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A Characterization of Homogeneous Fractional Hardy-Type Integrals on Variable Exponent Spaces

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Abstract: In this study, we establish boundedness of homogeneous fractional Hardy-type integral on variable exponent spaces.

Keywords: Boundedness, Fractional Hardy-Type integral, Variable exponent Lebesgue space.

1 Introduction

The theory of variable exponent Lebesgue spaces are started by Orlicz in 1931 and by Nakano in 1950 and 1951. However, the variable exponent function space, due to the failure of translation invariance and related properties, is very difficult to analysis. Nowadays there is an evident increase of investigations related to both the theory of the spaces $L^{q(.)}(R^n)$ themselves and the operator theory in these spaces (See[1-8]). This is caused by possible applications to models with non-standard local growth in elasticity theory, fluid mechanics, differential equations and is based on recent breakthrough result on boundedness of the Hardy-Littlewood maximal operator in these spaces.

Let S^{n-1} denote the unit sphere in Euclidean space R^n and $\Phi \in L^r(S^{n-1})$ $(r \geq 1)$ be homogeneous of degree zero on R^n. For $0 < \beta < n$, the homogeneous fractional integral is defined by

$$T^{\beta}_{\Phi} f(x) = \int_{R^n} \frac{\Phi(x-y)}{|x-y|^{n-\beta}} f(y) dy.$$

It is obvious that T^{β}_{Φ} just be the Riesz potential I^{β} when $\Phi \equiv 1$. Let E be a measurable set in R^n. We denote $p^-_E = inf_{x\in E} p(x)$ and $p^+_E = sup_{x\in E} p(x)$. Especially, we denote $p^- = p^-(R^n)$ and $p^+ = p^+(R^n)$. Let $p(.) : R^n \longrightarrow (0, \infty)$ be a measurable function with $0 < p^- \leq p^+ < \infty$ and $\Delta^0(R^n)$ be the set of all these $p(.)$. Let $\Delta(R^n)$ be the set of all measurable functions $p(.) : R^n \longrightarrow [1, \infty)$ such that $1 < p^- \leq p^+ < \infty$.

The variable Lebesgue space $L^{p(.)}(R^n)$ is defined as the set of all measurable function f for which the quantity $\int_{R^n} |\delta f(x)|^{p(x)} dx$ is finite for some $\delta > 0$ and

$$\|f\|_{L^{p(.)}(R^n)} = inf\{\lambda > 0 : \int_{R^n} (\frac{|f(x)|}{\lambda})^{p(x)} dx \leq 1\}.$$

As a special case of the theory of Nakano and Luxemburg, we see that $L^{p(.)}(R^n)$ is a quasi-normed space. Especially, when $p^- \geq 1$, $L^{p(.)}(R^n)$ is a Banach space. We say that(Log-Holder condition) $p(.) \in LH(R^n)$ if $p(.)$ satisfies

$$|p(x) - p(y)| \leq \frac{C}{-log(|x-y|)}, \quad |x-y| \leq \frac{1}{2}$$

and

$$|p(x) - p(y)| \leq \frac{C}{log|x| + e}, \quad |y| \leq |x|$$

Let $B_k = \{x \in R^n : |x| \leq 2^k\}, \quad A_k = B_k \backslash B_{k-1}, \quad k \in Z$ Let f be a locally integrable function on R^n. The n-dimensional Hardy operator is defined by

$$Hf(x) = \frac{1}{|x|^n} \int_{|t|<|x|} f(t) dt, \quad x \in R^n \backslash \{0\}.$$

In 1995, Christ and Grafakos [2] obtained the result for the boundedness of H on $L^p(R^n)$, $(1 < p < \infty)$ spaces, and they also found the exact operator norms of H on this space. In 2007, Fu et al. [8] gave the central BMO estimates for commutators of n-dimensional fractional and Hardy operators.

Now, we define the n-dimensional fractional Hardy-type operators of variable order $\beta(x)$ as follows.

Definition 1.1. Let f be a locally integrable function on R^n, $0 \leq \beta(x) < n$. The n-dimensional fractional Hardy-type operators of variable order $\beta(x)$ are defined by

$$H_{\beta(.)}f(x) = \frac{1}{|x|^{n-\beta(x)}} \int_{|t|<|x|} f(t)dt$$

and

$$H^*_{\beta(.)}f(x) = \int_{|t|\geq|x|} \frac{f(t)}{|t|^{n-\beta(x)}} dt$$

where $x \in R^n \backslash \{0\}$. Obviously, when $\beta(x) = 0$, $H_{\beta(.)}$ is just H and denote by $H^* = H^*_{\beta(.)} = H^*_0$. And when $\beta(x)$ is constant, $H_{\beta(.)}$ and $H^*_{\beta(.)}$ will become, H_β and H^*_β respectively.
We say that $\omega \in A(p,q)$ with $1 < p, q < \infty$, if there exists a constant $C > 0$, such that for any cube $Q \in R^n$,

$$(\frac{1}{|Q|}\int_Q \omega(x)^q dx)^{\frac{1}{q}} \ (\frac{1}{|Q|}\int_Q \omega(x)^{-p'} dx)^{\frac{1}{p'}} \leq C < \infty.$$

Let $\omega_r(\epsilon)$ be the integral modulus of continuity of order r of Φ defined by

$$\omega_r(\epsilon) = \sup_{|\rho|<\epsilon} (\int_{S^{n-1}} |\Phi(px') - \Phi(x')|^r d\sigma(x'))^{\frac{1}{r}},$$

where ρ is rotation in R^n and $|\rho| = \|\rho - I\|$.

Lemma 1.2. [1] Υ denote a family of ordered pairs of non-negative measurable functions (f, g). Assume that for some p_0 and q_0 with $0 < p_0 \leq q_0 < \infty$ and every weight $\omega \in A_1$,

$$(\int_{R^n} f(x)^{q_0}\omega(x)dx)^{\frac{1}{q_0}} \leq C_0(\int_{R^n} g(x)^{p_0}\omega(x)^{\frac{p_0}{q_0}}dx)^{\frac{1}{p_0}}, \ \ (f,g) \in \Upsilon$$

Given $p(.) \in \Delta^0(R^n)$ such that $p_0 < p^- \leq p^+ < p_0q_0 \backslash (q_0 - p_0)$, the function $q(.)$ is defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}$ for any $x \in R^n$. If $p(.) \in LH(R^n)$, then for any $(f, g) \in \Upsilon$ and $f \in L^{q(.)}(R^n)$, we have

$$\|f\|_{L^{q(.)}(R^n)} \leq C\|g\|_{L^{p(.)}(R^n)}.$$

Lemma1.3.[7] Suppose that $0 < \beta < n$, $1 \leq r' < p < n\backslash\beta$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. If $\Phi \in L^r(S^{n-1})$ and $\omega^{r'} \in A(\frac{p}{r'}, \frac{q}{r'})$, then there exists a constant C independent of f such that

$$(\int_{R^n} |T^\beta_\Phi f(x)\omega(x)|^q dx)^{\frac{1}{q}} \leq C(\int_{R^n} |f(x)\omega(x)|^p dx)^{\frac{1}{p}}.$$

2 Result and Discussion

Now let us declare and prove the theorem that gives boundedness of the fractional Hardy-Type integral.

Theorem 2.1. Let $p(.), q(.) \in \Delta(R^n)$, $0 < \beta < n$, $1 < p^- \leq p^+ < \frac{n}{\beta}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta}{n}$ for any $x \in R^n$. If $p(.) \in LH(R^n)$, $\Phi \in L^r(S^{n-1})$ and $1 \leq r' < p^-$, then

$$\|H^\beta_\Phi\|_{L^{q(.)}(R^n)} \leq C\|f\|_{L^{p(.)}(R^n)}$$

Proof. Choose $0 < p_0 \leq q_0 < \infty$ such that $r' < p_0 < p^-$ and $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\beta}{n}$. For any weight function $W(x) = \omega(x)^{q_0} \in A_1$ and any cube $Q \in R^n$ we have

$$\frac{1}{|Q|}\int_Q \omega(x)^{q_0} dx \leq C \inf_{x\in Q} \omega(x)^{q_0}$$

and

$$(\frac{1}{|Q|}\int_Q \omega(x)^{-r'(\frac{p_0}{r'})'} dx)^{\frac{1}{(\frac{p_0}{r'})'}} \leq \sup_{x\in Q} \omega(x)^{-r'} = (\inf_{x\in Q} \omega(x))^{-r'}.$$

These follow that

$$(\frac{1}{|Q|}\int_Q \omega(x)^(r'(\frac{q_0}{r'}))dx)^{\frac{r'}{q_0}} \ (\frac{1}{|Q|}\int_Q \omega(x)^{-r'(\frac{p_0}{r'})'} dx)^{\frac{1}{(\frac{p_0}{r'})'}} \leq C.$$

Thus we see that $\omega^{r'} \in A(\frac{p_0}{r'}, \frac{q_0}{r'})$. By Lemma 1.3, we obtain that

$$(\int_{R^n} |H^\beta_\Phi f(x)\omega(x|^{q_0} dx)^{\frac{1}{q_0}} \leq C(\int_{R^n} |f(x)\omega(x|^{p_0} dx)^{\frac{1}{p_0}}.$$

Finally, we choose the exponent function $p(.)$ and $q(.)$ such that $p_0 < p^- \leq p^+ < \frac{p_0 q_0}{(q_0 - p_0)}$, $\ p(.) \in LH(R^n)$ and for any $x \in R^n$

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}$$

By Lemma 1.2, we have

$$\|H_\Phi^\beta\|_{L^{q(\cdot)}(R^n)} \leq C\|f\|_{L^{p(\cdot)}(R^n)}.$$

This completes the proof of Theorem 2.1.

3 Conclusion

Under the given conditions, we obtained the boundary of the homogeneous fractional Hardy-type integral in variable exponential spaces. This method can also be applied to different operators and integrals.

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