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Neutrosophic Metric Spaces and Fixed Point Results

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Abstract: In this paper, we define the neutrosophic contraction mapping and give a fixed point theorem in neutrsophic metric spaces.

Keywords: Fixed point theorem, Neutrosophic contraction, Neutrosophic metric spaces.

1 Introduction

Fuzzy Sets (FSs) put forward by Zadeh [23] has influenced deeply all the scientific fields since the publication of the paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [1] initiated Intuitionistic fuzzy sets (IFSs) for such cases. Neutrosophic set (NS) is a new version of the idea of the classical set which is defined by Smarandache [17]. Examples of other generalizations are FS [23] interval-valued FS [19], IFS [1], interval-valued IFS [2], the sets paraconsistent, dialetheist, paradoxist, and tautological [18], Pythagorean fuzzy sets [21] .

Using the concepts Probabilistic metric space and fuzzy, fuzzy metric space (FMS) is introduced in [12]. Kaleva and Seikkala [8] have defined the FMS as a distance between two points to be a non-negative fuzzy number. In [5] some basic properties of FMS studied and the Baire Category Theorem for FMS proved. Further, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [6]. Afterward, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision-making et al. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [14] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [5] idea of applying t-norm and t-conorm to the FMS meanwhile defining IFMS and studying its basic features.

Bera and Mahapatra defined the neutrosophic soft linear spaces (NSLSs) [3]. Later, neutrosophic soft normed linear spaces(NSNLS) has been defined by Bera and Mahapatra [4]. In [4], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were studied.

New metric space was defined which is called Neutrosophic metric Spaces (NMS) from the idea of neutrosophic sets [11]. In [11], some properties of NMS such as open set, Hausdorff, neutrosophic bounded, compactness, completeness, nowhere dense are investigated. Also we give Baire Category Theorem and Uniform Convergence Theorem for NMSs.

In this paper, fixed point results for NMSs are given.

2 Preliminaries

Some definitions related to the fuzziness, intuitionistic fuzziness and neutrosophy are given as follows:

The fuzzy subset F of $\mathbb{R}$ is said to be a fuzzy number(FN). The FN is a mapping $F : \mathbb{R} \to [0, 1]$ that corresponds to each real number a to the degree of membership $F(a)$.
Let F is a FN. Then, it is known that [9]

- If $F(a_0) = 1$, for $a_0 \in \mathbb{R}$, F is said to be normal,
- If for each $\mu > 0$, $F^{-1}\{[0, \tau + \mu)\}$ is open in the usual topology $\forall \tau \in [0, 1)$, F is said to be upper semi continuous, ,
- The set $[F]^{\tau} = \{a \in \mathbb{R} : F(a) \geq \tau\}$, $\tau \in [0, 1]$ is called $\tau-$cuts of F.

Choose non-empty set F. An IFS in F is an object U defined by

$$U = \{< a, G_U(a), Y_U(a) >: a \in F\}$$

where $G_U(a) : F \to [0,1]$ and $Y_U(a) : F \to [0,1]$ are functions for all $a \in F$ such that $0 \leq G_U(a) + Y_U(a) \leq 1$ [1]. Let U be an IFN. Then,

- an IF subset of the $\mathbb{R}$,
- If $G_U(a_0) = 1$ and, $Y_U(a_0) = 0$ for $a_0 \in \mathbb{R}$, normal,
- If $G_U(\lambda a_1 + (1-\lambda)a_2) \geq \min(G_U(a_1), G_U(a_2))$, $\forall a_1, a_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, then the membership function(MF) $G_U(a)$ is called convex,
- If $Y_U(\lambda a_1 + (1-\lambda)a_2) \geq \min(Y_U(a_1), Y_U(a_2))$, $\forall a_1, a_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, then the nonmembership function(NMF)$Y_U(a)$ is concav,
- G_U is upper semi continuous and Y_U is lower semi continuous
- $suppU = cl(\{a \in F : Y_U(a) < 1\})$ is bounded.

An IFS $U = \{< a, G_U(a), Y_U(a) >: a \in F\}$ such that $G_U(a)$ and $1 - Y_U(a)$ are FNs, where $(1 - Y_U)(a) = 1 - Y_U(a)$, and $G_U(a) + Y_U(a) \leq 1$ is called an IFN.

Let's consider that F is a space of points(objects). Denote the $G_U(a)$ is a truth-MF, $B_U(a)$ is an indeterminacy-MF and $Y_U(a)$ is a falsity-MF, where U is a set in F with $a \in F$. Then, if we take $I =]0^-, 1^+[$

$$G_U(a) : F \to I,$$
$$B_U(a) : F \to I,$$
$$Y_U(a) : F \to I,$$

There is no restriction on the sum of $G_U(a)$, $B_U(a)$ and $Y_U(a)$. Therefore,

$$0^- \leq \sup G_U(a) + \sup B_U(a) + \sup Y_U(a) \leq 3^+.$$

The set U which consist of with $G_U(a)$, $B_U(a)$ and $Y_U(a)$ in F is called a neutrosophic sets(NS) and can be denoted by

$$U = \{< a, (G_U(a), B_U(a), Y_U(a)) >: a \in F, G_U(a), B_U(a), Y_U(a) \in I\} \tag{1}$$

Clearly, NS is an enhancement of $[0,1]$ of IFSs.

An NS U is included in another NS V, ($U \subseteq V$), if and only if,

$$\inf G_U(a) \leq \inf G_V(a), \quad \sup G_U(a) \leq \sup G_V(a),$$
$$\inf B_U(a) \geq \inf B_V(a), \quad \sup B_U(a) \geq \sup B_V(a),$$
$$\inf Y_U(a) \geq \inf Y_V(a), \quad \sup Y_U(a) \geq \sup Y_V(a).$$

for any $a \in F$. However, NSs are inconvenient to practice in real problems. To cope with this inconvenient situation, Wang et al [20] customized NS's definition and single-valued NSs (SVNSs) suggested.

To cope with this inconvenient situation, Wang et al [20] customized NS's definition and single-valued NSs suggested. Ye [22], described the notion of simplified NSs, which may be characterized by three real numbers in the $[0,1]$. At the same time, the simplified NSs' operations may be impractical, in some cases [22]. Hence, the operations and comparison way between SNSs and the aggregation operators for simplified NSs are redefined in [15].

According to the Ye [22], a simplification of an NS U, in (1), is

$$U = \{< a, (G_U(a), B_U(a), Y_U(a)) >: a \in F\},$$

which called an simplified NS. Especially, if F has only one element $< G_U(a), B_U(a), Y_U(a) >$ is said to be an simplified NN. Expressly, we may see simplified NSs as a subclass of NSs.

An simplified NS U is comprised in another simplified NS V ($U \subseteq V$), iff $G_U(a) \leq G_V(a)$, $B_U(a) \geq B_V(a)$ and $Y_U(a) \geq Y_V(a)$ for any $a \in F$. Then, the following operations are given by Ye[22]:

$$\begin{aligned}
U + V &= \langle G_U(a) + G_V(a) - G_U(a).G_V(a), B_U(a) + B_V(a) - B_U(a).B_V(a), Y_U(a) + Y_V(a) - Y_U(a).Y_V(a)\rangle,\\
U.V &= \langle G_U(a).G_V(a), B_U(a).B_V(a), Y_U(a).Y_V(a)\rangle,\\
\alpha.U &= \langle 1 - (1 - G_U(a))^\alpha, 1 - (1 - B_U(a))^\alpha, 1 - (1 - Y_U(a))^\alpha\rangle \qquad for \quad \alpha > 0,\\
U^\alpha &= \langle G_U^\alpha(a), B_U^\alpha(a), Y_U^\alpha(a)\rangle \qquad for \quad \alpha > 0.
\end{aligned}$$

Triangular norms (t-norms) (TN) were initiated by Menger [13]. In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers for distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations(intersections and unions).

Definition 1. *Give an operation $\circ : [0,1] \times [0,1] \to [0,1]$. If the operation $\circ$ is satisfying the following conditions, then it is called that the operation $\circ$ is* continuous TN*: For $s, t, u, v \in [0,1]$,*

i. $s \circ 1 = s$
ii. *If* $s \leq u$ *and* $t \leq v$, *then* $s \circ t \leq u \circ v$,
iii. $\circ$ *is continuous,*
iv. $\circ$ *is commutative and associative.*

Definition 2. *Give an operation* $\bullet : [0,1] \times [0,1] \rightarrow [0,1]$. *If the operation* $\bullet$ *is satisfying the following conditions, then it is called that the operation* $\bullet$ *is* continuous TC*:*

i. $s \bullet 0 = s$,
ii. *If* $s \leq u$ *and* $t \leq v$, *then* $s \bullet t \leq u \bullet v$,
iii. $\bullet$ *is continuous,*
iv. $\bullet$ *is commutative and associative.*

Form above definitions, we note that if we choose $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 > \varepsilon_2$, then there exist $0 < \varepsilon_3, \varepsilon_4 < 0, 1$ such that $\varepsilon_1 \circ \varepsilon_3 \geq \varepsilon_2$, $\varepsilon_1 \geq \varepsilon_4 \bullet \varepsilon_2$. Further, if we choose $\varepsilon_5 \in (0,1)$, then there exist $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \circ \varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_7 \bullet \varepsilon_7 \leq \varepsilon_5$.

Definition 3. *[11] Take* F *be an arbitrary set,* $V = \mathcal{N} = \{< a, G(a), B(a), Y(a) >: a \in F\}$ *be a NS such that* $\mathcal{N} : F \times F \times \mathbb{R}^+ \rightarrow [0,1]$. *Let* $\circ$ *and* $\bullet$ *show the continuous TN and continuous TC, respectively. The four-tuple* $(F, \mathcal{N}, \circ, \bullet)$ *is called neutrosophic metric space(NMS) when the following conditions are satisfied.* $\forall a, b, c \in F$,

i. $0 \leq G(a,b,\lambda) \leq 1, \quad 0 \leq B(a,b,\lambda) \leq 1, \quad 0 \leq Y(a,b,\lambda) \leq 1 \quad \forall \lambda \in \mathbb{R}^+$,
ii. $G(a,b,\lambda) + B(a,b,\lambda) + Y(a,b,\lambda) \leq 3$, *(for* $\lambda \in \mathbb{R}^+$*),*
iii. $G(a,b,\lambda) = 1$ *(for* $\lambda > 0$*) if and only if* $a = b$,
iv. $G(a,b,\lambda) = G(b,a,\lambda)$ *(for* $\lambda > 0$*),*
v. $G(a,b,\lambda) \circ G(b,c,\mu) \leq G(a,c,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$,
vi. $G(a,b,.) : [0,\infty) \rightarrow [0,1]$ *is continuous,*
vii. $lim_{\lambda \rightarrow \infty} G(a,b,\lambda) = 1 \quad (\forall \lambda > 0)$,
viii. $B(a,b,\lambda) = 0$ *(for* $\lambda > 0$*) if and only if* $a = b$,
ix. $B(a,b,\lambda) = B(b,a,\lambda)$ *(for* $\lambda > 0$*),*
x. $B(a,b,\lambda) \bullet B(b,c,\mu) \geq B(a,c,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$,
xi. $B(a,b,.) : [0,\infty) \rightarrow [0,1]$ *is continuous,*
xii. $lim_{\lambda \rightarrow \infty} B(a,b,\lambda) = 0 \quad (\forall \lambda > 0)$,
xiii. $Y(a,b,\lambda) = 0$ *(for* $\lambda > 0$*) if and only if* $a = b$,
xiv. $Y(a,b,\lambda) = Y(b,a,\lambda) \quad (\forall \lambda > 0)$,
xv. $Y(a,b,\lambda) \bullet Y(b,c,\mu) \geq Y(a,c,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$,
xvi. $Y(a,b,.) : [0,\infty) \rightarrow [0,1]$ *is continuous,*
xvii. $lim_{\lambda \rightarrow \infty} Y(a,b,\lambda) = 0$ *(for* $\lambda > 0$*),*
xviii. *If* $\lambda \leq 0$, *then* $G(a,b,\lambda) = 0$, $B(a,b,\lambda) = 1$ *and* $Y(a,b,\lambda) = 1$.

Then $\mathcal{N} = (G, B, Y)$ *is called Neutrosophic metric(NM) on* F.

The functions $G(a,b,\lambda), B(a,b,\lambda), Y(a,b,\lambda)$ denote the degree of nearness, the degree of neutralness and the degree of non-nearness between a and b with respect to λ, respectively.

Definition 4. *[11] Give* V *be a NMS,* $0 < \varepsilon < 1$, $\lambda > 0$ *and* $a \in F$. *The set* $O(a,\varepsilon,\lambda) = \{b \in F : G(a,b,\lambda) > 1 - \varepsilon, \quad B(a,b,\lambda) < \varepsilon, \quad Y(a,b,\lambda) < \varepsilon\}$ *is said to be the open ball (OB) (center* a *and radius* ε *with respect to* λ*).*

Lemma 1. *[11] Every OB* $O(a,\varepsilon,\lambda)$ *is an open set (OS).*

3 Fixed point results

Definition 5. *[7] Let* F *be a set. A non-negative real-valued function* f *on* $F \times F$ *is called as a quasi-metric on* F *if it satisfies the following axioms:*

i. $f(a,b) = f(b,a) = 0$ *if and only if* $a = b$,
ii. $f(a,b) \leq f(a,c) + f(c,b)$,

for all $a, b, c \in F$.

From this definition we can understand: It is possible $f(a,b) \neq f(b,z)$ for some $a, b \in F$.

A quasi-metric is a distance function which satisfies the triangle inequality but is not symmetric in general. Quasi-metrics are a subject of comprehensive investigation both in pure and applied mathematics in areas such as in functional analysis, topology and computer science.

Proposition 1. *Let* V *be the NMS. For any* $\varepsilon \in (0,1]$, *define* $h : F \times F \rightarrow R^+$ *as follows:*

$$h_\varepsilon(a,b) = \inf\{\lambda > 0 : G(a,b,\lambda) > 1 - \varepsilon, \quad B(a,b,\lambda) < \varepsilon, \quad Y(a,b,\lambda) < \varepsilon\}$$

Then,

i. $(F, h_\varepsilon : \varepsilon \in (0,1])$ *is a generating space of quasi-metric family.*
ii. *The topology* $\tau_{\mathcal{N}}$ *on* $(F, h_\varepsilon : \varepsilon \in (0,1])$ *coincides with the* $\mathcal{N}-$*topology on* V*, that is,* h_ε *is a compatible symmetric for* $\tau_{\mathcal{N}}$*.*

Definition 6. *Let* V *be a NMS. The mapping* $f : F \to F$ *is called neutrosophic contraction(NC) if there exists* $k \in (0,1)$ *such that*

$$\frac{1}{G(f(a), f(b), \lambda)} - 1 \leq k(\frac{1}{G(a,b,\lambda) - 1}), \quad B(f(a), f(b), \lambda) \leq kB(a,b,\lambda), \quad Y(f(a), f(b), \lambda) \leq kY(a,b,\lambda)$$

for each $a, b \in F$ *and* $\lambda > 0$*.*

Definition 7. *Let* V *be a NMS and let* $f : F \to F$ *be a NC mapping. Then there exists* $c \in F$ *such that* $c = f(c)$*. That is,* c *is called neutrosophic fixed point (NFP) of* f*.*

Generally, we claim that the contractions have fixed point. If all contractions(including NC) have fixed points, then we can easily say that f^2 should have a fixed point. In below proposition, we will show that if f^n is a NC then, f^n has fixed point.

Proposition 2. *Suppose that* f *is a NC. Then* f^n *is also a NC. Furthermore, if* k *is the constant for* f*, then* k^n *is the constant for* f^n*.*

Remark 1. *From Proposition* 2*, we can say that each* f^n *has the same fixed point. Because, if we take* $f(a) = a$*, then* $f^2 = f(f(a)) = f(a) = a$ *and by induction,* $f^n(a) = a$*.*

Proposition 3. *Let* f *be a NC and* $a \in F$*.* $f[O(a, \varepsilon, \lambda)] \subset O(a, \varepsilon, \lambda)$ *for large enough values of* ε*.*

Remark 2. *From Proposition* 3 *and the definitions neutrosophic open ball and neutrosophic closed ball, if the inclusion* $f[O(a, \varepsilon, \lambda)] \subset O(a, \varepsilon, \lambda)$ *is hold, then the inclusion also* $\overline{f[O(a, \varepsilon, \lambda)]} \subset \overline{O(a, \varepsilon, \lambda)}$ *is hold.*

Proposition 4. *The inclusion* $f^n[O(a, \varepsilon, \lambda)] \subset O(f^n(a), \epsilon, \lambda)$ *is hold for all* n*, where* $\epsilon = k^n \times \varepsilon$*.*

Remark 3. *It is fact that if the inclusion* $f^n[O(a, \varepsilon, \lambda)] \subset O(f^n(a), \epsilon, \lambda)$ *is hold, then the inclusion also* $\overline{f^n[O(a, \varepsilon, \lambda)]} \subset \overline{O(f^n(a), \epsilon, \lambda)}$ *is hold.*

Propositions 2-4 are proved as similar in [10].

Theorem 1. *Let* V *be a complete NMS. Let* $f : F \to F$ *be a NC mapping. Then,* f *has a unique NFP.*

Theorem 1 is a consequence of Theorem 3.6 in [16]. Hence, using the consept of neutrosophy, Theorem 1 is proved as similar Theorem 3.6 in [16].

4 Conclusion

The purpose of this paper is to apply the NMS which defined by Kirisci and Simsek [11]. NC mapping is defined. After the properties related to NC are proved, fixed point theorem is given.

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