \( c \in I \). Let \( T : E_{0} \to E \) be a function satisfying the following conditions:

(i) if there exist \( \xi \in \Psi, \eta \in \Phi, \alpha : E \times E \to \mathbb{R}^{+}, \mu : E \times E \to (0, \infty), \lambda \in (0, 1) \) and a lower semicontinuous function \( \varphi : E \to \mathbb{R}^{+} \) such that

\[
\alpha(\phi(c), \psi(c))\xi(||T\phi - T\psi||_{E} + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda \mu(\phi(c), \psi(c))\xi(||M(\phi, \psi) - \eta(N(\phi, \psi))) \leq 0
\]

for any \( \phi, \psi \in E_{0} \), where \( \xi(t) > \eta(t) \) for any \( t > 0 \),

\[
M(\phi, \psi) = \max \left\{ ||\phi - \psi||_{E_{0}} + ||\varphi(\phi(c)) + \varphi(\psi(c))||_{E} \right\},
\]

\[
N(\phi, \psi) = \max \left\{ ||\phi - \psi||_{E_{0}} + ||\varphi(\phi(c)) + \varphi(\psi(c))||_{E} \right\},
\]

(ii) \( T \) is a triangular \( \alpha_{c} \)-admissible mapping and triangular \( \mu_{c} \)-subadmissible mapping,

(iii) \( R_{c} \) is algebraically closed with respect to the difference,

(iv) if \( \{\phi_{n}\} \) is a sequence in \( E_{0} \) such that \( \phi_{n} \to \phi \) as \( n \to \infty \), \( \alpha(\phi_{n}(c), \phi_{n+1}(c)) \geq 1 \) and \( \mu(\phi_{n}(c), \phi_{n+1}(c)) \leq 1 \) for any \( n \in \mathbb{N} \cup \{0\} \) then \( \alpha(\phi_{n}(c), \phi_{n+1}(c)) \geq 1 \) and \( \mu(\phi_{n}(c), \phi_{n+1}(c)) \leq 1 \) for any \( n \in \mathbb{N} \cup \{0\} \) and

(v) there exists \( \phi_{0} \in R_{c} \) such that \( \alpha(\phi_{0}(c), T\phi_{0}) \geq 1 \) and \( \mu(\phi_{0}(c), T\phi_{0}) \leq 1 \). Then \( T \) has a PPF dependent fixed point \( \phi^{*} \in R_{c} \) such that \( \varphi(\phi^{*}(c)) = 0 \).

By choosing \( \xi(t) = t, t \in \mathbb{R}^{+} \) in Corollary 3.6 we get the following corollary.

Corollary 3.7. Let \( c \in I \). Let \( T : E_{0} \to E \) be a function satisfying the following conditions:
if there exist $\eta \in \Phi$, $\alpha : E \times E \to \mathbb{R}^+$, $\mu : E \times E \to (0, \infty)$, $\lambda \in (0, 1)$ and a lower semicontinuous function $\varphi : E \to \mathbb{R}^+$ such that
\[
\alpha(\phi(c), \psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi)))
\] (3.2)
for any $\phi, \psi \in E_0$, where $\eta(t) < t$ for any $t > 0$,
\[
M(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),
\]
Corollary 3.8. Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:
(i) if there exist $\eta \in \Phi$, $\alpha : E \times E \to \mathbb{R}^+$, $\mu : E \times E \to (0, \infty)$ and $\lambda \in (0, 1)$ such that
\[
\alpha(\phi(c), \psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi)))
\] (3.3)
for any $\phi, \psi \in E_0$, where $\eta(t) < t$ for any $t > 0$,
\[
M(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) - T\phi||_E, ||\phi(c) - T\phi||_E, \}
\]
\[
N(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) - T\phi||_E, ||\psi(c) - T\phi||_E, \}
\]
(ii) $T$ is a triangular $\alpha_c$-admissible mapping and triangular $\mu_c$-subadmissible mapping,
(iii) $R_c$ is algebraically closed with respect to the difference, 
(iv) if $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$, $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ then $\alpha(\phi_n(c), \phi(c)) \geq 1$ and $\mu(\phi_n(c), \phi(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ 
(v) there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$.
Then $T$ has a PPF dependent fixed point $\phi^* \in R_c$ such that $\varphi(\phi^*(c)) = 0$.

By choosing $\varphi(x) = 0$ for any $x \in E$ in Corollary 3.7 we get the following corollary.

Corollary 3.9. Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:
(i) if there exist $\xi \in \Psi, \eta \in \Phi, \lambda \in (0, 1)$ and a lower semicontinuous function $\varphi : E \to \mathbb{R}^+$ such that
\[
\xi(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq \lambda(\eta(M(\phi, \psi)) + \eta(N(\phi, \psi)))
\] (3.4)
We define Example 3.1. Let
\begin{equation}
  T \phi = \begin{cases}
    -2 & \text{if } \phi(c) \leq 0 \\
    \frac{3\phi(c) - 4}{2} & \text{if } 0 \leq \phi(c) < \frac{1}{2} \\
    -\frac{1}{2} & \text{if } \phi(c) \geq \frac{1}{2},
  \end{cases}
\end{equation}

\begin{equation}
  \alpha(x, y) = \begin{cases}
    1 & \text{if } x \geq y \\
    0 & \text{if } x < y,
  \end{cases}
\end{equation}

and
\begin{equation}
  \mu(x, y) = \begin{cases}
    \frac{1}{2} & \text{if } x \geq y \\
    -\frac{1}{2} & \text{if } x < y.
  \end{cases}
\end{equation}

Then \( T \) has a PPF dependent fixed point in \( R_c \) such that \( \varphi(\phi^*(c)) = 0 \).

By choosing \( \varphi(x) = 0 \) for any \( x \in E \) in Corollary 3.9 we get the following corollary.

**Corollary 3.10.** Let \( c \in I \). Let \( T : E_0 \to E \) be a function satisfying the following conditions:

(i) if there exist \( \xi \in \Psi, \eta \in \Phi \) and \( \lambda \in (0, 1) \) such that

\begin{equation}
  \xi(||T \phi - T \psi||_E) \leq \lambda(\xi(M(\phi, \psi)) - \eta(N(\phi, \psi)))
\end{equation}

for any \( \phi, \psi \in E_0 \), where \( \xi(t) > \eta(t) \) for any \( t > 0 \),

\begin{equation}
  M(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\phi(c) - T \phi||_E, ||\psi(c) - T \psi||_E, \frac{1}{2}[||\phi(c) - T \phi||_E + ||\psi(c) - T \psi||_E]\},
\end{equation}

\begin{equation}
  N(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\phi(c) - T \phi||_E, ||\psi(c) - T \psi||_E, \frac{1}{2}[||\phi(c) - T \phi||_E + ||\psi(c) - T \psi||_E]\},
\end{equation}

(ii) \( R_c \) is algebraically closed with respect to the difference.

Then \( T \) has a PPF dependent fixed point in \( R_c \).

By choosing \( \xi(t) = t \) for any \( t \in \mathbb{R}^+ \) in Corollary 3.10 we get the following corollary.

**Corollary 3.11.** Let \( c \in I \). Let \( T : E_0 \to E \) be a function satisfying the following conditions:

(i) if there exist \( \eta \in \Phi \) and \( \lambda \in (0, 1) \) such that

\begin{equation}
  ||T \phi - T \psi||_E \leq \lambda(M(\phi, \psi) - \eta(N(\phi, \psi)))
\end{equation}

for any \( \phi, \psi \in E_0 \), where \( \eta(t) < t \) for any \( t > 0 \),

\begin{equation}
  M(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\phi(c) - T \phi||_E, ||\psi(c) - T \psi||_E, \frac{1}{2}[||\phi(c) - T \phi||_E + ||\psi(c) - T \psi||_E]\},
\end{equation}

\begin{equation}
  N(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\phi(c) - T \phi||_E, ||\psi(c) - T \psi||_E, \frac{1}{2}[||\phi(c) - T \phi||_E + ||\psi(c) - T \psi||_E]\}
\end{equation}

and

(ii) \( R_c \) is algebraically closed with respect to the difference.

Then \( T \) has a PPF dependent fixed point in \( R_c \).

We present the following example in support of Theorem 2.1 which suggests that under the hypotheses of Theorem 2.1 \( T \) may have more than one fixed point.

**Example 3.1.** Let \( E = \mathbb{R} \), \( c = 1 \in I = [\frac{1}{2}, 2] \subseteq \mathbb{R} \), \( E_0 = C(I, E) \).

We define \( T : E_0 \to E, \alpha : E \times E \to \mathbb{R}^+, \mu : E \times E \to (0, \infty) \) by

\begin{equation}
  T \phi = \begin{cases}
    -2 & \text{if } \phi(c) \leq 0 \\
    \frac{3\phi(c) - 4}{2} & \text{if } 0 \leq \phi(c) < \frac{1}{2} \\
    -\frac{1}{2} & \text{if } \phi(c) \geq \frac{1}{2},
  \end{cases}
\end{equation}

\begin{equation}
  \alpha(x, y) = \begin{cases}
    1 & \text{if } x \geq y \\
    0 & \text{if } x < y,
  \end{cases}
\end{equation}

and

\begin{equation}
  \mu(x, y) = \begin{cases}
    \frac{1}{2} & \text{if } x \geq y \\
    -\frac{1}{2} & \text{if } x < y.
  \end{cases}
\end{equation}
We first prove that $T$ is an $\alpha_{\varphi}$-admissible mapping.

For any $\phi, \psi \in E_0$, we suppose that $\alpha(\phi(c), \psi(c)) \geq 1$.

From the definition of $\alpha$, we get $\phi(c) \geq \psi(c)$.

Case (i): Suppose that $0 \leq \phi(c), \psi(c) < \frac{1}{2}$.

Clearly $3\phi(c) - 4 \geq 3\psi(c) - 4$ and which implies that $\frac{3\phi(c)-4}{2} \geq \frac{3\psi(c)-4}{2}$.

Therefore $T\phi \geq T\psi$ and hence $\alpha(T\phi, T\psi) \geq 1$.

Case (ii): Suppose that $\phi(c), \psi(c) \geq \frac{1}{2}$.

Clearly $T\phi = -\frac{1}{2} = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.

Case (iii): Suppose that $\phi(c), \psi(c) \leq 0$.

Clearly $T\phi = -2 = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.

Case (iv): Suppose that $0 \leq \phi(c) < \frac{1}{2}$ and $\psi(c) \leq 0$.

Since $\phi(c) \geq 0$ we have $T\phi = \frac{3\phi(c)-4}{2} \geq -2 = T\psi$
and which implies that $\alpha(T\phi, T\psi) \geq 1$.

Case (v): Suppose that $\phi(c) \geq \frac{1}{2}$ and $0 \leq \psi(c) < \frac{1}{2}$.

Since $\psi(c) \leq 1$ we have $T\phi = -\frac{1}{2} \geq \frac{3\phi(c)-4}{2} = T\psi$ and
which implies that $\alpha(T\phi, T\psi) \geq 1$.

From the above cases, we get that $T$ is an $\alpha_{\varphi}$-admissible mapping.

For any $\phi, \psi, \gamma \in E_0$, we suppose that $\alpha(\phi(c), \psi(c)) \geq 1$ and $\alpha(\psi(c), \gamma(c)) \geq 1$.

From the definition of $\alpha$, we get $\phi(c) \geq \psi(c) \geq \gamma(c)$.

Therefore $\phi(c) \geq \gamma(c)$ and hence $\alpha(\phi(c), \gamma(c)) \geq 1$.

Therefore $T$ is a triangular $\alpha_{\varphi}$-admissible mapping.

Similarly, we can prove that $T$ is a triangular $\mu_{\psi}$-subadmissible mapping.

Let $\lambda = \frac{1}{\sqrt{2}}$. Then $\lambda \in (0, 1)$.

We define $\varphi : E \to \mathbb{R}^+$ by

\[
\varphi(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
 x & \text{if } 0 \leq x < \frac{1}{2} \\
0 & \text{if } x \geq \frac{1}{2}.
\end{cases}
\]

Clearly $\varphi$ is a lower semicontinuous function.

We define $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ by $\eta(t) = \frac{t}{2}$ for any $t \in \mathbb{R}^+$. Clearly $\eta \in \Phi$.

Let $\phi, \psi \in E_0$.

If $\phi(c) < \psi(c)$ then from the definition of $\alpha$, the inequality $[3, 2]$ trivially holds.

Without loss of generality, we assume that $\phi(c) \geq \psi(c)$.

From the definition of $\alpha$, we get $T\phi \geq T\psi$.

We consider

\[ ||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi) \leq T\phi - T\psi + T\phi + T\psi = 2T\phi. \]

Therefore

\[ \alpha(\phi(c), \psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq 2T\phi. \]  (3.6)

Also we have

\[ M(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi), \frac{1}{2}||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\psi) + ||\psi(c) - T\phi||_E + \varphi(\psi(c)) + \varphi(T\phi)\} \]

\[ \geq \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi)\} \]

which implies that...
Therefore
\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max \{ \| \psi - \psi \|_{E_0} + \phi(\phi(c)) + \varphi(\psi(c)), \| \psi(c) - T\psi \|_E + \varphi(\psi(c)) + \varphi(T\psi) \}. \]

Case (i): Suppose that \( T\psi = \psi \).
\( \text{If } \psi \in R_c \text{ then } \psi \text{ is a PPF dependent fixed point of } T \text{ and hence the result holds.} \)
Let us suppose \( \psi \notin R_c \).
We define \( \psi_1 : I \rightarrow E \) by \( \psi_1(x) = \psi(c), \ x \in I \). Clearly \( \psi_1 \in R_c \).
From the definition of \( T \), we have
\[ T\psi_1 = \begin{cases} -2 & \text{if } \psi_1(c) \leq 0 \\ \frac{3\psi_1(c) - 4}{2} & \text{if } 0 \leq \psi_1(c) < \frac{1}{2} \\ -\frac{1}{2} & \text{if } \psi_1(c) \geq \frac{1}{2}. \end{cases} \]
That is
\[ T\psi_1 = \begin{cases} -2 & \text{if } \psi(c) \leq 0 \\ \frac{3\psi(c) - 4}{2} & \text{if } 0 \leq \psi(c) < \frac{1}{2} \\ -\frac{1}{2} & \text{if } \psi(c) \geq \frac{1}{2}. \end{cases} \]
Therefore \( T\psi_1 = T\psi = \psi(c) = \psi_1(c) \).
Hence \( \psi_1 \) is a PPF dependent fixed point of \( T \) in \( R_c \) and the result follows.
Case (ii): Suppose that \( \phi(c) < T\psi \).
From the definition of \( T \) we have \( \psi(c) < -2 \) and hence \( T\psi = -2 \).
Since \( \phi(c) \geq \psi(c) \) we have \( \phi(c) \leq 0 \) or \( 0 \leq \phi(c) < \frac{1}{2} \) or \( \phi(c) \geq \frac{1}{2} \).
Suppose that \( \phi(c) \leq 0 \). Clearly \( T\phi = -2 \).
From (3.7) we have
\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max \{ \phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)), \| \psi(c) - T\psi \|_E + \varphi(\psi(c)) + \varphi(T\psi) \}, \]
\[ = \frac{1}{2} \max \{ \phi(c) - \psi(c), T\psi - \psi(c) \} \]
\[ = \frac{1}{2} \max \{ 0, T\psi - \psi(c) \} \geq \frac{1}{2} \max \{ 0, T\psi - \phi(c) \}. \]
\( (\text{since } \phi(c) \geq \psi(c) \implies \psi(c) \geq -\phi(c)). \)
If \( \phi(c) < T\psi \) then \( T\psi - \phi(c) > 0 \) and hence
\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \left( -\frac{\phi(c)}{2} \right). \]
Clearly
\[ \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq -\frac{1}{2} - \frac{\phi(c)}{4} \geq 2 T\phi, \]
(\text{since } \lambda \mu(\phi(c), \psi(c)) \geq -4 \iff \phi(c) \leq 14) \]
If \( \phi(c) > T\psi \) then \( T\psi - \phi(c) < 0 \) and hence
\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq 0 > -4 = 2 \cdot (-2) = 2 T\phi. \]
Suppose that \( 0 \leq \phi(c) < \frac{1}{2} \). Clearly \( T\phi = \frac{3\phi(c) - 4}{2} \).
From (3.7) we have
\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max \{ \phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)), \| \psi(c) - T\psi \|_E + \varphi(\psi(c)) + \varphi(T\psi) \}. \]
Sub-case (i):

Suppose that 

\[ M \]

From (3.7) we have

\[ \text{Clearly} \]

\[ \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq -\frac{1}{2} - \frac{\phi(c)}{4} \geq 2 T \phi. \quad \text{(since } \frac{-1}{2} - \frac{\phi(c)}{4} \geq 3\phi(c) - 4 \iff \phi(c) \leq \frac{14}{11}) \]

Suppose that \( \phi(c) \geq \frac{1}{2} \). Clearly \( T \phi = -\frac{1}{2} \).

From (3.7) we have

\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max\{\phi(c) - \psi(c) + \varphi(\psi(c)) + \varphi(\psi(c)), \]

\[ ||\psi(c) - T \psi||_E + \varphi(\psi(c)) + \varphi(T \psi) \}

\[ = \frac{1}{2} \max\{\phi(c) - \psi(c), T \psi - \psi(c)\} \]

\[ = \frac{1}{2}(T \psi - \psi(c)) = -1 - \frac{\phi(c)}{2} \geq -1 - \frac{\phi(c)}{2}, \]

\[ \text{(since } \psi(c) < -2 \text{ and } T \psi - \psi(c) > 0) \]

\[ = \frac{1}{2} \max\{\phi(c) - \psi(c), T \psi - \psi(c)\} \]

\[ = \frac{1}{2}(T \psi - \psi(c)) = -1 - \frac{\phi(c)}{2} \geq -1 - \frac{\phi(c)}{2}, \]

\[ \text{(since } \psi(c) < -2 \text{ and } T \psi - \psi(c) > 0) \]

\[ > 0. \]

Clearly

\[ \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) > 0 > -1 = 2(-\frac{1}{2}) = 2 T \phi. \]

Case (iii): Suppose that \( \psi(c) > T \psi. \)

From the definition of \( T \) we have \( 0 \leq \psi(c) < \frac{1}{2} \) or \( -2 < \psi(c) \leq 0 \) or \( \psi(c) \geq \frac{1}{2} \).

Sub-case (i): Suppose that \( 0 \leq \psi(c) < \frac{1}{2} \). Clearly \( T \psi = \frac{3\psi(c)-4}{2} < 0. \)

Since \( \phi(c) \geq \psi(c) \) we have either \( 0 \leq \phi(c) < \frac{1}{2} \) or \( \phi(c) \geq \frac{1}{2} \).

Suppose that \( 0 \leq \phi(c) < \frac{1}{2} \). Clearly \( T \phi = \frac{3\phi(c)-4}{2} \)

From (3.7) we have

\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max\{\phi(c) - \psi(c) + \varphi(\psi(c)) + \varphi(\psi(c)), \]

\[ ||\psi(c) - T \psi||_E + \varphi(\psi(c)) + \varphi(T \psi) \}

\[ = \frac{1}{2} \max\{\phi(c) - \psi(c), T \psi - \psi(c)\} \]

\[ = \frac{1}{2}(T \psi - \psi(c)) \]

\[ = \psi(c) - \frac{T \phi}{2}. \quad \text{(since } T \psi < 0) \]

\[ \text{Clearly} \]

\[ \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c)}{2} - \frac{T \psi}{4} = \frac{\psi(c)}{2} - \frac{3\psi(c)-4}{8} = \frac{\psi(c)+4}{8} \geq 2 T \phi. \]

\[ \text{(since } \phi(c) \geq \psi(c) \text{ and } \frac{\psi(c)+4}{8} \geq 3\phi(c) - 4 \iff \psi(c) \leq \frac{36}{25}) \]

Suppose that \( \phi(c) \geq \frac{1}{2} \). Clearly \( T \phi = -\frac{1}{2} \).

From (3.7) we have

\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max\{\phi(c) - \psi(c) + \varphi(\psi(c)) + \varphi(\psi(c)), \]

\[ ||\psi(c) - T \psi||_E + \varphi(\psi(c)) + \varphi(T \psi) \}

\[ = \frac{1}{2} \max\{\phi(c) - \psi(c), T \psi - \psi(c)\} \]
Clearly \(M\) and \(T\)

Suppose that \(M\) and \(T\)

From (3.7) we have

\[
\lambda_\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c)}{2} - \frac{T\psi}{2} = \frac{\psi(c)}{2} - \frac{3\psi(c) - 4}{8} = \frac{\psi(c) + 4}{8} \geq -1 \iff \psi(c) \geq -12
\]

Sub-case (ii): Suppose that \(-2 < \psi(c) \leq 0\). Clearly \(T\phi = -2\).

Since \(\phi(c) \geq \psi(c)\) we have either \(-2 < \phi(c) \leq 0\) or \(0 \leq \phi(c) < \frac{1}{2}\) or \(\phi(c) \geq \frac{1}{2}\).

Suppose that \(-2 < \phi(c) \leq 0\). Clearly \(T\phi = -2\).

From (3.7) we have

\[M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max\{\phi(c) - \psi(c) + \psi(c) + \psi(c), 0, \psi(c) + 2\}.\]

\[\forall (\psi(c) - T\psi) \geq \frac{\psi(c) + 4}{2} \geq -4 \iff \psi(c) \geq -18\]

Suppose that \(0 \leq \phi(c) < \frac{1}{2}\). Clearly \(T\phi = \frac{3\phi(c) - 4}{2}\).

From (3.7) we have

\[M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max\{\phi(c) - \psi(c) + \psi(c) + \psi(c), 0, \psi(c) + 2\}.\]

\[\forall (\psi(c) - T\psi) \geq \frac{\psi(c) + 4}{2} \geq -4 \iff \psi(c) \geq \frac{\psi(c) + 4}{2}.\]

Clearly

\[\lambda_\mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c) + 4}{4} \geq 2 T\phi.\]

\[\forall (\psi(c) \geq \psi(c)) and \frac{\psi(c) + 4}{4} \geq 3\phi(c) - 4 \iff \psi(c) \leq \frac{18}{11}\]

Suppose that \(\phi(c) \geq \frac{1}{2}\). Clearly \(T\phi = -\frac{1}{2}\).

From (3.7) we have

\[M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max\{\phi(c) - \psi(c) + \psi(c) + \psi(c), 0, \psi(c) + 2\}.\]

\[\forall (\psi(c) - T\psi) \geq \frac{\psi(c) + 4}{2} \geq -4 \iff \psi(c) \geq \frac{\psi(c) + 4}{2}.\]

Clearly
\[ \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c) + 2}{4} \geq 2 \lambda \mu \]  

(since \( \frac{\psi(c) + 2}{4} \geq -1 \iff \psi(c) \geq -\frac{2}{3} \))

**Sub-case (iii): Suppose that \( \psi(c) \geq \frac{1}{2} \). Clearly \( T \psi = -\frac{1}{2} \).

Since \( \phi(c) \geq \psi(c) \) we have \( \phi(c) \geq \frac{1}{2} \). Clearly \( T \phi = -\frac{1}{2} \).

From (3.7) we have

\[ M(\phi, \psi) - \eta(N(\phi, \psi)) \geq \frac{1}{2} \max \{ \phi(c) - \psi(c) + \varphi(\phi(c)) + \varphi(\psi(c)), \]  

\[ ||\psi(c) - T \psi||_E + \varphi(\psi(c)) + \varphi(T \psi) \} \]

(since \( T \psi \leq 0 \) and \( \psi(c), \phi(c) \geq \frac{1}{2} \) we have \( \varphi(T \psi) = \varphi(\phi(c)) = \varphi(\psi(c)) = 0 \)  

\[ \geq \frac{1}{2} \max \{ 0, \psi(c) + \frac{1}{2} \} = \frac{\psi(c) + 1}{2} \]  

(since \( \psi(c) + \frac{1}{2} > 0 \)).

Clearly

\[ \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \frac{\psi(c)}{4} + \frac{1}{8} \geq 2 \lambda \mu \]

\[ \phi(c) \geq -\frac{2}{3} \]

From all the above cases, we get

\[ \lambda \mu(\phi(c), \psi(c))(M(\phi, \psi) - \eta(N(\phi, \psi))) \geq \alpha(\phi(c), \psi(c))(||T \phi - T \psi||_E + \varphi(T \phi) + \varphi(T \psi)). \]

Therefore the inequality (5.2) holds.

Let \( \{ \phi_n \} \) be a sequence in \( E_0 \) such that \( \alpha(\phi_n(c), \phi_n + 1(c)) \geq 1 \) and \( \mu(\phi_n(c), \phi_n + 1(c)) \leq 1 \) for any \( n \in \mathbb{N} \cup \{0\} \).

Then from the definition of \( \alpha \), we have \( \phi_n(c) \geq \phi_n + 1(c) \) for any \( n \in \mathbb{N} \cup \{0\} \) and hence convergent. Since \( \mathbb{R} \) is complete, there exists \( r \in \mathbb{R} \) such that \( \phi_n(c) \to r \) as \( n \to \infty \).

We define \( \gamma : I \to E \) by \( \gamma(x) = r, x \in I \). Then \( \gamma \in R_c \) and \( \gamma(c) = r \).

Therefore \( \phi_n(c) \to \gamma(c) \) as \( n \to \infty \). Clearly \( \phi_n(c) \geq \gamma(c) \) for any \( n \in \mathbb{N} \cup \{0\} \).

From the definition of \( \alpha \) and \( \mu \), we get \( \alpha(\phi_n(c), \gamma(c)) \geq 1 \) and \( \mu(\phi_n(c), \gamma(c)) \leq 1 \) for any \( n \in \mathbb{N} \cup \{0\} \). Therefore the condition (iv) is satisfied.

For any \( n \in \mathbb{N} \), we define \( \phi_n : I \to E \) by

\[ \phi_n(x) = \begin{cases} \frac{nx^2}{2} & \text{if } x \in [\frac{1}{2}, 1] \\ \frac{nx}{2} & \text{if } x \in [1, 2]. \end{cases} \]

Clearly \( \phi_n \in E_0, ||\phi_n||_E = ||\phi_n(c)||_E \) and hence \( \phi_n \in R_c \) for any \( n \in \mathbb{N} \).

Let \( F_0 = \{ \phi_n \mid n \in \mathbb{N} \} \). Then \( F_0 \subseteq R_c \) and \( F_0 \) is algebraically closed with respect to the difference.

Clearly \( \phi_2(c) \geq T \phi_2 \) and hence \( \alpha(\phi_2(c), T \phi_2) \geq 1 \) and \( \mu(\phi_2(c), T \phi_2) \leq 1 \).

Therefore the condition (v) is satisfied.

Therefore \( T \) satisfies all the hypotheses of Corollary 3.7 which in turn \( T \) satisfies all the hypotheses of Theorem 2.4 with \( \zeta(t, s) = \lambda s - t, G(s, t) = s - t, \xi(t) = \alpha \) for any \( s, t \in \mathbb{R}^+ \), \( C \) is 0 and \( \lambda = \sqrt{2} \in (0, 1) \) and hence \( \phi_{-2} \in R_c \) is a PPF dependent fixed point of \( T \) such that \( \varphi(\phi_{-2}(c)) = 0 \).

We define \( \gamma_1 : I \to E \) by

\[ \gamma_1(x) = \begin{cases} -2x & \text{if } x \in [\frac{1}{2}, 1] \\ 2x - 4 & \text{if } x \in [1, 2]. \end{cases} \]
Clearly \(|\gamma_1|\|E_0^\gamma = 2 = |\gamma_1(c)|\|E and hence \gamma_1 \in R_c.\)

We observe that \(T\gamma_1 = \gamma_1(c).\) (since \(\gamma_1(c) = -2 < 0,\) we have \(T\gamma_1 = -2 = \gamma_1(c)\))

Therefore \(\gamma_1 \in R_c\) is another PPF dependent fixed point of \(T\) such that \(\varphi(\gamma_1(c)) = 0.\)

References


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