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On Spectral Properties of Discontinuous Differential Operator with Second Order

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Abstract: In this work, we consider the spectral problem for a second-order discontinuous differential operator with a spectral parameter in the boundary condition in $L_p, 1 < p < \infty$. We study a method for establishing the basicity of eigenfunctions for such a problem. Such spectral problems arise while one solves the problem of a loaded string fixed at both ends with a load placed in the between ends of the string by the Fourier method.

Keywords: Completeness, Eigenfunctions, Minimality, Spectral problem.

1 Introduction

The spectral problems with discontinuity conditions inside the interval play an important role in mathematics, mechanics, physics and other fields of science. The applications of boundary value problems are related to discontinuous material properties.

The study of spectral properties of many discrete differential operators requires new methods for constructing a basis. This was the motivation for many mathematicians to study intensively the basis properties (such as completeness, minimality, basicity) of the systems of special functions mostly eigen and associated functions of differential operators. For this purpose, various methods were developed for these properties [1]-[8]. However, in the case of a discontinuous differential operator, a system of eigenfunctions emerges, which cannot be demonstrated the basicity properties by standard methods. An example of this situation has been the subject of our study.

In this paper, we consider the following spectral problem with a point of discontinuity

$$y''(x) + \lambda y(x) = 0, \qquad x \in (-1, 0) \cup (0, 1), \tag{1}$$

$$\begin{aligned} y(-1) = y(1) = 0 \\ y(-0) = y(+0) \\ y'(-0) - y'(+0) = \lambda m y(0) \end{aligned} \tag{2}$$

where λ is the spectral parameter,m is a non-zero complex number. This problem comes from the problem of vibrations of a loaded string with the fixed ends with a load placed in the middle of a string when the problem was solved by applying Fourier methods [[9]-[11]]. For these methods, basis properties of the eigenfunctions system should be studied suitable spaces of functions (generally Lebesgue spaces or Sobolev space).

Grand Lebesgue Spaces introduced by Iwaniec and Sbordone come from integrability properties of the Jacobian determinant ([12]), and the spaces play an important role in PDEs theory (see e.g. [13]) and in Functions Spaces Theory (see e.g. [14]). There are many applications in analysis, see [12]-[19]. These spaces attracted the interest of many researchers, either in Harmonic Analysis (see [20],[21]) and Interpolation-Extrapolation Theory ([22]) or in P.D.Es ([23],[24]).

In subsequent years, quite a number of problems in Harmonic Analysis and the theory of non- linear differential equations were studied in these spaces (see, e.g., the papers [25]-[29]). So, in this work, we study the basicity properties of the eigenfunctions system of the problem (1),(2) in grand Lebesgue spaces. For this purpose, at first, we find corresponding spaces dense in grand Lebesgue spaces. Then we denote that the eigenfunctions system of (1),(2) form a basis on these spaces.

2 Auxiliary informations

Let $\Omega \subset R^n, n \geq 2$ be a measurable set of Lebesgue measure $|\Omega| < +\infty$. In 1992, grand Lebesgue space $L^{n}(\Omega)$ was established by Iwaniec and Sbordone [12] as space such that

$$|Df| \in L^{n)}(\Omega) \Rightarrow |Jf| \in L^1_{loc}(\Omega)$$

for all Sobolev mappings $f : \Omega \to \mathbb{R}^n, f = (f_1, \ldots, f_n)$. After that we will use letter p instead of n, supposing $1 < p < +\infty$. Grand Lebesgue spaces are defined by

$$L^{p)}(\Omega) = \left\{ f \in M_0 : f_{p)} = \varrho(|f|) = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_\Omega |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < +\infty \right\},$$

where M_0 is the set of all real-valued measurable functions on Ω, M_0^+ show the subset nonnegative functions of M_0 and $\varrho : M_0^+ \to [0, +\infty]$ satisfy the following conditions for all $f, g, f_n (n = 1, 2, 3, \ldots), \lambda \leq 0$ constants, measurable subsets $E \subset \Omega$

- $\varrho(f) = 0 \Leftrightarrow f = 0$ a.e. in Ω,
- $\varrho(\lambda f) = \lambda \varrho(f)$,
- $\varrho(f + g) \leq \varrho(f) + \varrho(g)$
- $0 \leq g \leq f$ a.e. in $\Omega \Rightarrow \varrho(g) \leq \varrho(f)$,
- $0 \leq f_n \uparrow f$ a.e. in $\Omega \Rightarrow \varrho(f_n) \uparrow \varrho(f)$,
- $E \subset \Omega \Rightarrow \varrho(\chi_E) < +\infty$,
- $E \subset \Omega \Rightarrow \int_E f d\chi \leq C_E \varrho(f)$,

where $C_E, 0 < C_E < \infty$ depend on E and ϱ but not to f. Grand Lebesgue spaces are a special category of Banach Function spaces: The spaces are rearrangement-invariant:

$$\mu_f(\lambda) = |\{\chi \in \Omega : |f(\chi)| > \lambda\}|, \qquad for \quad all \quad \lambda \geq 0$$

it is $\varrho(f) = \varrho(g)$ if $\mu_f = \mu_g$. $L_{p)}$ is nonseparable spaces. Because for $\alpha \in \mathbb{R}$

$$f_\alpha(x) = \begin{cases} x^{-1/p}; & x \in [0, \alpha) \\ 0; & x \in [\alpha, 1] \end{cases}$$

functions satisfy the following inequality. For all $\alpha, \beta \in \mathbb{R}, (\alpha \neq \beta)$ there exists $\varepsilon_0 > 0$:

$$\|f_\alpha - f_\beta\|_{p)} \geq \varepsilon_0 > 0,$$

so $L_{p)}(0, 1)$ is nonseparable spaces. But these spaces must be separable so that we can talk about basicity properties. That's why we should study on separable subspaces of $L_{p)}$. Thus, for $\delta > 0$ we give shift operatör in $L_{p)}$,

$$(T_\delta f)(x) = \begin{cases} f(x + \delta); & x + \delta \in [0, 1] \\ 0; & x + \delta \notin [0, 1] \end{cases},$$

where $f \in L_{p)}(0, 1)$. Let us define the following set

$$\widetilde{G}^{p)}(0, 1) = \{f \in L_{p)}(0, 1) : \|T_\delta f - f\|_{p)} \to 0, \delta \to 0\}$$

then it is evident that

$$\overline{(\widetilde{G}^{p)}, \|.\|_{p)})} = G_{p)} \subset L_{p)}.$$

Hence we can express the following lemma.

Lemma 1. *For $1 < p < \infty$, the following expressions are true.*

1. $\overline{(C_0^\infty, \|.\|_{p)})} = G_{p)}$;
2. $\overline{(C_0^\infty, \|.\|_p)} = L_p$

The proof of Lemma 1 can be easily shown.

Let us mention the continuous embedding and we can give the following inclusions from [30]

$$L_p \subsetneq L_{p)} \subsetneq L_{p-\varepsilon}, \qquad 0 < \varepsilon < p - 1.$$

Then we conclude that

$$L_p \subsetneq G_{p)} \subsetneq L_{p)} \subsetneq L_{p-\varepsilon}, \qquad 0 < \varepsilon < p - 1.$$

Because we have the following example. Let us consider the series

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{n^2},$$

where $f_n(x) = \begin{cases} x^{-1/p}, & x \in (e^{-n^{2p}}; 1] \\ 0, & x \notin (e^{-n^{2p}}; 1] \end{cases}$ .Here $f \notin L_p(0, 1)$ since $\|f_n\|_p^p = n^{2p}$.

Let us denote $f \in G_{p)}(0,1)$.

$$\|f_n\|_{p)} \le \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1 x^{\frac{-1}{p}(p-1)}dx\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1} p^{\frac{1}{p-\varepsilon}} = p,$$

and we use the partial sum of the series,

$$S_m(x) = \sum_{n=1}^{m} \frac{f_n(x)}{n^2}.$$

From here

$$\left\|\sum_{n=m}^{m+p} \frac{f_n(x)}{n^2}\right\|_{p)} \le \sum_{n=m}^{m+p} \frac{\|f_n(x)\|_{p)}}{n^2} < p\sum_{n=m}^{m+p}\frac{1}{n^2} < +\infty,$$

then $f \in G_{p)}(0,1)$. Thus

$$L_p(0,1) \subsetneq G_{p)}(0,1)$$

and from definitions

$$G_{p)}(0,1) \subsetneq L_{p)}(0,1)$$

We conclude that $\overline{G_{p)}} = L_{p)}$ and from Lemma 1, $G_{p)}$ is separable for $1 < p < \infty$.

Let us recall the definition of completeness, minimality, basicity and theirs criterions from [31] in any Banach space. Let X be a Banach space.

"A system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called **complete** in X if $\overline{L[\{x_n\}_{n\in\mathbb{N}}]} = X$."

Completeness Criterion. Let X be a normed space. A system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is complete in X if and only if for all $f \in X^* :< x_n, f >= 0$ for each $n \in \mathbb{N}$ implies $f = 0$.

"A system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called **minimal** in X if $x_k \notin \overline{L[\{x_n\}_{n\in\mathbb{N}_k}]}$ for all $k \in \mathbb{N}$, where $\mathbb{N}_k = \mathbb{N}\backslash\{k\}$."

"Systems $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{x_n^*\}_{n\in\mathbb{N}} \subset X^*$ are called **biorthogonal** if $< x_m, x_n^* >= \delta_{nm}$ for all $n, m \in \mathbb{N}$."

Minimality Criterion. *A system in a Banach space is minimal if and only if it has a biorthogonal system.*

Minimality Criterion. *A system $\{x_n\}_{n\in\mathbb{N}} \subset X$ form a basis for X if and only if the following conditions are satisfied:*

1. $\{x_n\}_{n\in\mathbb{N}}$ *is complete in* X*;*
2. $\{x_n\}_{n\in\mathbb{N}}$ *is minimal in* X*;*
3. *The projectors* $P_m(.) = \sum_{n=1}^m < ., x_n^* > x_n$ *are uniformly bounded, i.e., there exists* $M > 0$ *such that*

$$\|P_m x\|_X \le M\|x\|_X, \qquad \forall x \in X,$$

where $\{x_n^*\}_{n\in\mathbb{N}} \subset X^*$ *is a system biorthogonal to* $\{x_n\}_{n\in\mathbb{N}}$.

Let's give the Dirac delta functional that can find from many sources.

$$\delta(x) = \begin{cases} +\infty; & x = 0 \\ 0; & x \neq 0 \end{cases}$$

imposing that

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1$$

For δ to satisfy the above property, we define δ_ε as

$$\int_{-\infty}^{+\infty} \delta(x)dx = \lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty} \delta_\varepsilon(x)dx,$$

where δ_ε is a generic function of both x and ε such that

$$\lim_{\varepsilon\to 0^+} \delta_\varepsilon(x) = \begin{cases} +\infty; & x = 0 \\ 0; & x \neq 0 \end{cases},$$

and

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(x)dx = 1.$$

From here

$$\int_{-\infty}^{+\infty} \delta(x) f(x) dx = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} \delta_\varepsilon(x) f(x) dx,$$

for any function $f(x)$ and

$$\int_{-\infty}^{+\infty} \delta(x-c) f(x) dx = f(c).$$

Also, we will use the Muckenhoupt condition [32] in this work. So we mention Hardy's inequality for $1 \le p \le \infty$ and $bp < -1$,

$$\left[\int_0^\infty \left| x^b \int_0^x f(t)dt \right|^p dx\right]^{\frac{1}{p}} \le \frac{-p}{bp+1} \left[\int_0^\infty |x^{b+1} f(x)|^p dx\right]^{\frac{1}{p}}.$$

Later several authors such as Tomaselli, Talenti and Artola investigated the problem of for what functions, $U(x)$ and $V(x)$, there is a finite constant C such that

$$\left[\int_0^\infty \left| U(x) \int_0^x f(t)dt \right|^p dx\right]^{\frac{1}{p}} \le C \left[\int_0^\infty |V(x) f(x)|^p dx\right]^{\frac{1}{p}}. \tag{3}$$

where $U(x)$ and $V(x)$ are weight functions. In 1972, Muckenhoupt gives a condition for the inequality (3):

Theorem 1. *[32] If $1 \le p \le \infty$, there is a finite C for which* (3) *is true if and only if*

$$\sup_{r>0} \left[\int_r^\infty |U(x)|^p dx\right]^{\frac{1}{p}} \left[\int_0^r |V(x)|^{-p'} dx\right]^{\frac{1}{p'}} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Now we need to give some notation and results from [33] that will use throughout the paper.

Let us take $\lambda = \rho^2$ and denote the following designation for boundary forms of (2)

$$U_v(y) = U_{v1}(y) + U_{v2}(y), \qquad v = \overline{1,4}$$

where

$$\begin{aligned} U_{11} = y(-1) \qquad U_{12} \equiv 0, \\ U_{21} \equiv 0 \qquad U_{22} = y(1), \\ U_{31} = y(0-) \qquad U_{32} = y(0+), \\ U_{41} = y'(0-) \qquad U_{42} = -y'(0+) - \lambda m y(0). \end{aligned}$$

Lemma 2. *[33] Spectral problem* (1),(2) *has two series of simple eigenvalues:*
$\lambda_{1,n} = (\pi n)^2, \quad n = 1, 2, ...$ *and* $\lambda_{2,n} = (\rho_{2,n})^2, \quad n = 0, 1, 2, ...$ *where* $\rho_{2,n}$*has asymptotic form*

$$\rho_{2,n} = \pi n + \frac{2}{\pi m n} + o\left(\frac{1}{n^2}\right).$$

The eigenfunctions $u_n(x), \quad n = 0, 1, 2, \ldots$ *prescribed by formula*

$$u_{2n-1}(x) = \sin \pi n x, \qquad n = 1, 2, ...,$$

$$u_{2n}(x) = \begin{cases} \sin \rho_{2,n}(1+x) & at x \in [-1, 0] \\ \sin \rho_{2,n}(1-x) & at x \in [0, 1] \end{cases}$$

correspond to them.

Lemma 3. *[33] For Green function components* $G_{kj}(x, \xi, \rho)$ *the following expressions*

$$G_{11}(x,\xi,\rho) = \begin{cases} -\frac{1}{\rho}\sin\rho(x-\xi) + \frac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1+\xi) - \frac{1}{\rho\sin\rho}\sin\rho(1+x)\sin\rho\xi, & -1 \le \xi < x \le 0 \\ \frac{1}{\rho}\sin\rho(x-\xi) + \frac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1+\xi) - \frac{1}{\rho\sin\rho}\sin\rho x\sin\rho(1+\xi), & -1 \le x \le \xi \le 0 \end{cases};$$

$$G_{22}(x,\xi,\rho) = \begin{cases} -\frac{1}{\rho}\sin\rho(x-\xi) + \frac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1-\xi) + \frac{1}{\rho\sin\rho}\sin\rho x\sin\rho(1-\xi), & 0 \le \xi < x \le 1 \\ \frac{1}{\rho}\sin\rho(x-\xi) + \frac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1-\xi) - \frac{1}{\rho\sin\rho}\sin\rho(1-x)\sin\rho\xi, & 0 \le x \le \xi \le 1 \end{cases};$$

$$G_{12}(x,\xi,\rho) = \frac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1-\xi), \quad x \in [-1,0], \xi \in [0,1];$$

$$G_{21}(x,\xi,\rho) = \frac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1+\xi), \quad x \in [0,1], \xi \in [-1,0].$$

Let (a,b) be an interval on $\mathbb{R}$ and let us define G-Sobolev spaces

$$GW^{p)}(a,b)=\left\{f,f'\in G^{p)}(a,b):\|f\|_{W_{p)}}=\|f\|_{p)}+\|f'\|_{p)}\right\}.$$

$GW^{p)}(-1,0)\times GW^{p)}(0,1)$ denotes a space functions whose shrinkages on intervals $[-1,0]$ and $[0,1]$ belong respectively to G-Sobolev Spaces $GW^{p)}(-1,0)$ and $GW^{p)}(0,1)$. We define the operator L in $G_{p)}(-1,1)$ spaces as

$$D(L)=\left\{\hat{u}\in G_{p)}(-1,1)\oplus\mathbb{C}:\hat{u}=(u,mu(0));u\in W_G^{p)};u(-1)=u(1)=0;u(0-)=u(0+)\right\} \tag{4}$$

where $W_G^{p)}=GW^{p)}(-1,0)\times GW^{p)}(0,1)$ and for $\hat{u}\in D(L)$

$$L\hat{u}=\left(-u'';u'(0-)-u'(0+)\right). \tag{5}$$

Let us take the following equation to construct the resolvent of L.

$$L\hat{u}-\lambda\hat{u}=\hat{f}, \tag{6}$$

where $\hat{u}\in D(L),\hat{f}=(f;\beta)\in G_{p)}(-1,1)\oplus\mathbb{C}$. This equation can be expressed as follows.

$$\begin{cases}-u''=\lambda u+f,\\ u'(0-)-u'(0+)-\lambda mu(0)=\beta,\\ U_v(u)=0,\quad v=1,2,3\end{cases} \tag{7}$$

We shall use the following Lemma to prove basicity in grand Lebesgue spaces.

Lemma 4. *[33] For solution $\hat{u}=(u;mu(0))$ of the equation (6) it holds the following representations*

$$\begin{aligned}u(x,\rho)=&\frac{\beta\sin\rho(1+x)}{\rho(2\cos\rho-\rho m\sin\rho)}-\frac{1}{\rho}\int_{-1}^{x}f(\xi)\sin\rho(x-\xi)d\xi+\frac{1}{\rho}\int_{x}^{0}f(\xi)\sin\rho(x-\xi)d\xi+\\&+\frac{1}{\Delta(\rho)}\int_{-1}^{0}f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)d\xi-\frac{1}{\rho\sin\rho}\int_{-1}^{x}f(\xi)\sin\rho(1+x)\sin\rho\xi d\xi-\\&-\frac{1}{\rho\sin\rho}\int_{x}^{0}f(\xi)\sin\rho x\sin\rho(1+\xi)d\xi+\frac{1}{\Delta(\rho)}\int_{0}^{1}f(\xi)\sin\rho(1+x)\sin\rho(1-\xi)d\xi,\end{aligned} \tag{8}$$

if $x\in[-1,0]$;

$$\begin{aligned}u(x,\rho)=&\frac{\beta\sin\rho(1-x)}{\rho(2\cos\rho-\rho m\sin\rho)}-\frac{1}{\rho}\int_{0}^{x}f(\xi)\sin\rho(x-\xi)d\xi+\frac{1}{\rho}\int_{x}^{1}f(\xi)\sin\rho(x-\xi)d\xi+\\&+\frac{1}{\Delta(\rho)}\int_{0}^{1}f(\xi)\sin\rho(1-x)\sin\rho(1-\xi)d\xi+\frac{1}{\rho\sin\rho}\int_{0}^{x}f(\xi)\sin\rho x\sin\rho(1-\xi)d\xi+\\&+\frac{1}{\rho\sin\rho}\int_{x}^{1}f(\xi)\sin\rho(1-x)\sin\rho\xi d\xi+\frac{1}{\Delta(\rho)}\int_{-1}^{0}f(\xi)\sin\rho(1-x)\sin\rho(1+\xi)d\xi,\end{aligned} \tag{9}$$

if $x\in[0,1]$;

$$u(0,\rho)=\frac{1}{\rho(2\cos\rho-\rho m\sin\rho)}\left[\beta\sin\rho+\int_{-1}^{0}f(\xi)\sin\rho(1+\xi)d\xi+\int_{0}^{1}f(\xi)\sin\rho(1-\xi)d\xi\right]. \tag{10}$$

Finally, let us give the Riesz theorem, which we will apply to the Hilbert transformation. This theorem can be reached from many sources.

Theorem 2. *(Riesz Theorem) Let $\Gamma\in L^p(X,\mu)^*$, where $1\le p<\infty$ and μ is $\sigma-$finite. Then if $\frac{1}{p}+\frac{1}{q}=1$, there exists a unique $g\in L^q(X,\mu)^*$ such that*

$$\Gamma(f)=\int_X fgd\mu=\Phi_g(f).$$

Moreover $\|\gamma\|=\|g\|_q$.

3 Main results

Lemma 5. *The operator defined by* (4),(5) *is a linear closed operator with dense definitional domain in* $G_{p)}(-1,1)\oplus\mathbb{C}$. *Eigenfunctions of the operator L and problem* (1),(2) *overlap, and* $\hat{u}_k$ *are eigenvectors of the operator L, where* $\hat{u}_{2n-1}=(u_{2n-1}(x);0)\hat{u}_{2n}=(u_{2n}(x);m\sin\rho_{2,n})$.

Proof: For the proof of dense, we take $\hat{u}=(u;\alpha)\in G_{p)}(-1,1)\oplus\mathbb{C}$ and define functional $F(\hat{u})$ as follows

$$F(\hat{u})=mu(0)-\alpha.$$

Assume that

$$U_v(\hat{u})=U_v(u),\quad v=1,2,3.$$

Let us show that F and U_v are bounded linear functionals on $W_G^{p)}\oplus\mathbb{C}$, but unbounded on $G_{p)}(-1,1)\oplus\mathbb{C}$. For boundedness of F and $U_v, v=1,2,3$ it is sufficient to prove that $\delta_{x_0}(f)=f(x_0)$ Dirac functional is bounded on $W_G^{p)}$ where $x_0\in(-1,1)$ is any fixed point. For any $f\in W_G^{p)}$,

$$|f(x_0)|=\left|\int_{x_0}^{x}f'(t)dt-f(x)\right|\le\int_{x_0}^{x}|f'(t)|dt+|f(x)|$$

$$2|f(x_0)|\le\int_{-1}^{1}\int_{x_0}^{x}|f'(t)|dtdx+\int_{-1}^{1}|f(x)|dx\le 2\int_{-1}^{1}|f'(t)|dt+\int_{-1}^{1}|f(x)|dx\le(2\|f'\|_{p-\varepsilon_0}+$$

$$\|f\|_{p-\varepsilon_0}2^{1-\frac{1}{p-\varepsilon_0}}\le 2^{2-\frac{1}{p-\varepsilon_0}}\varepsilon_0^{\frac{1}{p-\varepsilon_0}}(\|f'\|_{p)}+\|f\|_{p)}),$$

then

$$|\delta_{x_0}(f)|\le 2^{2-\frac{1}{p-\varepsilon_0}}\varepsilon_0^{\frac{1}{p-\varepsilon_0}}\|f\|_{W_G^{p)}}.$$

So δ_{x_0} is bounded on $W_G^{p)}$ but unbounded on $G_{p)}(-1,1)$ because for $f\in G_{p)}(-1,1)$,

$$\|f\|_{p)}\le(p-1)2^{\frac{p-1}{p}}\|f\|_p=C_p\|f\|_p,$$

then for $g\in L_p(-1,1)$

$$\sup_{\|g\|_p\le 1}|\delta_{x_0}(g)|=\sup_{C_p\|f\|_p\le 1}|\delta_{x_0}(C_pf)|=C_p\sup_{C_p\|f\|_p\le 1}|\delta_{x_0}(f)|\le C_p\sup_{\|f\|_{p)}\le 1}|\delta_{x_0}(f)|.$$

We conclude that δ_{x_0} is unbounded on $W_G^{p)}(-1,1)$ since it is unbounded on $L_p(-1,1)$ [10]. It is evident that $F,U_v,v=1,2,3$ are bounded on $W_G^{p)}\oplus\mathbb{C}$ and unbounded on $G^{p)}(-1,1)\oplus\mathbb{C}$. Therefore the set

$$D(L)=\left\{\hat{u}=(u,\alpha):u\in W_G^{p)};F(\hat{u})=U_v(\hat{u})=0,v=1,2,3\right\}$$

is everywhere dense in $G^{p)}(-1,1)\oplus\mathbb{C}$ and L is a closed operator as a contraction of the corresponding closed maximal operator. The second part of the lemma is certified directly. □

Theorem 3. *Eigenvectors of operator L form a basis in spaces* $G^{p)}(-1,1)\oplus\mathbb{C},1<p<\infty$.

4 Conclusion

In this study, the problem (1),(2) is discussed in grand spaces and basic properties are examined. It is foreseen that these properties can be examined in more general cases of this problem as arbitrary point for discontinuity.

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